Learning Seminar at PCMI

Richard Derryberry

July 14, 2019

These notes consist of two lectures on Hitchin's paper *The Self Duality Equations on a Riemann Surface* [Hit87]. The first will discuss the derivation of the equation and construction of the moduli space; the second will discuss its topological and geometric properties. I will focus on the results as they are found in Hitchin's paper, but also occasionally hint at how these results generalise.

For those interested in learning more, Andy Neitzke has notes on his website from a course he taught on the moduli of Higgs bundles. A direct link is: `https://web.ma.utexas.edu/users/neitzke/teaching/392C-higgs-bundles/higgs-bundles.pdf`

Contents

1 Lecture 1: Construction of the moduli space

1.1 Derivation of Hitchin's equations

For most of these lectures I will restrict to the groups $SU(2)/SO(3)$ (as in [Hit87]). For the derivation of the equations, however, no such restriction is necessary.

Our initial goal is to derive Hitchin's equations from four-dimensional gauge theory. Start with the data of

$$\begin{array}{c}(P,\nabla)\\ \big\downarrow \\ \mathbb{R}^4\end{array} \qquad \text{a principal } G\text{-bundle with connection.}$$

Denote the curvature 2-form by $F_\nabla \in \Omega^2_{\mathbb{R}^4}(\mathrm{ad}(P))$. Give $\mathbb{R}^4$ its standard Euclidean metric g, and consider the Hodge star

$$\star : \Omega^2_{\mathbb{R}^4}(\mathrm{ad}(P)) \to \Omega^2_{\mathbb{R}^4}(\mathrm{ad}(P))$$
$$\mathrm{Tr}(\alpha \wedge \star\beta) = \mathrm{Tr}(g(\alpha,\beta))d\mathrm{vol}.$$

We can split F_∇ into its *self dual* and *anti-self dual* components,

$$F^\pm_\nabla = \frac{1}{2}(F_\nabla \pm \star F_\nabla), \qquad \star F^\pm_\nabla = \pm F^\pm_\nabla.$$

Now consider the action for 4d pure Yang-Mills,

$$S = \frac{1}{2e^2}\int \mathrm{Tr}(F_\nabla \wedge \star F_\nabla).$$

We want to find critical points for this action functional. Observe that

$$\frac{1}{e^2}\int F^\mp \wedge F^\mp = \frac{1}{2e^2} \underbrace{\int \mathrm{Tr}(F\wedge F)}_{\text{topological – “instanton number”}} \mp \underbrace{\frac{1}{2e^2}\int \mathrm{Tr}(F\wedge\star F)}_{=S} = \frac{8\pi^2}{e^2}k \mp S$$

for some $k \in \mathbb{Z}$.[1] Since $\star F^\pm = \pm F^\pm$, this gives us the inequality

$$S \geq \frac{8\pi^2}{e^2}|k|$$

with equality iff

$$F = F^+ \qquad (k > 0), \qquad (1.1)$$
$$F = F^- \qquad (k < 0). \qquad (1.2)$$

These are the *self dual* (1.1) and *anti-self dual* (1.2) *Yang-Mills equations*.[2]

Let's focus on (1.1), i.e. $F = \star F$. Following Hitchin, we want to dimensionally reduce this equation to 2d by considering solutions which are translationally invariant along a plane. Choosing a trivialisation of P and taking the standard coordinates (x^1, x^2, x^3, x^4) on $\mathbb{R}^4$, we can write

$$\nabla = d + A\cdot,$$
$$A = A_\mu dx^\mu$$

for $A_\mu : \mathbb{R}^4 \to \mathfrak{g}$. We can then express the curvature form as

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

Now, let's restrict to solutions which are invariant in the x^3x^4-plane. Write

$$A = a + \phi_1 dx^3 + \phi_2 dx^4$$

[1] Up to my possibly having messed up the positive constant that multiplies k.

[2] In Manolescu's general lecture, these were termed "BPS states", and he claimed that they minimise the energy. Now you see why!

where a defines a connection ∇_a in 2d, and $\phi_i : \mathbb{R}^2 \to \mathfrak{g}$. The curvature becomes

$$F_A = F_a + \nabla_a(\phi_1 dx^3 + \phi_2 dx^4) + [\phi_1, \phi_2] dx^3 \wedge dx^4.$$

Write $a = a_i dx^i$, $\nabla_a^{(i)} = \partial_i + [a_i, -]$. Then $F = \star F$ becomes the system of equations

$$\begin{cases} F_a &= [\phi_1, \phi_2] dx^1 \wedge dx^2, \\ \nabla_a^{(1)} \phi_1 &= -\nabla_a^{(2)} \phi_2 \\ \nabla_a^{(2)} \phi_1 &= \nabla_a^{(1)} \phi_2 \end{cases} \tag{1.3}$$

Problem: (1.3) is not invariant under coordinate transformations.

Solution: Consider $\mathfrak{g} \otimes \mathbb{C}$, set $z = x^1 + ix^2$, and take

$$\Phi = \frac{\phi_1 - i\phi_2}{2} dz \in \Omega^{1,0}_{\mathbb{R}^2}(\mathrm{ad}(P)_{\mathbb{C}})$$
$$\Phi^* = \frac{\phi_1^* + i\phi_2^*}{2} d\bar{z} \in \Omega^{0,1}_{\mathbb{R}^2}(\mathrm{ad}(P)_{\mathbb{C}})$$

where $(-)^*$ is the involution on $\mathfrak{g} \otimes \mathbb{C}$ defining the compact real form. Then the equations (1.3) become *Hitchin's equations*

$$\begin{cases} F_a + [\Phi, \Phi^*] &= 0, \\ \bar{\partial}_a \Phi &= 0 \end{cases} \tag{1.4}$$

These make sense on an *arbitrary* Riemann surface C, and can be partially interpreted as follows: the 2d connection a defines a holomorphic structure on the bundle $\mathrm{ad}(P)_{\mathbb{C}}$, and Φ is a *holomorphic* section of $\mathrm{ad}(P)_{\mathbb{C}} \otimes K_C$ (K_C the canonical bundle – holomorphic 1-forms). This motivates the following definition (terminology due to Simpson [Sim92]).

Definition 1.1. A $G_{\mathbb{C}}$*-Higgs bundle* is a pair (P, Φ) of a *holomorphic* principal $G_{\mathbb{C}}$-bundle and $\Phi \in H^0(C; \mathrm{ad}(P) \otimes K_C)$.

Remark 1.1. Note that in (1.4) the curvature is necessarily tracefree. As such when we start with a 2d connection we should really take the *trace-free* part of the curvature, i.e.

$$F^{\perp} + [\Phi, \Phi^*] = 0.$$

(This will be important for some of our calculations later.)

From now on: Assume that C is compact.

Example 1.1. If $\Phi \equiv 0$ then (1.4) becomes $F_a \equiv 0$, whose solutions are flat unitary G-connections on C. By results of Narasimhan and Seshadri [NS65] these are equivalent to stable holomorphic $G_{\mathbb{C}}$-bundles on C (up to appropriate notions of equivalence – more on this later).

1.2 Stability conditions for Higgs bundles

From now on:

- C is a compact Riemann surface
- $G = SU(2)$ or $SO(3)$
- The 2d connection will be denoted by capital A

- Will mostly work with the rank 2 complex vector bundle associated to the principal $SU(2)/SO(3)$ bundle, denoted V

In Haydys' *Intro to Gauge Theory* course in week 1, we saw that the way the gauge theory game is played is to take (roughly)

- An infinite dimensional space of solutions (or potential solutions), $\mathcal{S}$.
- An infinite dimensional group of symmetries $\mathcal{G}$ acting on $\mathcal{S}$ – tells us when solutions are equivalent.
- The moduli space of solutions up to equivalence, $\mathcal{M} = \mathcal{S}/\mathcal{G}$. *Hopefully get:* finite dimensional, smooth manifold, perhaps with other nice features (compact/complete metric/symplectic/etc.)

We also saw that in order to find a "nice" moduli space we might have to remove "bad" points from $\mathcal{S}$, usually because they have "too many symmetries" (i.e. non-minimal stabilizer under the gauge group). To distinguish the good and bad points in our situation, make the following definition.

Definition 1.2. A rank 2 Higgs bundle (V, Φ) is *stable* if for all Φ-invariant line subbundles $L \subset V$, $\deg(L) < \frac{1}{2}\deg(V)$.

Remark 1.2. In the above definition, if $\deg(L) = \frac{1}{2}\deg(V)$ we call (V, Φ) *strictly semistable.*

The above definition of stability is justified by the following results:

(1) The definition of stability is "built in" to Hitchin's equations (1.4) – if (A, Φ) is a solution to Hitchin's equations the corresponding Higgs bundle is semistable; if the Higgs bundle is strictly semistable, (A, Φ) reduces to $U(1)$ [Hit87, Thm.2.1].

(2) In line with our "good points have minimal symmetries" mantra, the only endomorphisms of a stable pair are given by scalar multiplication [Hit87, Prop.3.15].

(3) Existence theorem: Suppose that C has genus $g > 1$, let $(P, A) \to C$ be a principal $SO(3)$-bundle with connection and let V be the associated rank 2 complex vector bundle. Choose $\Phi \in \Omega^{1,0}_C(\mathrm{ad}(P)_{\mathbb{C}})$ satisfying $\bar{\partial}_A \Phi = 0$. If (V, Φ) is a stable pair, then there is an automorphism of V of determinant 1, unique up to $SO(3)$ gauge transformations, which sends (A, Φ) to a solution of Hitchin's equations [Hit87, Thm.4.3].

The proofs of (1) and (2) are relatively straightforward. The proof of (3) is more involved, and involves the moment map for the gauge group action (which we'll discuss in a minute) and the Uhleckbeck compactness theorem [Uhl82] (which we won't discuss).

Example 1.2. The details of the following examples are left as exercises (or you can read [Hit87]):

- There are no stable pairs on $\mathbb{P}^1$ – use that vector bundles decompose as sums of line bundles $\mathcal{O}(n)$ and that $K_{\mathbb{P}^1} \simeq \mathcal{O}(-2)$.
- There is a unique stable pair on an elliptic curve – need to use Atiyah's classification of vector bundles on an elliptic curve [Ati57] and triviality of the canonical bundle.
- For $g(C) > 1$, choose a square root of the canonical bundle $K^{1/2}$ and set $V = K^{1/2} \oplus K^{-1/2}$. For each $q \in H^0(K^2)$,
$$\Phi := \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix} : K^{1/2} \oplus K^{-1/2} \to K^{3/2} \oplus K^{1/2}$$
defines a stable pair.

Taking into account the above examples, we'll assume from now on that $g(C) > 1$.

1.3 Construction of the moduli space

Following Hitchin, let's now construct the moduli space of solutions to (1.4). I will ignore the issues that arise from the fact that various objects are infinite dimensional: in Haydys' lectures you saw how to deal with such issues in Seiberg-Witten theory, and they can be similarly resolved for the following construction.

1.3.1 Symplectic quotients and moment maps

Our method of construction will make use of a generalisation of the symplectic quotient.

Recall the definition of a *moment map*: a group G acting on a symplectic manifold (M, ω) by symplectomorphisms yields a map from $\mathfrak{g}$ symplectic vector fields. Contracting each $X \in \mathfrak{g}$ with the symplectic form this yields a 1-form, which is closed since

$$0 = \mathcal{L}_X \omega = (d\iota_X + \iota_X d)\omega = d(\iota_X \omega).$$

If the 1-form is also exact, $\iota_X \omega = d\mu_X$, we obtain a function $\mu_X : M \to \mathbb{R}$. Dualising this one obtains a function

$$\mu : M \to \mathfrak{g}^*,$$

the *moment map* for the action.[3]

Example 1.3. Consider $M = \operatorname{End}(\mathbb{C}^n)$ with Kähler metric $g(A, B) = \operatorname{Re}\operatorname{Tr}(AB^*)$, acted on by $G = U(n)$. The Kähler form is

$$\omega(A, B) = g(iX, B) = \operatorname{Re}\operatorname{Tr}(iAB^*) = -\operatorname{Im}\operatorname{Tr}(AB^*).$$

The vector field induced by a matrix $X \in \mathfrak{u}(n)$ is $X_A = [X, A] \in T_A \operatorname{End}(\mathbb{C}^n) = \operatorname{End}(\mathbb{C}^n)$, so

$$\begin{aligned} \omega(X_A, B) &= -\operatorname{Im}\operatorname{Tr}([X, A]B^*) \\ &= \frac{i}{2}\operatorname{Tr}([X, A]B^* - [X, A^*]B) \\ &= \frac{i}{2}\operatorname{Tr}([B, A^*]X + [A, B^*]X) = (d\mu_X)_A(B) \end{aligned}$$

for $\mu_X(A) = \frac{i}{2}\operatorname{Tr}([A, A^*]X)$. Hence the moment map for the action is

$$\mu(A) = \frac{i}{2}[A, A^*]. \tag{1.5}$$

Given the setup above, one can take the *symplectic quotient* of M by G,

$$M /\!\!/ G := \mu^{-1}(0)/G;$$

if G acts freely on $\mu^{-1}(0)$, this naturally inherits the structure of a symplectic manifold.

In the situation we're considering, let $M = \mathcal{A} \times \Omega$ where

- $\mathcal{A}$ is the affine space of unitary connections on a fixed bundle P, and
- $\Omega := \Omega^{1,0}_C(\operatorname{ad}(P)_{\mathbb{C}})$.

The tangent space to $\mathcal{A} \times \Omega$ at any point is given by

$$\Omega^{0,1}_C(\operatorname{ad}(P)_{\mathbb{C}}) \oplus \Omega^{1,0}_C(\operatorname{ad}(P)_{\mathbb{C}}) \simeq \Omega^1(\operatorname{ad}(P)) \oplus \Omega^1(\operatorname{ad}(P))$$

[3] Beware: moment maps do not always exist!

(isomorphism as real vector spaces), and using this second description we define a symplectic form on $\mathcal{A} \times \Omega$ by

$$\omega_1((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_C \mathrm{Tr}\left(-\dot{A}_1 \wedge \dot{A}_2 + \dot{\Phi}_1 \wedge \dot{\Phi}_2\right).$$

The group of G-gauge transformations acts on $\mathcal{A} \times \Omega$ by symplectomorphisms – specifically, as a gauge transformation on the connection 1-form, and the adjoint action on the Higgs field – and there is a moment map for this action given by

$$\mu_1(A, \Phi) = F_A + [\Phi, \Phi^*],$$

the first equation in (1.4). As an exercise you should derive this (the second term should be essentially Example 1.3).

1.3.2 Hyperkähler quotients

Having encoded one part of Hitchin's equations (1.4) in a moment map, it is reasonable to ask whether there is a similarly natural way to also encode the holomorphicity of Φ. The answer is yes, and will lead us to a construction of the moduli space as a *hyperkähler quotient*.

Definition 1.3. A *hyperkähler metric* g on a manifold M is a Riemannian metric which is simultaneously Kähler for three complex structures, I, J, K, which satisfy the quaternionic relations

$$I^2 = J^2 = K^2 = -1, \qquad IJ = K.$$

In such a situation we then have three symplectic forms $\omega_{1,2,3}$ and given a G action on M *potentially* three moment maps $\mu_{1,2,3}$. The *hyperkähler quotient* of M by G is then

$$M /\!/\!/\!/ G := \left(\bigcap_{i=1}^{3} \mu_i^{-1}(0) \right) / G;$$

as its name suggests, this naturally inherits the structure of a hyperkähler manifold (provided the G-action is free on the joint zero locus).

In order to describe the moduli space of Higgs bundles as a hyperkähler quotient, we need to find the other two Kähler forms on $\mathcal{A} \times \Omega$ and describe the corresponding moment maps. The Kähler forms[4] arise as the real and imaginary parts of the holomorphic symplectic form

$$\omega((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = 2i \int_C \mathrm{Tr}(\dot{A}_1 \wedge \dot{\Phi}_2 - \dot{A}_2 \wedge \dot{\Phi}_1)$$

where we have used the description of the tangent space as $\Omega^{0,1}_C(\mathrm{ad}(P)_{\mathbb{C}}) \oplus \Omega^{1,0}_C(\mathrm{ad}(P)_{\mathbb{C}})$. The corresponding moment maps are

$$(\mu_2 + i\mu_3)(A, \Phi) = 2i\bar{\partial}_A \Phi,$$

and so we recover the second part of Hitchin's equations (1.4). Finally, we obtain the *moduli space of Higgs bundles* as the hyperkähler quotient

$$\mathbf{Higgs} := \mathcal{A} \times \Omega /\!/\!/\!/ \mathcal{G}.$$

[4] The corresponding complex structures are given by

$$I(\dot{A}, \dot{\Phi}) = (i\dot{A}, i\dot{\Phi})$$
$$J(\dot{A}, \dot{\Phi}) = (i\dot{\Phi}^*, -i\dot{A}^*)$$
$$K(\dot{A}, \dot{\Phi}) = (-\dot{\Phi}^*, \dot{A}^*)$$

where we use the description of the tangent space where $\dot{A} \in \Omega^{0,1}_C(\mathrm{ad}(P)_{\mathbb{C}})$ and $\dot{\Phi} \in \Omega^{1,0}_C(\mathrm{ad}(P)_{\mathbb{C}})$.

Remark 1.3. To be entirely transparent, the only space that the above construction applies to without caveats is

$$\mathbf{Higgs}^1_{SL_2}(C) = \left\{ \begin{array}{c} \text{moduli of } (V,\Phi),\ V \text{ a rank 2} \\ \text{holomorphic bundle with fixed} \\ \text{determinant and degree 1,} \\ \Phi \in H^0(C; \mathrm{End}_0(V) \otimes K_C) \end{array} \right\} \Big/ \sim$$

where $\mathrm{End}_0(V)$ are the *trace-free* endomorphisms of V. The other possibilities *do* exist, but have the following issues I have not explored:

- $\mathbf{Higgs}^0_{SL_2}(C)$ contains strictly semistable pairs – i.e. connections reducible to $U(1)$ – which are singular points. Removing these makes the hyperkähler metric incomplete.
- $\mathbf{Higgs}^1_{PGL_2}(C)$ has orbifold singularities (despite the fact that it has no strictly semistable points).
- $\mathbf{Higgs}^0_{PGL_2}(C)$ is the worst of both worlds.

2 Lecture 2: Geometry and topology of the moduli space

Last lecture we constructed the moduli of Higgs bundles – this lecture we'll explore some of the geometry and topology of this space. We really will focus on the rank 2 case in this lecture. Today C will always be a compact Riemann surface of genus $g(C) > 1$.

Reminder: Hitchin's equations are (1.4)

$$\left\{ \begin{array}{rcl} F^\perp + [\Phi, \Phi^*] &=& 0, \\ \bar\partial_A \Phi &=& 0 \end{array} \right.$$

2.1 Topology of the moduli space

Following Hitchin [Hit87, §7] we will use the $U(1)$ action on the moduli space to understand the topology of

$$\mathbf{Higgs}^1_{SL_2}(C) = \left\{ \begin{array}{c} \text{moduli of } (V,\Phi),\ V \text{ a rank 2} \\ \text{holomorphic bundle with fixed} \\ \text{determinant and degree 1,} \\ \Phi \in H^0(C; \mathrm{End}_0(V) \otimes K_C) \end{array} \right\} \Big/ \sim$$

There is a $U(1)$-action by isometries on $\mathcal{A} \times \Omega$ given by "rotating the Higgs field", i.e. the connection is unchanged and

$$\Phi \to e^{i\theta}\Phi, \qquad \Phi^* \to e^{-i\theta}\Phi^*.$$

This $U(1)$ action preserves Hitchin's equations (1.4) and commutes with gauge transformations, and so descends to the moduli space.

It also preserves the symplectic form ω_1 (though not $\omega_{2/3}$), so we can look for a moment map for the action. The vector field generated by the action is

$$X_{(A,\Phi)} = \left.\frac{d}{d\theta}\right|_{\theta=0} (A, e^{i\theta}\Phi) = (0, i\Phi),$$

and so,

$$(\iota_X \omega_1)_{(A,\Phi)}(Y) = g(IX, Y) = g(-\Phi, Y) = -\frac{1}{2} dg(\Phi,\Phi)(Y), \tag{2.1}$$

i.e. the moment map for the $U(1)$ action is $-\frac{1}{2}g(\Phi,\Phi)$. So, set

$$\mu(A,\Phi) = \frac{i}{2}\int_C \mathrm{Tr}(\Phi\Phi^*). \tag{2.2}$$

By (2.1) the critical points of μ are precisely the fixed points of the $U(1)$-action. We will investigate the topology of $\mathbf{Higgs}^1_{SL_2}(C)$ using this Morse function – for this we need to know some of the properties of μ.

Proposition 2.1. *The following is [Hit87, Prop.7.1]:*

(i) μ *is proper.*

(ii) μ *has critical values* 0 *and* $\left(d-\frac{1}{2}\right)\pi$, $d \in [1, g-1] \cap \mathbb{Z}$.

(iii) $\mu^{-1}(0)$ *is a non-degenerate critical manifold of index 0, diffeomorphic to the moduli space of stable rank 2 bundles of odd degree and fixed determinant over* C.

(iv) $\mu^{-1}\left(\left(d-\frac{1}{2}\right)\pi\right)$ *is a non-degenerate critical manifold of index* $2(g+2d-2)$*, diffeomorphic to a* 2^{2g}*-fold covering of the symmetric product* $S^{2g-2d-1}C$*; the covering is defined by the following pullback square*

$$\begin{array}{ccccc} \widetilde{S^{2g-2d-1}}C & \longrightarrow & \mathrm{Jac}(C) & & x \\ \downarrow & & \downarrow & & \downarrow \\ S^{2g-2d-1}C & \longrightarrow & \mathrm{Jac}(C) & & 2x \end{array}$$

$$(p_i)_{i=1,\dots,2g-2d-1} \longmapsto \left(\textstyle\sum_i p_i\right) - (2g-2d-1)c$$

where $c \in C$ *is a choice of basepoint.*

Proof. (i) Follows by bounding the curvature and applying Uhlenbeck compactness.

(iii) 0 is an absolute minimum of μ, and this occurs if and only if $\Phi \equiv 0$. Then Hitchin's equations become $F_A = 0$, and by Narasimhan and Seshadri [NS65] this gives the moduli space of stable holomorphic rank 2 vector bundles of fixed determinant and odd degree. The index is the rank of the subbundle of the normal bundle on which $U(1)$ acts by negative weights – since 0 is an absolute minumum, this is 0.

(ii) The fixed points of the $U(1)$ action on $\mathcal{A}\times\Omega$ are precisely the points where $\Phi \equiv 0$, so you might be tempted to think that we are now done. Remember, however, that $\mathbf{Higgs}^1_{SL_2}(C)$ is the quotient of a subspace of $\mathcal{A}\times\Omega$ by the group of gauge transformations – as such, there are more fixed points corresponding to Higgs bundles which are *gauge equivalent* to the other points in their $U(1)$ orbit. Explicitly:

$$(A,\Phi) \text{ is a fixed point } \iff \text{ there are gauge transformations } g(\theta) \text{ such that } \left.\begin{array}{rcl} g(\theta)^{-1}\Phi g(\theta) & = & e^{i\theta}\Phi \\ g(\theta)^{-1} d_A g(\theta) & = & d_A \end{array}\right\}$$

(d_A the covariant derivative associated to A). Since $g(\theta)$ is nonconstant and preserved by parallel transport, d_A preserves its eigenspaces, and so A reduces to a $U(1)$-connection. So, the associated vector bundle V decomposes into holomorphic line subbundles

$$V = L \oplus (L^* \otimes \wedge^2 V)$$

the eigenlines of g_θ. Since Φ is an eigenvector of the adjoint action, without loss of generality it is strictly lower triangular,

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}, \qquad \phi \in \Omega^0_C(L^{-2}\otimes K_C \otimes \wedge^2 V),$$

(i.e. $\phi : L \to L^* \otimes \wedge^2 V \otimes K_C$). Since (A, Φ) is a solution to Hitchin's equations, ϕ is holomorphic and

$$0 = F^\perp + [\Phi, \Phi^*] = \begin{pmatrix} F_1 - \phi\phi^* & 0 \\ 0 & -F_1 + \phi\phi^* \end{pmatrix}$$

where $F^\perp = \begin{pmatrix} F_1 & 0 \\ 0 & -F_1 \end{pmatrix}$ is the trace-free part of the curvature of the reducible connection. The curvature of the connection on L can be expressed as[5]

$$F(L) = F_1 + \frac{1}{2}F(\wedge^2 V) = \phi\phi^* + \frac{1}{2}F(\wedge^2 V),$$

and integrating this expression gives

$$\begin{aligned} \deg(L) &= \frac{i}{2\pi}\int_C F(L) \\ &= \frac{i}{2\pi}\int_C \phi\phi^* + \frac{i}{2\pi}\int_C \frac{1}{2}F(\wedge^2 V) \\ &= \frac{i}{2\pi}\int_C \mathrm{Tr}(\Phi\Phi^*) + \frac{1}{2}\deg(\wedge^2 V) \\ &= \frac{1}{\pi}\mu(A, \Phi) + \frac{1}{2} \end{aligned}$$

since we have fixed $\deg(\wedge^2 V) = 1$. Setting $d = \deg(L)$ we have

$$\mu(A, \Phi) = \pi\left(d - \frac{1}{2}\right)$$

for some integer d. Stability places constraints on the degree of the line bundle L,[6] – these give the constraint $d - \frac{1}{2} \leq g - 1$.

(iv) Choose a basepoint $c \in C$ and consider the degree 1 line bundle $\mathcal{O}(c)$. For ease of notation, denote this by $\mathcal{O}(1) := \mathcal{O}(c)$ and $L(1) := L \otimes \mathcal{O}(c)$. From part (iii) we see that $\mu^{-1}\left(\pi\left(d - \frac{1}{2}\right)\right)$ is diffeomorphic to the moduli space of stable pairs (V, Φ) where

- $V = L \oplus L^*(1)$, L a holomorphic line bundle of degree d
- Φ is determined by $\phi \in H^0(C; L^{-2}K_C(1))$, i.e. a holomorphic bundle map $\phi : L \to L^* \otimes K_C(1)$.

$\deg(L^{-2}K_C(1)) = (-2d) + (2g - 2) + 1 = 2g - 2d - 1$, so the vanishing locus of ϕ is a positive divisor of degree $2g - 2d - 1$ – i.e. an element of the symmetric product $S^{2g-2d-1}C$.

Conversely, if we are given a positive divisor of degree $2g - 2d - 1$ we get a holomorphic line bundle U of degree $2d$ together with a section ϕ of $U^{-1}K_C(1)$ vanishing on this divisor. To build a Higgs bundle from this, we need to choose a square root of U – there are 2^{2g} possibilities (think about why this is true), and since destabilising line bundles are unique[7] each choice uniquely determines a rank 2 holomorphic bundle V. ϕ is only determined up to nonzero scalar multiple, but all such multiples are related by an automorphism of V, and so are gauge equivalent.

[5] Since the trace of the curvature is $F(\wedge^2 V)$.
[6] Note that if it weren't for the Higgs field, L would be a destabilising subbundle for V.
[7] Something that we haven't shown, but is true.

So we have shown that $\mu^{-1}\left(\pi\left(d-\frac{1}{2}\right)\right)$ is diffeomorphic to the pullback

$$\begin{array}{ccccc}
\widetilde{S^{2g-2d-1}C} & \longrightarrow & \mathrm{Jac}(C) & & x \\
\downarrow & & \downarrow & & \downarrow \\
S^{2g-2d-1}C & \longrightarrow & \mathrm{Jac}(C) & & 2x
\end{array}$$

$$(p_i)_{i=1,\ldots,2g-2d-1} \longmapsto \left(\textstyle\sum_i p_i\right) - (2g-2d-1)c$$

It remains to calculate the value of the index. We're not going to do this calculation – as for $\mu^{-1}(0)$ the first step is to use the fact that when a Morse function arises as the moment map for a circle action on a Kähler manifold, the subbundle of the normal bundle on which the Hessian acts with negative eigenvalues is also the subbundle on which the circle action acts by negative weights. □

Having understood various properties of μ, we can prove results about the topology of $\mathbf{Higgs}^1_{SL_2}(C)$ using Morse theory. For instance:

Theorem 2.2 ([Hit87, Thm 7.6.iv])**.** *The Poincaré polynomial of $\boldsymbol{Higgs}^1_{SL_2}(C)$ is*

$$\begin{aligned}
P_t(\boldsymbol{Higgs}^1_{SL_2}(C)) = &\ \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} \\
&- \frac{t^{4g-4}}{4(1-t^2)(1-t^4)}\left((1+t^2)^2(1+t)^{2g} - (1+t)^4(1-t)^{2g}\right) \\
&- (g-1)t^{4g-3}\frac{(1+t)^{2g-2}}{(1-t)} + 2^{2g-1}t^{4g-4}\left((1+t)^{2g-2} - (1-t)^{2g-2}\right)
\end{aligned}$$

We're not going to run through this computation, but you can probably guess the method of proof: the Morse function μ is perfect, so calculating the Poincaré polynomial for the full moduli space by calculating the Poincaré polynomials of the critical submanifolds. These are of two types:

- The index zero submanifold, $\mu^{-1}(0)$, is the moduli of rank 2 stable bundles of fixed determinant and on degree – one can consult Atiyah and Bott [AB83] to learn its Poincaré polynomial.
- The other critical manifolds are covering spaces of symmetric products of the curve C.

2.1.1 A very short word on higher rank Higgs bundles

If V is a holomorphic vector bundle of rank > 2, the above analysis can be performed up to a point. In particular, for a $U(1)$-fixed (V,Φ), the splitting of V into a sum of line bundles with Φ strictly lower triangular becomes a decomposition of V into subbundles with respect to which Φ is block subdiagonal, i.e.

$$\Phi = \begin{pmatrix} 0 & 0 & & \\ \phi_1 & 0 & & \\ 0 & \phi_2 & & \\ & & \ddots & \end{pmatrix}$$

– the terminology to google is "system of Hodge bundles" [Sim92, §4]. It becomes much harder to give a description of the critical manifolds that you can actually compute with (the analog of the covers of symmetric powers of the curve); see [Got94] for the rank 3 case.

2.2 Preview of nonabelian Hodge theory

Recall from Section 1.3.2 that the moduli of Higgs bundles is a hyperkähler manifold. The description we have given of it so far (i.e. the description as literally the moduli space of *Higgs bundles*) has been relative to the complex structure I – let's now ask whether we can naturally interpret the moduli space in the *other* complex structures.

We'll begin with complex structure J. Recall that with respect to the description $T_p(\mathcal{A}\times\Omega) = \Omega^{0,1}_C(\mathrm{ad}(P)_\mathbb{C})\oplus\Omega^{1,0}_C(\mathrm{ad}(P)_\mathbb{C})$, J is defined by

$$J(\dot{A},\dot{\Phi}) = (i\dot{\Phi}^*, -i\dot{A}^*).$$

Let's find a more natural expression for this complex structure. Define an isomorphism

$$\begin{aligned} \mathcal{A}\times\Omega &\xrightarrow{\quad\alpha\quad} \mathcal{A}\times\overline{\mathcal{A}} \\ (A,\Phi) &\longmapsto (\bar{\partial}_A+\Phi^*, \partial_A+\Phi) \end{aligned}$$

where $d_A = \partial_A + \bar{\partial}_A$ is the covariant derivative corresponding to the unitary connection A. The derivative of α is

$$\alpha_*(\dot{A},\dot{\Phi}) = (\dot{A}+\dot{\Phi}^*, -\dot{A}^*+\dot{\Phi}),$$

and so

$$\alpha_*(J(\dot{A},\dot{\Phi})) = \alpha_*(i\dot{\Phi}^*, -i\dot{A}^*) = (i\dot{\Phi}^*+i\dot{A}, i\dot{\Phi}-i\dot{A}^*) = i\alpha_*(\dot{A},\dot{\Phi}).$$

So $\alpha : (\mathcal{A}\times\Omega, J)\simeq(\mathcal{A}\times\overline{\mathcal{A}}, i)$, the natural complex structure on the latter space. We can interpret elements of $\mathcal{A}\times\overline{\mathcal{A}}$ as *complex* $PSL(2,\mathbb{C})$ connections by taking

$$(\bar{\partial}_1,\partial_2)\mapsto\partial_2+\bar{\partial}_1.$$

Suppose that (A,Φ) solves Hitchin's equations (1.4), and consider the corresponding $PSL(2,\mathbb{C})$-connection $D_{A,\Phi} := d_A+\Phi+\Phi^*$. We compute that

$$\begin{aligned} F(D_{A,\Phi}) &= D_{A,\Phi}\circ D_{A,\Phi} \\ &= d_A^2 + d_A(\Phi) - \Phi d_A + d_A(\Phi^*) - \Phi^* d_A + \Phi d_A + \underbrace{\Phi\Phi}_{=0} + \Phi\Phi^* + \Phi^* d_A + \Phi^*\Phi + \underbrace{\Phi^*\Phi^*}_{=0} \\ &= \underbrace{F(A)+[\Phi,\Phi^*]}_{=0\ (1.4)} + \underbrace{\bar{\partial}_A\Phi+\partial_A\Phi^*}_{=0\ (1.4)} + \underbrace{\partial_A\Phi+\bar{\partial}_A\Phi^*}_{=0\ (\text{degree reasons})} = 0 \end{aligned}$$

so that $D_{A,\Phi}$ is a *flat* $PSL(2,\mathbb{C})$-connection. Hitchin then performs an analysis to show that in fact the corresponding quotient is – almost – the moduli space of (irreducible) flat $PSL(2,\mathbb{C})$-connections on C.

Why almost? This has to do with the fact that in order to achieve a smooth moduli space we had to study $\mathbf{Higgs}^1_{SL_2}(C)$ instead of $\mathbf{Higgs}^1_{PGL_2}(C)$. To describe the moduli space $\mathbf{Higgs}^1_{SL_2}(C)$ in the J complex structure, we'll use the description of the moduli space of flat connections as a character variety:

The fundamental group of C,

$$\pi_1(C) = \left\langle A_1,\ldots,A_g,B_1,\ldots,B_g \;\middle|\; \prod_{i=1}^{g}[A_i,B_i]=1 \right\rangle,$$

has a universal central extension

$$0\to\mathbb{Z}\to\Gamma\to\pi_1(C)\to 1$$

generated by the A_i and B_j, together with a central element c and relation $\prod_{i=1}^g[A_i,B_i]=c$. Representations of $\Gamma\to SL(2,\mathbb{C})$ can either send $c\mapsto 1$ – in which case they factor through $\pi_1(C)$ – or $c\mapsto -1$, which is the situation resulting in a flat $PSL(2,\mathbb{C})$ connection with nontrivial Stiefel-Whitney class w_2.

The resulting diffeomorphism is the first hint of the *non-abelian Hodge theorem*:

$$\mathbf{Higgs}^1_{SL_2}(C) \xrightarrow[\text{diffeo.}]{\sim} \mathrm{Hom}(\Gamma, SL(2,\mathbb{C}))^{c\mapsto -1,\mathrm{irr}}/SL(2,\mathbb{C})$$

Remark 2.1. For those who are interested: the reference for the non-abelian Hodge theorem is Simpson's paper [Sim92].

References

[AB83] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[Ati57] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[Got94] Peter B. Gothen. The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface. *Internat. J. Math.*, 5(6):861–875, 1994.

[Hit87] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[NS65] M. S. Narasimhan and C. S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 82:540–567, 1965.

[Sim92] Carlos T. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, (75):5–95, 1992.

[Uhl82] Karen K. Uhlenbeck. Connections with L^p bounds on curvature. *Comm. Math. Phys.*, 83(1):31–42, 1982.