Fourier Transforms and Physical Dualities.

(UIC Seminar, 17 Feb 2020)

Say, don't write.

Context: T-duality.

In physics, two theories are said to be "dual" to each other if they are secretly just different mathematical presentations of the same physical system. The translation scheme between the two descriptions can be highly nontrivial, and therefore useful - intractible calculations on one side may become easy on the other.

The duality relevant to the work I'll discuss today is "T-duality". This relates

"some theory" on a compact torus $\longleftrightarrow$ "some theory" on the dual torus

Omit unless asked

[e.g. string theory on S^1_R $\longleftrightarrow$ string theory on $S^1_{1/R}$]

and the translation scheme is given by "some kind" of Fourier transform.

replace → mirror symmetry $\overset{\text{SYZ}}{\underset{?}{=}}$ T-duality in families

In '96, Strominger, Yau and Zaslow conjectured that a mysterious duality called "mirror symmetry" could be understood as arising from T-duality in families.

I.e. two spaces X & Y are (SYZ) mirror dual – and so yield equivalent theories – if they are integrable systems over a common base

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

and generically, $p^{-1}(b)$ and $q^{-1}(b)$ are torsors for dual tori.

I'm interested in taking dualities predicted by physics and proving them mathematically.

In particular, by the end of the talk we'll have seen a more precise version of the following statement from my thesis (notation explained later)

Thm / Conjecture [–, '18]:

[There are interesting self-dual spaces $\mathcal{M}_g(\mathbb{C})/\Gamma$ which are the 3d Coulomb branches for theories of class S.]

To get to this, we need to discuss duality and Fourier transforms.

What is a Fourier Transform?

Back when we were all little tikes, we learned about the Fourier transform. This was defined by

$$\hat{f}(p) = \int_{-\infty}^{+\infty} f(x)\, e^{-ixp}\, dx$$

~~and yields an isomorphism~~ omit

$$\cancel{L^2(\mathbb{R}_x) \xrightarrow{\sim} L^2(\mathbb{R}_p).}$$

Q: What is the natural framework for the Fourier transform?

Let A be a Hausdorff locally compact abelian (lca) group, and set

$$A^\vee := \mathrm{Hom}_{cts}(A, U(1)) \quad \text{"Pontryagin dual group"}$$

There is a canonical evaluation map

$$\begin{aligned} ev: A \times A^\vee &\longrightarrow U(1) \\ a, \phi &\longmapsto \phi(a) \end{aligned}$$

and so we can construct a canonical integral transform on functions

treat f as a $f^{\underline{n}}$ on product, const. in $A^\vee$-directions

$$\begin{tikzcd}
 & \pi_1^* f \cdot ev \arrow[ddrr, bend left] & & \\
 & A \times A^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
f \arrow[uur, bend left] \quad A & & A^\vee & \pi_{2*}\left(\pi_1^* f \cdot ev\right) =: \hat{f}
\end{tikzcd}$$

$f \in L^2(A, \mu_{Haar})$

integrate along A w.r.t. Haar measure

Compare w/the original def$^{\underline{n}}$:

$$\begin{tikzcd}
 & \mathbb{R}_x \times \mathbb{R}_x^\vee \arrow[dl] \arrow[dr] & \\
\mathbb{R}_x & & \mathbb{R}_x^\vee
\end{tikzcd}$$

omit unless asked

$$\left(\begin{array}{ccc} \mathbb{R}_x^\vee & \overset{\sim}{\longleftrightarrow} & \mathbb{R}_p \\ e^{-ip(\cdot)} & \longleftrightarrow & p \end{array}\right)$$

link these two

$$L^2(\mathbb{R}_x, dx) \ni f \longmapsto \hat{f}(p) := \int_{-\infty}^{+\infty} dx\, f\big(\pi_1(x,p)\big)\, e^{-ixp}$$

Theorems: For any lca group A:

① There is a functorial isomorphism ("Pontrjagin duality")

$$\begin{aligned} A &\xrightarrow{\sim} (A^\vee)^\vee \\ a &\longmapsto ev(a,-) \end{aligned}$$

② The Fourier transform yields an isomorphism

$$\begin{aligned} L^2(A) &\xrightarrow{\sim} L^2(A^\vee) \\ f &\longmapsto \hat{f} \end{aligned}$$

Omit altogether

~~Mottos: ① "$A^\vee$ determines A"~~
~~② "$A^\vee$ and A have 'the same' function theory"~~

What is a Fourier-Mukai transform?

Cartier duality — the algebraic analogue — holds in less generality. E.g.

- $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ may not be a scheme
- $\mathrm{Hom}(A, \mathbb{C}^\times) = 0$ for an abelian variety A

Partial solution: categorify characters.

Say all this while erasing the board.

I.e. for an algebraic $\underbrace{\text{commutative group (stack)}}_{\text{(cgs)}}$ $\mathcal{A}$ define

$$\mathcal{A}^D := \mathcal{H}om(\mathcal{A}, B\mathbb{C}^\times) = \text{moduli of multiplicative line bundles on } \mathcal{A}$$

$\otimes$-product of line bundles $\Rightarrow$ $\mathcal{A}^D$ is a cgs

<u>E.g.</u>:
- A an abelian variety $\Rightarrow$ A^D usual dual ab. var.
- $\mathbb{Z}^D = B\mathbb{C}^\times$, $(B\mathbb{C}^\times)^D = \mathbb{Z}$
- K finitely generated $\Rightarrow$ $K^D \simeq B(K^\vee)$, $(BK)^D \simeq K^\vee$

There is a canonical evaluation map

$$\mathcal{A} \times \mathcal{A}^D \longrightarrow B\mathbb{C}^\times$$

classifying a line bundle $\mathcal{P} \longrightarrow \mathcal{A} \times \mathcal{A}^D$.

$\Rightarrow$ <u>integral transform of sheaves</u>

$$\begin{array}{ccc} & \mathcal{A} \times \mathcal{A}^D & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathcal{A} & & \mathcal{A}^D \end{array}$$

$$D^b_c(\mathcal{A}) \xrightarrow{FM} D^b_c(\mathcal{A}^D)$$

$$\mathcal{F} \longmapsto R\pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{P})$$

"Fourier-Mukai transform"

(First discovered by Mukai for abelian varieties.)

Mukai ('81)

Theorem-ish [Arinkin ('08?), Brochard ('14?), Chen-Zhu ('17?), ...]:

For a large class of cgs $\mathcal{A}$ (including all examples above)

$$\mathcal{A} \xrightarrow{\sim} (\mathcal{A}^D)^D \quad \& \quad D^b_c(\mathcal{A}) \xrightarrow[\sim]{FM} D^b_c(\mathcal{A}^D).$$

In what follows, when I discuss SYZ mirror symmetry I will mean an algebraic analogue where X and Y are generically torsors for dual cgs.

Diagram:

$$\begin{tikzcd}
\text{4d: } SYM_G \arrow[r, leftrightarrow, "S"] \arrow[d, rightsquigarrow] & SYM_{{}^LG} \arrow[d, rightsquigarrow] \\
\text{2d: } \begin{array}{c}\sigma\text{-model} \\ \text{to } \mathrm{Higgs}_G(C)\end{array} \arrow[r, leftrightarrow, "T"] & \begin{array}{c}\sigma\text{-model} \\ \text{to } \mathrm{Higgs}_{{}^LG}(C)\end{array}
\end{tikzcd}$$

$\mathrm{Higgs}_G(C) = ----$

Warmup: 4d super Yang-Mills.

Physics [Bershadsky-Johansen-Sadov-Vafa ('95);
Harvey-Moore-Strominger ('95);
Kapustin-Witten ('07)]:

Give less detail here - very rough, just want a 4d-to-2d procedure.

- Compactifying a 4d supersymmetric G-gauge theory on a torus and studying the resulting theory on a Riemann surface C yields a theory with fields valued in

"~~Compactifying to 2d~~, putting resulting thy on C yields...."
Just say "there is a procedure", and mention S-duality $\underset{\text{descends}}{\rightsquigarrow}$ T-duality of targets

say "σ-model with target..."

the "moduli of Higgs bundles"

$$\mathcal{H}iggs_G(C) = \left\{ (P, \phi) \;\middle|\; \begin{array}{l} P \text{ a holomorphic } G_{\mathbb{C}}\text{-bdle on } C \\ \phi \in H^0(ad(P) \otimes K_C) \end{array} \right\}$$

- There is a 4d "S-duality" exchanging G with ${}^LG$ (Langlands dual group), descending to T-duality.

- Prediction: $\mathcal{H}iggs_G(C)$ and $\mathcal{H}iggs_{{}^LG}(C)$ are an SYZ mirror pair.

Mathematics

$\mathcal{H}iggs_G(C)$ famously has the structure of an integrable system via the Hitchin map, (a generalization of the characteristic polynomial of a matrix.)

write $\mathcal{B}_{\mathfrak{g}} := \cdots$

"Hitchin base"

$$h : \mathcal{H}iggs_G(C) \longrightarrow H^0(C; \mathfrak{h} \otimes K_C / W) =: \mathcal{B}_{\mathfrak{g}} \quad \mathfrak{g} = \mathrm{Lie}(G)$$

$\mathfrak{h} = \mathrm{Lie}(H)$, $H \subset G$ max'l torus

Away from a locus $\Delta_{\mathfrak{g}} \subset \mathcal{B}_{\mathfrak{g}}$, $\mathcal{H}iggs_G(C)$ is (a torsor for) a cgs over $\mathcal{B}_{\mathfrak{g}}$, locally of the form

$$\begin{pmatrix} \text{abelian} \\ \text{variety} \end{pmatrix} \times Z(G) \times BZ(G)$$

<u>Theorem [Hausel-Thaddeus (SL_n / PGL_n); Donagi-Panter (reductive G)]:</u>

$\mathcal{B}_{\mathfrak{g}} \simeq \mathcal{B}_{{}^L\mathfrak{g}}$ exchanging $\Delta_{\mathfrak{g}} \longleftrightarrow \Delta_{{}^L\mathfrak{g}}$, and there is an isomorphism of cgs

$$\mathcal{H}iggs_G(C)\Big|_{\mathcal{B}_{\mathfrak{g}} \setminus \Delta_{\mathfrak{g}}} \simeq \left(\mathcal{H}iggs_{{}^LG}(C)\Big|_{\mathcal{B}_{{}^L\mathfrak{g}} \setminus \Delta_{{}^L\mathfrak{g}}} \right)^D .$$

Main event: Class $\mathcal{S}$. (Theories of class $\mathcal{S}$)

Physics [Gaiotto-Moore-Neitzke]:

Studied a family of 4d theories called "theories of class $\mathcal{S}$".

Observation 1: Torus reduction to 2d is σ-model with target that looks like $\mathcal{H}iggs_G(C)$.

Observation 2 [Neitzke]: Target should be self SYZ mirror dual.

Contradiction: $\mathcal{H}iggs_G(C)$ is usually not self dual.

Mathematics

Build candidate spaces as follows (thesis construction):

$\pi_1 = 0$ $\qquad$ $\det^d$ by $Z(\tilde{G})$

① Form $\tilde{G}_\tau := \dfrac{\tilde{G} \times (\mathbb{C}^\times)^s}{Z(\tilde{G})}$. $\qquad$ E.g. $GL_n = \dfrac{SL_n \times \mathbb{C}^\times}{\mu_n}$

② $\det : \tilde{G}_\tau \to (\mathbb{C}^\times)^s / Z(\tilde{G})$ $\qquad$ lit. det for matrices & bundles

$\quad \Big\{$ induces

$\det : \mathcal{H}iggs_{\tilde{G}_\tau}(C) \to \mathcal{P}ic(C)^{\times s}$

Cut a "fixed det" moduli space out via hom. $\pi_0(\mathcal{H}iggs_{\widehat{G}_T}(C)) \hookrightarrow Pic(C)^{\times S}$

$$\begin{tikzcd} \mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{H}iggs_{\widehat{G}_T}(C) \arrow[d] \\ \pi_0 \arrow[r, hook] & Pic(C)^{\times S} \end{tikzcd}$$

fix the det. line of the vector bdles

$$\begin{tikzcd} \mathcal{H}iggs^{\bullet}_{SL_n}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{H}iggs_{GL_n}(C) \arrow[d] \\ \mathbb{Z} \arrow[r, hook] & Pic(C) \end{tikzcd}$$

$$1 \longmapsto \mathcal{O}(x)$$

(choose a pt $x \in C$)

(3) Quotient by action of a lattice that removes redundant geometric information

$$\mathcal{M}_{\mathfrak{g}}(C) := \mathcal{H}iggs^{\bullet}/_{\text{lattice}}$$

action

$$(E, \phi) \mapsto (E \otimes \mathcal{O}(x), \phi \otimes 1)$$

(4) $\mathcal{M}_{\mathfrak{g}}(C)$ carries a residual action of $H^1(C; Z(\widetilde{G}))$, $\otimes$-ing by a $Z(\widetilde{G})$-torsor

action

$\otimes$ with n-torsion line bundles

(5) $\mathcal{M}_{\mathfrak{g}}(C)$ inherits a Hitchin map to $\mathcal{B}_{\mathfrak{g}}$ (and the above action is fibrewise).

Thm [—]: Let $\Gamma \subset H^1(C; Z(\tilde{G}))$ be a subgp. There is an iso (of cgs)

$$\left(\frac{\mathcal{M}_{\mathfrak{g}}(C)}{\Gamma} \Bigg|_{B_{\mathfrak{g}} \setminus \Delta_{\mathfrak{g}}} \right)^{D} \simeq \frac{\mathcal{M}_{{}^L\mathfrak{g}}(C)}{\mathrm{ann}(\Gamma)} \Bigg|_{B_{{}^L\mathfrak{g}} \setminus \Delta_{{}^L\mathfrak{g}}}$$

W.r.t. natural skew pairing on $H^1(C; Z(\tilde{G}))$

If $\mathfrak{g} \simeq {}^L\mathfrak{g}$ (e.g. simply-laced) and $\Gamma = \mathrm{ann}(\Gamma)$, the above iso. yields a self-dual space with correct properties to be the target of the 2d reduction of a class $\mathcal{S}$ theory.

Conj. [—]: These actually are the desired spaces.