

Fourier Transforms and Physical Dualities.

(UIC Seminar, 17 Feb 2020)

← Say, don't write.

Context: T-duality:

In physics, two theories are said to be "dual" to each other if they are secretly just different mathematical presentations of the same physical system. The translation scheme between the two descriptions can be highly nontrivial, and therefore useful - intractable calculations on one side may become easy on the other.

The duality relevant to the work I'll discuss today is "T-duality". This relates

"some theory" on a compact torus



"some theory" on the dual torus

[e.g. string theory on S^1_R



string theory on $S^1_{1/R}$

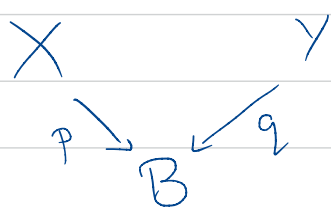
Don't unless asked

and the translation scheme is given by "some kind" of Fourier transform.

replace \rightarrow mirror symmetry $\stackrel{\text{SYZ}}{?}$ T-duality in families

In '96, Strominger, Yau and Zaslow conjectured that a mysterious duality called "mirror symmetry" could be understood as arising from T-duality in families.

i.e. two spaces X & Y are (SYZ) mirror dual - and so yield equivalent theories - if they are integrable systems over a common base



and generically, $p^{-1}(b)$ and $q^{-1}(b)$ are torsors for dual tori.

I'm interested in taking dualities predicted by physics and proving them mathematically.

In particular, by the end of the talk we'll have seen a more precise version of the following statement from my thesis (notation explained later)

Th^m / Conjecture [-, '18]:

There are interesting self-dual spaces $\text{Mog}(\mathbb{C})/\Gamma$ which are the 3d Coulomb branches for theories of class S.

To get to this, we need to discuss duality and Fourier transforms.

What is a Fourier Transform?

Back when we were all little tikes, we learned about the Fourier transform. This was defined by

$$\hat{f}(p) = \int_{-\infty}^{+\infty} f(x) e^{-ixp} dx$$

and yields an isomorphism

$$L^2(\mathbb{R}_x) \xrightarrow{\sim} L^2(\mathbb{R}_p).$$

unit

Q: What is the natural framework for the Fourier transform?

Let A be a Hausdorff locally compact abelian (lca) group, and set

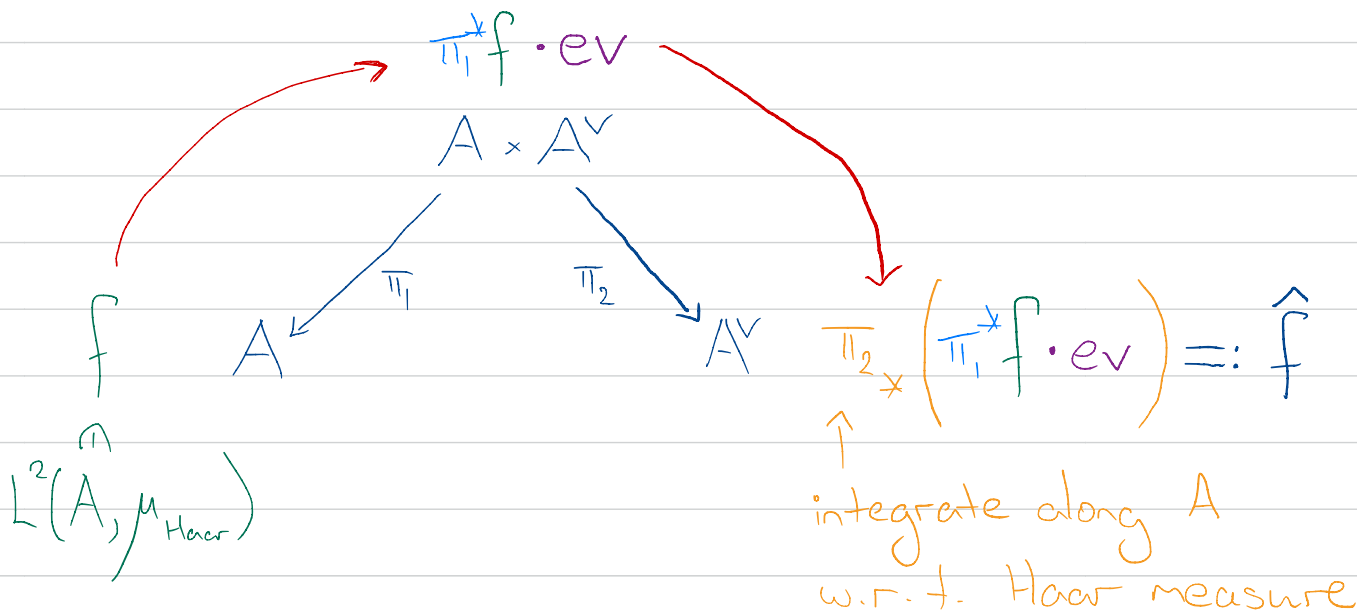
$$A^\vee := \text{Hom}_{\text{cts}}(A, \mathbb{U}(1)) \quad \text{"Pontryagin dual group"}$$

There is a canonical evaluation map

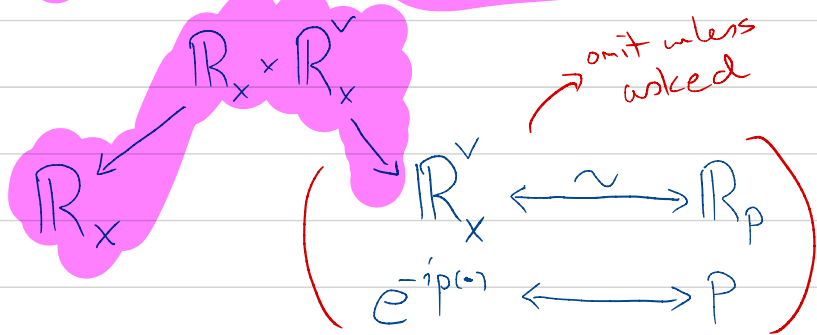
$$\begin{aligned} \text{ev}: A \times A^\vee &\longrightarrow \mathbb{U}(1) \\ a, \phi &\longmapsto \phi(a) \end{aligned}$$

and so we can construct a canonical integral transform on functions

treat f as a f^{pre} on product, const. in A^{\vee} -directions



Compare w/ the original defⁿ:



link these two

$$L^2(\mathbb{R}_x, dx) \ni f \mapsto \hat{f}(p) := \int_{-\infty}^{+\infty} dx f(\pi_1(x, p)) e^{-ixp}$$

Theorems: For any lca group A :

① There is a functorial isomorphism ("Pontrjagin duality")

$$\begin{array}{ccc} A & \xrightarrow{\sim} & (A^\vee)^\vee \\ a & \longmapsto & \text{ev}(a, -) \end{array}$$

② The Fourier transform yields an isomorphism

$$\begin{array}{ccc} L^2(A) & \xrightarrow{\sim} & L^2(A^\vee) \\ f & \longmapsto & \hat{f} \end{array}$$

Unit altogether

Mottos: ① " A^\vee determines A "

② " A^\vee and A have 'the same' function theory"

What is a Fourier-Mukai transform?

Cartier duality — the algebraic analogue — holds in less generality. E.g.

• $G^\vee := \text{Hom}(G, \mathbb{C}^\times)$ may not be a scheme

• $\text{Hom}(A, \mathbb{C}^\times) = 0$ for an abelian variety A

Partial solution: categorify characters.

Say all this while erasing the board.

i.e. for an algebraic commutative group (stack) A
 define (egs)

$$A^{\mathcal{D}} := \text{Hom}(A, \mathbb{B}\mathbb{C}^{\times}) = \text{moduli of multiplicative line bundles on } A$$

\otimes -product of line bundles $\Rightarrow A^{\mathcal{D}}$ is a egs

- E.g.:
- A an abelian variety $\Rightarrow A^{\mathcal{D}}$ usual dual ab. var.
 - $\mathbb{Z}^{\mathcal{D}} = \mathbb{B}\mathbb{C}^{\times}$, $\cup (\mathbb{B}\mathbb{C}^{\times})^{\mathcal{D}} = \mathbb{Z}$
 - K finitely generated $\Rightarrow K^{\mathcal{D}} \simeq \mathbb{B}(K^{\vee})$, $(\mathbb{B}K)^{\mathcal{D}} \simeq K^{\vee}$

There is a canonical evaluation map

$$A \times A^{\mathcal{D}} \longrightarrow \mathbb{B}\mathbb{C}^{\times}$$

classifying a line bundle $\mathcal{P} \rightarrow A \times A^{\mathcal{D}}$.

\Rightarrow integral transform of sheaves

$$\begin{array}{ccc} & A \times A^{\mathcal{D}} & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ A & & A^{\mathcal{D}} \end{array}$$

$$\mathcal{D}_c^b(A) \xrightarrow{\text{FM}} \mathcal{D}_c^b(A^{\mathcal{D}})$$

$$F \mapsto R\pi_{2,*}(\pi_1^* F \otimes \mathcal{P})$$

"Fourier-Mukai transform"

(First discovered by Mukai for abelian varieties.)

Mukai ('81)

Theorem-ish [Arikin ('08?), Bockland ('14?), Chen-Zhu ('17?), ...]:

For a large class of cgs \mathcal{A} (including all examples above)

$$\mathcal{A} \xrightarrow{\sim} (\mathcal{A}^{\mathbb{D}})^{\mathbb{D}} \quad \varepsilon \quad \mathcal{D}_c^b(\mathcal{A}) \xrightarrow[\sim]{\text{FM}} \mathcal{D}_c^b(\mathcal{A}^{\mathbb{D}}).$$

In what follows, when I discuss SYZ mirror symmetry I will mean an algebraic analogue where X and Y are generically torsors for dual cgs.

Diagram: 4d: SYM_G \xleftrightarrow{S} SYM_{L_G}



2d: σ -model to $\text{Higgs}_G(\mathbb{C})$ \xleftrightarrow{T} σ -model to $\text{Higgs}_{L_G}(\mathbb{C})$

$\text{Higgs}_G(\mathbb{C}) = \dots$



Warmup: 4d super Yang-Mills.

Physics [Bershadsky-Johansen-Sadov-Vafa (1995);
Harvey-Moore-Straninger (1995);
Kapustin-Witten (107)]:

Give less detail
here - very
rough, just
want a 4d-to-2d
procedure.

- Compactifying a 4d supersymmetric G -gauge theory on a torus and studying the resulting theory on a Riemann surface C yields a theory with fields valued in

"Compactifying to 2d, putting resulting theory on C yields....
Just say "there is a procedure", and mention S-duality means T-duality of targets

say "G-model with target.."

the "moduli of Higgs bundles"

$$\text{Higgs}_G(C) = \left\{ (P, \phi) \mid \begin{array}{l} P \text{ a holomorphic } G_C\text{-bundle on } C \\ \phi \in H^0(\text{ad}(P) \otimes K_C) \end{array} \right\}$$

- There is a 4d "S-duality" exchanging G with ${}^L G$ (Langlands dual group), descending to T-duality.
- Prediction: $\text{Higgs}_G(C)$ and $\text{Higgs}_{{}^L G}(C)$ are an SYZ mirror pair.

Mathematics

$\text{Higgs}_G(C)$ famously has the structure of an integrable system via the Hitchin map, (a generalization of the characteristic polynomial of a matrix.)

write \mathcal{B}_{reg} = ...

"Hitchin base"

$$h: \text{Higgs}_G(C) \longrightarrow H^0(C; \mathfrak{h} \otimes K_C/W) =: \mathcal{B}_{\text{reg}} \subset \text{Lie}(G)$$

$\mathfrak{h} = \text{Lie}(H)$, $H \subset G$
max¹ torus

Away from a locus $\Delta_{\text{reg}} \subset \mathcal{B}_{\text{reg}}$, $\text{Higgs}_G(C)$ is (a torsor for) a cgs over \mathcal{B}_{reg} , locally of the form

$$(\text{abelian variety}) \times Z(G) \times BZ(G)$$

Theorem [Hausel-Thaddeus (SL_n/PGL_n); Donagi-Pantzer (reductive G)]:

$\mathcal{B}_{\text{reg}} \simeq \mathcal{B}_{\text{reg}}$ exchanging $\Delta_{\text{reg}} \leftrightarrow \Delta_{\text{reg}}$, and there is an isomorphism of cgs

$$\text{Higgs}_G(C) \Big|_{\mathcal{B}_{\text{reg}} \setminus \Delta_{\text{reg}}} \simeq \left(\text{Higgs}_G(C) \Big|_{\mathcal{B}_{\text{reg}} \setminus \Delta_{\text{reg}}} \right)^{\mathbb{D}}$$

Main event: Class \mathcal{S} . (Theories of class \mathcal{S})

Physics [Gaiotto-Moore-Neitzke]:

Studied a family of 4d theories called "theories of class \mathcal{S} ".

Observation 1: Torus reduction to 2d is \mathcal{G} -model with target that looks like $\text{Higgs}_{\mathcal{G}}(\mathbb{C})$.

Observation 2 [Neitzke]: Target should be self SYZ mirror dual.

Contradiction: $\text{Higgs}_{\mathcal{G}}(\mathbb{C})$ is usually not self dual.

Mathematics

Build candidate spaces as follows (thesis construction):

① Form $\tilde{\mathcal{G}}_{\tau} := \frac{\tilde{\mathcal{G}} \times (\mathbb{C}^{\times})^s}{Z(\tilde{\mathcal{G}})}$. E.g. $GL_n = \frac{SL_n \times \mathbb{C}^{\times}}{\mu_n}$

Annotations: $\tau_r = 0$ (pointing to $\tilde{\mathcal{G}}$), \det^s by $Z(\tilde{\mathcal{G}})$ (pointing to denominator)

② $\det: \tilde{\mathcal{G}}_{\tau} \rightarrow (\mathbb{C}^{\times})^s / Z(\tilde{\mathcal{G}})$

\downarrow induces

$$\det: \text{Higgs}_{\tilde{\mathcal{G}}_{\tau}}(\mathbb{C}) \rightarrow \text{Pic}(\mathbb{C})^{*s}$$

lit. det for matrices of bundles

Cut a "fixed det" moduli space
out via hom. $\pi_0(\text{Higgs}_{\tilde{G}_r}(C)) \hookrightarrow \text{Pic}(C)^{xs}$

$$\begin{array}{ccc} \text{Higgs}_{\tilde{G}}^{\bullet}(C) & \rightarrow & \text{Higgs}_{\tilde{G}_r}(C) \\ \downarrow & \nearrow & \downarrow \\ \pi_0 & \hookrightarrow & \text{Pic}(C)^{xs} \end{array}$$

fix the det. line
of the vector bundles

$$\begin{array}{ccc} \text{Higgs}_{\text{SL}_n}^{\bullet}(C) & \rightarrow & \text{Higgs}_{\text{GL}_n}(C) \\ \downarrow & \nearrow & \downarrow \\ \mathbb{Z} & \hookrightarrow & \text{Pic}(C) \\ \downarrow & & \downarrow \\ 1 & \hookrightarrow & \mathcal{O}(x) \end{array}$$

(choose a pt $x \in C$)

- ③ Quotient by action of
a lattice that
removes redundant
geometric information

action

$$(E, \phi) \mapsto (E \otimes \mathcal{O}(x), \phi \otimes 1)$$

$$\mathcal{M}_{\text{og}}(C) := \text{Higgs}^{\bullet} / \text{lattice}$$

- ④ $\mathcal{M}_{\text{og}}(C)$ carries a residual
action of $H^1(C; \mathbb{Z}(\tilde{G}))$, \otimes -ing
by a $\mathbb{Z}(\tilde{G})$ -torsor

action

\otimes with n -torsion
line bundles

- ⑤ $\mathcal{M}_{\text{og}}(C)$ inherits a Hitchin
map to \mathcal{B}_{og} (and the
above action is fibrewise).

Thm [-]: Let $\Gamma \subset H^1(C; \mathbb{Z}(\tilde{G}))$ be a subgroup. There is an iso (of cgs)

$$\left(\frac{\mathcal{M}_{\text{log}}(C)}{\Gamma} \right)^{\mathbb{D}} \Big|_{\mathcal{B}_{\text{log}}^{-1} \Delta_{\text{log}}} \simeq \frac{\mathcal{M}_{\text{log}}(C)}{\text{ann}(\Gamma)} \Big|_{\mathcal{B}_{\text{log}}^{-1} \Delta_{\text{log}}}$$

↑
W.r.t. natural skew pairing on $H^1(C; \mathbb{Z}(\tilde{G}))$

If $\mathfrak{g} \simeq \mathfrak{L}\mathfrak{g}$ (e.g. simply-laced) and $\Gamma = \text{ann}(\Gamma)$, the above iso. yields a self-dual space with correct properties to be the target of the 2d reduction of a class \mathfrak{g} theory.

Conj. [-]: These actually are the desired spaces.