INTRODUCTION TO MIRROR SYMMETRY (SUPERSCHOOL ON DERIVED CATEGORIES AND D-BRANES, EDMONTON, JULY 2016).

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1. A guide to the references.

- The material on complex geometry, Hodge theory and Kähler geometry can be found in [5].
- A derivation of the Hodge diamond symmetries from mirror symmetry can be found in [4].
- An explanation of the numerology for the quintic threefold can be found in the corresponding lecture of [1].
- The homotopy version of the Lefschetz hyperplane theorem is due to [2].
- The book [3] is a good general reference, as well as containing a lot of content overlap with the above.

2. Review of differential forms and complex geometry.

2.1. **Differential topology.** We begin by recalling the definition of *de Rham cohomology*: on a smooth n-manifold M we assign the dg-algebra of *differential forms* on M, $(\Omega^\bullet(M), d_{\mathrm{dR}})$. These are sections of the exterior algebra on the bundle T^*M, with grading given by form degree and differential the de Rham differential $d = d_{\mathrm{dR}}$. Concretely, recall that in local coordinates $(x^1, \ldots, x^n)$ the de Rham differential is defined on functions by

$$df = \frac{\partial f}{\partial x^i} dx^i,$$

where we have used the Einstein convention of summing over pairs of raised and lowered indices, and is extended to higher degree forms via the Leibniz rule. We define the *de Rham cohomology* of M to be the cohomology of $(\Omega^\bullet(M), d_{\mathrm{dR}})$, i.e.

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k \to \Omega^{k+1})}{\mathrm{im}\ (d : \Omega^{k-1} \to \Omega^k)}.$$

We will define the *Betti numbers* of a closed manifold M to be $b_k = \dim H^k_{\mathrm{dR}}(M)$ – this is a nonstandard definition, but for a closed smooth manifold it is equivalent to the standard definition from algebraic topology. There is in this case a symmetry on the Betti numbers $b_k = b_{n-k}$, implied by the stronger theorem of *Poincaré duality.*

2.2. **Complex geometry.** We now review some facts and definitions from complex geometry. Let (X, J) be a complex d-manifold[1]. Unless required for clarity we will omit the complex structure operator J, and we will call X a *(complex) d-fold* for short.

We will write TX and T^*X for the *holomorphic* tangent and cotangent bundles of X, obtained by the natural identification of the tangent bundle with the $+i$-eigenspace of J in the complexified tangent bundle

$$TX \otimes \mathbb{C} \cong \underbrace{T^{1,0}X}_{+i} \oplus \underbrace{T^{0,1}X}_{-i}.$$

Dualizing this decomposition similarly decomposes the complexified cotangent bundle into $(1, 0)$ and $(0, 1)$ summands, and we define

$$(T^{p,q}X)^* = \bigwedge^p (T^{1,0}X)^* \otimes \bigwedge^q (T^{0,1}X)^*$$

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[1]The dimension d here is the *complex* dimension of the manifold.

the sections of which are the differential (p,q)-forms $\Omega^{p,q}(X)$. By composing the de Rham differential with the projections maps $\Omega^1(X;\mathbb{C}) \to \Omega^{p,q}(X)$, $p+q=1$, we obtain a decomposition

$$d_{\mathrm{dR}} = \partial + \bar{\partial}, \quad \partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \quad \bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X).$$

By taking cohomology with respect to the $\bar{\partial}$-operator, we arrive at the *Dolbeault cohomology groups*

$$H^{p,q}(X) = \frac{\ker(\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{\mathrm{im}\,(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})},$$

and we define the *Hodge numbers* of X to be the dimensions $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$.

Remark The *Dolbeault theorem* allows us to identify $H^{p,q}(X)$ with the sheaf cohomology groups $H^q(\Omega^p)$, where Ω^p is the sheaf of *holomorphic* p-forms on X.

We may present the Hodge numbers graphically in the *Hodge diamond* of X; e.g. the Hodge diamond of a 3-fold is

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & & \\
 & & \nwarrow\partial & & \bar{\partial}\nearrow & &
\end{array}$$

(Hodge diamond for a complex 3-fold)

We remark that although it is not displayed in the above diamond, there is a symmetry between Hodge numbers of the form $h^{p,q} = h^{d-p,d-q}$. This symmetry can in particular be derived via the theorem of *Serre duality*.

3. Kähler geometry.

3.1. **Definition and examples.** Recall that a *symplectic form* on a manifold M is a closed and non-degenerate 2-form $\omega \in \Omega^2(M)$.

Definition 1. A *Kähler manifold* is the data of a complex manifold equipped with a symplectic form, (X, J, ω), satisfying the condition that the symmetric 2-tensor g defined by $g(V,W) = \omega(V, JW)$ is a Riemannian metric for which J is orthogonal.

Remark A Riemannian metric g for which J is orthogonal is called a *Hermitian metric*. We note in passing that it would have been equivalent to require the Hermitian metric g be data and $d\omega = 0$ be a condition.

Observe that the above also implies that $(J^*\omega)(V,W) = \omega(JV, JW) = \omega(V,W)$, i.e. that ω is a $(1,1)$-form.

Example 1. Consider the projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times$, and let $s : \mathbb{CP}^n \to \mathbb{C}^{n+1} - \{0\}$ be a local section of the projection map (i.e. a set of local coordinates). Define a 2-form locally by

$$\omega_{FS} = -i\bar{\partial}\partial \log |s|.$$

It is an exercise to show that this is independent of the choice of section, and that the global 2-form obtained is a Kähler form. The form ω_{FS} is called the *Fubini-Study form.*

The example of projective space now provides us with a wealth of further examples of Kähler manifolds via the following proposition.

Proposition 3.1. *Any complex submanifold of a Kähler manifold is naturally Kähler.*

Proof. This follows immediately from the fact that the restriction of a Riemannian metric to a submanifold is a Riemannian metric, and the invariance of the tangent bundle of a complex submanifold under the complex structure operator. □

3.2. **Hodge theory.** On a closed Riemannian manifold (M, g) there is a deep relationship between the de Rham cohomology of M and solutions to the Laplace equation–*harmonic forms*–which goes by the name of Hodge theory. Specifically, letting $\mathcal{H}^k(M)$ denote the space of harmonic k-forms, Hodge theory provides an isomorphism

$$\mathcal{H}^k(M) \cong H^k_{\mathrm{dR}}(M).$$

On closed complex manifolds, a Kähler structure allows us to refine this to a statement about Dolbeault cohomology and its relation to de Rham cohomology, leading to the following extra relations between Hodge and Betti numbers:

$$h^{p,q} = h^{q,p}, \qquad b_k = \sum_{p+q=k} h^{p,q}.$$

Introducing these symmetries and the symmetry obtained from Serre duality, we can refine the Hodge diamond for a *Kähler* 3-fold to the following (we include the Betti numbers also):

$$\begin{array}{ccccccc|l} & & & h^{0,0} & & & & b_0 = h^{0,0} \\ & & h^{1,0} & & h^{1,0} & & & b_1 = 2h^{1,0} \\ & h^{2,0} & & h^{1,1} & & h^{2,0} & & b_2 = 2h^{2,0} + h^{1,1} \\ h^{3,0} & & h^{2,1} & & h^{2,1} & & h^{3,0} & b_3 = 2h^{3,0} + 2h^{2,1} \\ & h^{2,0} & & h^{1,1} & & h^{2,0} & & b_2 = 2h^{2,0} + h^{1,1} \\ & & h^{1,0} & & h^{1,0} & & & b_1 = 2h^{1,0} \\ & & & h^{0,0} & & & & b_0 = h^{0,0} \end{array}$$

(Hodge diamond for a Kähler 3-fold)

4. Mirror symmetry and Calabi-Yau manifolds.

4.1. **Statement of mirror symmetry.** The version of mirror symmetry that we will discuss has the following, rough principal at its core:

Two manifolds X and $X^\vee$ are mirror dual if there is a correspondence between the parameters deforming the Kähler structures of one manifold and the parameters deforming the complex structures of the other manifold.

Note that in the above formulation there are many structures that X and $X^\vee$ must have that we have failed to make explicit.

4.2. **Calabi-Yau manifolds.** We will partially remedy the omission of any necessary structures on X and $X^\vee$ now.

Definition 2. The *canonical bundle* of a complex d-fold X is $K_X := \bigwedge^d T^*X$, the bundle of holomorphic d-forms on X.

Definition 3. A compact Kähler manifold X is called *Calabi-Yau* if it has trivial canonical bundle.

From here on out we will assume that all manifolds are **connected and simply connected Calabi-Yau**.

Remark The simple connectedness assumption implies that the Calabi-Yau condition is equivalent to the vanishing of the first Chern class $c_1(X)$, as there are then no topologically trivial but holomorphically nontrivial line bundles.

Our assumptions allow us to further reduce the number of parameters in the Hodge diamond for a 3-fold as follows:

(1) Triviality of the canonical bundle implies that $h^{3,0} = 1$ (in words: up to scaling there is a unique holomorphic top form).

(2) Connectedness implies $b_0 = 1$.
(3) Simply connected implies $b_1 = 0$, hence $h^{0,1} = 0$.
(4) Serre duality *together with* triviality of the canonical bundle implies that $h^{0,2} = h^{0,1} = 0$.

This leaves two parameters remaining in the Hodge diamond:

$$\begin{array}{ccccccc|c} & & & 1 & & & & 1 \\ & & 0 & & 0 & & & 0 \\ & 0 & & h^{1,1} & & 0 & & h^{1,1} \\ 1 & & h^{2,1} & & h^{2,1} & & 1 & 2(h^{2,1}+1) \\ & 0 & & h^{1,1} & & 0 & & h^{1,1} \\ & & 0 & & 0 & & & 0 \\ & & & 1 & & & & 1 \end{array}$$

(Hodge diamond for a connected, simply connected Calabi-Yau 3-fold)

4.3. **Mirror symmetry for simply connected Calabi-Yau 3-folds.** We may interpret the parameters $h^{1,1}$ and $h^{2,1}$ as follows.

First, recall that a Kähler form ω on X is a closed (1,1)-form. The converse is clearly not true for two reasons: an arbitrary (1,1)-form need not come from a real 2-form, and even if it does it may not satisfy the required positivity condition. Under certain assumptions however–including our case of a simply-connected Calabi-Yau 3-fold–the space of admissible Kähler forms is an *open* cone inside of $H^2_{\mathrm{dR}}(X)$, and so the space of (1,1)-forms whose real part is Kähler is open inside of $H^{1,1}(X)$. We therefore say that the number of *Kähler parameters* is given by $h^{1,1}(X)$.

Second, recall that infinitesimal deformations of the complex structure of a manifold X are parametrized by $H^1(TX)$. Triviality of K_X implies triviality of its dual $\bigwedge^3 TX$, and so the wedge pairing

$$\wedge : TX \otimes \bigwedge^2 TX \to \bigwedge^3 TX$$

induces an identification $TX \cong \bigwedge^2 T^*X$. Hence $H^1(TX) = H^1(\bigwedge^2 T^*X) = H^{2,1}(X)$, and so $h^{2,1}(X)$ is the number of *complex structure parameters* for X.

The rough principal given above now leads us to make the following prediction:

If two simply connected Calabi-Yau 3-folds X and $X^\vee$ are mirror dual, then

$$h^{1,1}(X) = h^{2,1}(X^\vee) \quad \textit{and} \quad h^{1,1}(X^\vee) = h^{2,1}(X).$$

Remark This prediction may be refined to $h^{p,q}(X) = h^{d-p,q}(X^\vee)$ on higher dimensional Calabi-Yau manifolds with $H^2(\mathcal{O}) = 0$.

5. Canonical Example: The quintic threefold.

Consider the zero set $Q \subset \mathbb{CP}^4$ of a degree 5 polynomial p, i.e. p is a section of $\mathcal{O}(5)$. Since Q is cut out of $\mathbb{CP}^4$ by a single equation, it is a 3-fold. Assuming Q is nonsingular, it inherits a Kähler structure from the Fubini-Study metric for $\mathbb{CP}^4$.

Q is simply connected by the Lefschetz hyperplane theorem ($\pi_1(Q) \xrightarrow{\sim} \pi_1(\mathbb{CP}^4) = 0$) and by the adjunction formula

$$c(Q) = \frac{(1 + c_1(H))^5}{1 + 5c_1(H)} = 1 + O(c_1(H)^2)$$

where H is the hyperplane bundle on $\mathbb{CP}^4$, we see that $c_1(Q) = 0$. Hence Q is a Calabi-Yau 3-fold.

There are 126 degree 5 monomials in 5 variables, hence the dimension of the space $H^0(\mathcal{O}(5)|_Q)$ of homogeneous degree 5 polynomials *not* vanishing on Q is 125 (one simply excludes p). Counting (infinitesimal) deformations of $\mathbb{CP}^4$ gives

$$\dim H^0(T\mathbb{CP}^4) = \dim(PGL_5\mathbb{C}) = \dim(GL_5\mathbb{C}) - \dim(\mathbb{C}^\times) = 5^2 - 1 = 24.$$

Recalling that we have $TQ \cong \bigwedge^2 T^*Q$ (since Q is Calabi-Yau),

$$H^0(TQ) = H^0(\Omega^2_Q) = H^{2,0}(Q) = 0$$

by calculations we have already performed for the Hodge diamond of a simply-connected threefold. Taking the long exact sequence associated to the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^4} \to \mathcal{O}_{\mathbb{CP}^4}(1)^{\oplus(5)} \to T\mathbb{CP}^4 \to 0$$

gives us $H^1(T\mathbb{CP}^4) = 0$; hence the long exact sequence associated to the adjunction short exact sequence

$$0 \to TQ \to T\mathbb{CP}^4 \to \mathcal{O}(5)|_Q \to 0$$

yields

$$H^1(TQ) = H^0(\mathcal{O}(5)|_Q)/H^0(T\mathbb{CP}^4).$$

Via dimension counting we see that $\dim H^1(TQ) = 125 - 24 = 101$. Hence, $h^{2,1}(Q) = 101$. We also have that $h^{1,1} = 1$ (this is another consequence of the Lefschetz hyperplane theorem: $H_2(Q) = H_2(\mathbb{CP}^4) = \mathbb{Z}$), and so the Hodge diamond for the quintic 3-fold is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(Hodge diamond for the quintic 3-fold)

We want to construct a mirror to the quintic. We consider the family of quintics

$$Q_\psi = \{[X_0 : \cdots : X_4] \in \mathbb{CP}^4 \mid f_\psi = X_0^5 + \cdots + X_4^5 - 5\psi X_0X_1X_2X_3X_4 = 0\}.$$

Calculating the derivative of f_ψ, one shows that Q_ψ is smooth provided ψ is not a fifth root of unity.

Let $G = \{(a_0, \ldots, a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 \mid \sum a_i = 0\}/\langle (a, a, a, a, a)\rangle \cong (\mathbb{Z}/5\mathbb{Z})^3$. G acts on Q_ψ via

$$(a_0, a_1, a_2, a_3, a_4) \cdot [X_0 : X_1 : X_2 : X_3 : X_4] = [X_0\xi^{a_0} : X_1\xi^{a_1} : X_2\xi^{a_2} : X_3\xi^{a_3} : X_4\xi^{a_4}]$$

where $\xi = e^{\frac{2\pi i}{5}}$. This action is not free, and the points with nontrivial stabiliser–where at least two of the homogeneous coordinates vanish–produce singularities in Q_ψ/G. We may find a construct a "good" (in this case meaning *crepant*) resolution of the singularities of this quotient using techniques from toric geometry to obtain a nonsingular space $Q_\psi^\vee$ – as a part of this process, the singularities are replaced by new algebraic cycles which introduce 100 new $h^{1,1}$ parameters. Together with the original hyperplane class, we find that $h^{1,1}(Q_\psi^\vee) = 101$.

Furthermore, we see that we have at least a one parameter family of deformations in complex structure given by the coordinate ψ^5. It is possible to show that this is the only family of deformations, hence $h^{2,1}(Q_\psi^\vee) = 1$, and so the Hodge diamond for $Q_\psi^\vee$ is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(Hodge diamond for $Q_\psi^\vee$)

which is as predicted for the mirror to the quintic.

References

[1] D. Auroux, "Lecture notes from the spring 2009 topics in geometry class." [Online]. Available: https://math.berkeley.edu/~auroux/18.969-S09/

[2] R. Bott, "On a theorem of Lefschetz," *Michigan Math. J.*, vol. 6, pp. 211–216, 1959.

[3] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, ser. Clay Mathematics Monographs. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2003, vol. 1, with a preface by Vafa.

[4] C. Voisin, *Mirror symmetry*, ser. SMF/AMS Texts and Monographs. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 1999, vol. 1, translated from the 1996 French original by Roger Cooke.

[5] ——, *Hodge theory and complex algebraic geometry. I*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002, vol. 76, translated from the French original by Leila Schneps.