The Grothendieck Spectral Sequence
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1 Preliminaries on derived functors.

1.1 A computational definition of right derived functors.

We begin by recalling that a functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$ is called *left exact* if it takes short exact sequences (SES) in $\mathcal{A}$

$$0 \to A \to B \to C \to 0$$

to exact sequences

$$0 \to FA \to FB \to FC$$

in $\mathcal{B}$. If in fact F takes SES in $\mathcal{A}$ to SES in $\mathcal{B}$, we say that F is *exact*.

Question. *Can we measure the "failure of exactness" of a left exact functor?*

The answer to such an obviously leading question is, of course, yes: the *right derived functors* R^pF, which we will define below, are in a precise sense the unique extension of F to an exact functor.

Recall that an object $I \in \mathcal{A}$ is called *injective* if the functor

$$\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to \mathrm{Ab}$$

is exact. An *injective resolution* of $A \in \mathcal{A}$ is a quasi-isomorphism in $\mathrm{Ch}(\mathcal{A})$

$$A \to I^\bullet = (I^0 \to I^1 \to I^2 \to \cdots)$$

where all of the I^i are injective, and where we think of A as a complex concentrated in degree zero. If every $A \in \mathcal{A}$ embeds into some injective object, we say that $\mathcal{A}$ has *enough injectives* – in this situation it is a theorem that every object admits an injective resolution.

So, for $A \in \mathcal{A}$ choose an injective resolution $A \to I^\bullet$ and define the p^{th} *right derived functor of* F *applied to* A by

$$R^pF(A) := H^p(F(I^\bullet)).$$

Remark
- You might worry about whether or not this depends upon our choice of injective resolution for A – it does not, up to canonical isomorphism.
- Since $0 \to FA \to FI^0 \to FI^1$ is exact, $R^0F(A) \cong FA$.

If $R^pF(A) = 0$ for $p \neq 0$, we say that A is F*-acyclic*.

1.2 Some examples of derived functors.

I claim that everybody here knows at least one example of a derived functor, since in his talk yesterday on the Eilenberg-Moore Spectral Sequence, Richard W. introduced Tor groups.[1] As some other examples, we have:

- Sheaf cohomology: $H^p(X;\mathcal{F}) = R^p\Gamma(\mathcal{F})$, the right derived functors of global sections
$$\Gamma : \mathrm{Sh}(X) \to \mathrm{Ab}$$
- Group cohomology: $H^p(G;M) = R^p(-)^G(M)$, the right derived functors of taking G-invariants
$$(-)^G : G - \mathrm{Mod} \to \mathrm{Ab}$$
- Ext groups: $\mathrm{Ext}^p_R(M,N) = R^p\mathrm{Hom}_R(M,-)(N)$, the right derived functors of
$$\mathrm{Hom}_R(M,-) : R - \mathrm{Mod} \to \mathrm{Ab}$$

2 The Grothendieck Spectral Sequence.

2.1 Statement of the Grothendieck Spectral Sequence.

We now know enough to explain what the Grothendieck Spectral Sequence (GSS) is, and what problemn it is trying to solve. Suppose that $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ are abelian categories, and that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives: for instance, Ab, $R-\mathrm{Mod}$, $\mathrm{Sh}(X)$, $\mathcal{O}_X - \mathrm{Mod}$ for $(X,\mathcal{O}_X)$ a ringed space. Suppose further that we are given left exact functors
$$F : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad G : \mathcal{B} \to \mathcal{C}$$
such that F sends injective objects to G-acyclic objects. Then the GSS tells us how to *compose* the derived functors of F and G. Explicitly:

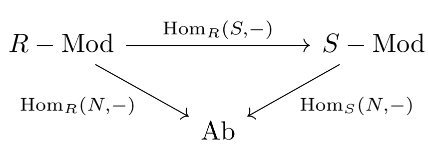
Theorem 2.1. *Given the above setup, there exists a convergent (1st quadrant) spectral sequence for every object A of $\mathcal{A}$,*
$$E_2^{p,q} = (R^pG \circ R^qF)(A) \Rightarrow R^{p+q}(G \circ F)(A).$$

Remark There is a dual (homological) version of this spectral sequence dealing with left-derived functors of right exact functors between categories with enough projectives, etc.

2.2 Base change for Ext.

In order to actually compute examples of the GSS we will need to specify specific functors to work with. As a first example, we will consider the **base change spectral sequence for Ext**.

Suppose that we have a ring homomorphism $R \to S$, choose $N \in S - \mathrm{Mod}$, and consider the functors



[1] This is an example of a *left* derived functor of the right exact functor of taking tensor product.

This triangle, commutes, since by tensor-hom adjunction

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_R(N \otimes_S S, M) = \mathrm{Hom}_R(N, M)$$

and since $\mathrm{Hom}_R(S, -)$ is right adjoint to an exact functor, it preserves injectives. Thus the hypotheses of the GSS are satisfied, and we have for each $M \in R - \mathrm{Mod}$ a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_S(N, \mathrm{Ext}^q_R(S, M)) \Rightarrow \mathrm{Ext}^{p+q}_R(N, M).$$

Example 1. Suppose that S is projective as an R-module. Then $\mathrm{Ext}^q_R(S, M) = 0$ for $q \neq 0$, and the spectral sequence collapses at the E_2-page to yield

$$\mathrm{Ext}^n_R(N, M) \cong \mathrm{Ext}^n_S(N, \mathrm{Hom}_R(S, M)).$$

Example 2. Consider the quotient map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, let $N \in \mathbb{Z}/n\mathbb{Z} - \mathrm{Mod}$, and suppose that M is an abelian group with no n-torsion (so that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M) = 0$). Then the Ext spectral sequence again collapses at the E_2-page (since there is only the $q = 1$ row), and we obtain

$$\mathrm{Ext}^p_{\mathbb{Z}/n\mathbb{Z}}(N, \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M)) \cong \mathrm{Ext}^{p+1}_{\mathbb{Z}}(N, M).$$

Since $\mathrm{Ext}^{\geq 2}_{\mathbb{Z}} = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M) = M/nM$, this gives

$$\mathrm{Ext}^1_{\mathbb{Z}}(N, M) \cong \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(N, M/nM).$$

We can interpret this statement as follows: the group extensions of an n-torsion abelian group N by an n-torsion free abelian group M are parametrised by the homomorphisms[2] $\mathrm{Hom}(N, M/nM)$.

2.3 Proof of the Grothendieck Spectral Sequence.

We break the proof of the GSS up into three parts. Many details have been discussed in previous talks, and so are omitted here.

2.3.1 Spectral Sequence of a Filtered Complex.

Suppose that we have a complex $C^\bullet \in \mathrm{Ch}(\mathcal{A})$ and a filtration by subcomplexes

$$\cdots \supseteq F^p C^\bullet \supseteq F^{p+1} C^\bullet \supseteq \cdots, \quad d(F^p C^n) \subseteq F^p C^{n+1}.$$

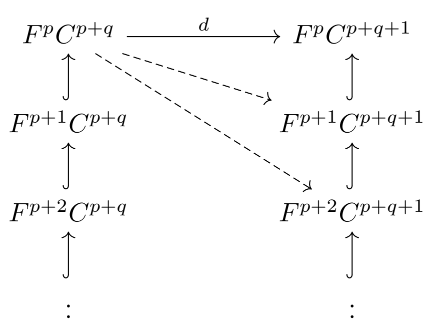
Assume that the filtration is exhaustive and bounded below. Then there is a spectral sequence

$$E_1^{p,q} = H^{p+q}(F^p C^\bullet / F^{p+1} C^\bullet) \Rightarrow H^{p+q}(C^\bullet).$$

This is obtained from the E_0-page

$$E_0^{p,q} = F^p C^{p+q} / F^{p+1} C^{p+q}$$

with all differentials induced by the differentials of the complex $C^\bullet$. We picture this as follows:



where the dashes are the partially defined maps which yield the higher differentials.

[2]Of abelian groups.

2.3.2 Spectral Sequence of a Double Complex.

A *double complex* is a collection of objects $C^{p,q} \in \mathcal{A}$ with differentials

$$\begin{aligned} d^h : C^{p,q} &\to C^{p+1,q} \\ d^v : C^{p,q} &\to C^{p,q+1} \\ (d^h)^2 = (d^v)^2 = d^h d^v + d^v d^h &= 0 \end{aligned}$$

The condition on the differentials ensures that

$$\mathrm{Tot}(C^{\bullet,\bullet})^n = \bigoplus_{p+q=n} C^{p,q}, \quad D = d^h + d^v$$

is a complex. There are two natural filtrations we an put on this, by truncating (c)olumns or (r)ows of $C^{\bullet,\bullet}$:

- Truncation of columns:

$$({}^{(c)}\tau_{\geq p} C)^{i,j} = \begin{cases} 0 & i < p, \\ C^{i,j} & i \geq p \end{cases}$$

  I.e. the truncation looks like

$$\begin{matrix} 0 & C^{p,q+1} & C^{p+1,q+1} \\ 0 & C^{p,q} & C^{p+1,q} \\ 0 & C^{p,q-1} & C^{p+1,q-1} \\ < p & \geq p & \end{matrix}$$

  and yields the filtration

$${}^{(c)}F^p \mathrm{Tot}(C)^n = \bigoplus_{\substack{i+j=n \\ i \geq p}} C^{i,j} = \mathrm{Tot}\left({}^{(c)}\tau_{\geq p} C\right)$$

- Truncation of rows:

$$({}^{(r)}\tau_{\geq q} C)^{i,j} = \begin{cases} 0 & j < q, \\ C^{i,j} & j \geq q \end{cases}$$

  I.e. the truncation looks like

$$\begin{matrix} C^{p-1,q+1} & C^{p,q+1} & C^{p+1,q+1} & \\ C^{p-1,q} & C^{p,q} & C^{p+1,q} & \geq q \\ 0 & 0 & 0 & < q \end{matrix}$$

  and yields the filtration

$${}^{(r)}F^q \mathrm{Tot}(C)^n = \bigoplus_{\substack{i+j=n \\ j \geq q}} C^{i,j} = \mathrm{Tot}\left({}^{(r)}\tau_{\geq q} C\right)$$

Let's focus on the spectral sequence coming from the column filtration. We have d^0 differential induced by $D = d^h + d^v$,

$$\begin{array}{ccc} {}^{(c)}E_0^{p,q} = {}^{(c)}F^p C^{p+q} / {}^{(c)}F^{p+1} C^{p+q} & \xrightarrow{d^0} & {}^{(c)}E_0^{p,q+1} \\ \Big| \simeq & & \Big| \simeq \\ C^{p,q} & \xrightarrow{\quad d^v \quad} & C^{p,q+1} \end{array}$$

So, taking cohomology,

$${}^{(c)}E_1^{p,q} = H^q(C^{p,\bullet}, d^v) =: H^q_{(v)}(C^{p,\bullet}).$$

Since $d^v \equiv 0$ on E_1, the differential d^2 is induced by d^h,and so we find that

$${}^{(c)}E_2^{p,q} = H^p(H^q_{(v)}(C^{\bullet,\bullet}), d^h) =: H^p_{(h)}(H^q_{(v)}(C^{\bullet,\bullet})).$$

Similarly, for the row truncated filtration, we find

$$\begin{array}{ccc} {}^{(r)}E_0^{p,q} & \xrightarrow{d^0} & {}^{(r)}E_0^{p,q+1} \\ \Big| \simeq & & \Big| \simeq \\ C^{q,p} & \xrightarrow{d^h} & C^{q+1,p} \end{array}$$

so that

$${}^{(r)}E_1^{p,q} = H^q(C^{\bullet,p}, d^h) = H^q_{(h)}(C^{\bullet,p})$$

and

$${}^{(r)}E_2^{p,q} = H^p(H^q_{(h)}(C^{\bullet,\bullet}), d^v) = H^p_{(v)}(H^q_{(h)}(C^{\bullet,\bullet})).$$

So applying the spectral sequence of a filtered complex, we get that under nice circumstances (e.g. $C^{\bullet,\bullet}$ first quadrant), there are spectral sequences

$$\begin{aligned} {}^{(c)}E_2^{p,q} &= H^p_{(h)}(H^q_{(v)}(C^{\bullet,\bullet})) \Rightarrow H^{p+q}(\mathrm{Tot}(C)) \\ {}^{(r)}E_2^{p,q} &= H^p_{(v)}(H^q_{(h)}(C^{\bullet,\bullet})) \Rightarrow H^{p+q}(\mathrm{Tot}(C)) \end{aligned}$$

2.3.3 Derivation of the GSS.

We will now construct the GSS by playing these two spectral sequences against each other. Recall the setup: $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ left exact functors, $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and F sends injective objects to G-acyclic objects.

Choose for $A \in \mathcal{A}$ an injective resolution, $A \to I^\bullet$. Apply F to get $F(I^\bullet) \in \mathrm{Ch}^-(\mathcal{B})$. Now, we resolve *again* – specifically, we choose a *first quadrant Cartan-Eilenberg resolution* of $F(I^\bullet)$, $J^{\bullet,\bullet}$, i.e. we choose a double complex such that

- $F(I^\bullet) \to J^{\bullet,0}$ is a map inducing a quasi-isomorphism for each $F(I^p)$,
- $J^{\bullet,\bullet}$ is a complex of injectives, and
- the induced maps on coboundaries/cohomology induce injective resolutions of the $B^p(F(I^\bullet))$ and $H^p(F(I^\bullet))$ (e.g. $H^q_{(h)}(J^{\bullet,\bullet})$ is an injective resolution of $R^qF(A)$).

It is a theorem that such resolutions exist. We have

$${}^{(c)}E_2^{p,q} = H^p_{(h)}(H^q_{(v)}(G(J^{\bullet,\bullet})) \Rightarrow H^{p+q}(\mathrm{Tot}(G(J)))$$

But $H^q_{(v)}(G(J^{\bullet,\bullet}) = 0$ unless $q = 0$ since $F(I^p)$ is G-acyclic, so the sequence collapses at E_2, and converges to $R^p(G \circ F)(A)$ with no extension problems. So the other sequence is

$${}^{(r)}E_2^{p,q} = H^p_{(v)}(H^q_{(h)}(G(J^{\bullet,\bullet})) \Rightarrow R^{p+q}(G \circ F)(A)$$

Now,

$$\begin{aligned} H^p_{(v)}(H^q_{(h)}(G(J^{\bullet,\bullet})) &= H^p_{(v)}(G(H^q_{(h)}(J^{\bullet,\bullet})) \quad \text{(since in the CE resolution the kernels of the } d^h \text{ are injective)} \\ &= R^pG(R^qF(A)) \end{aligned}$$

where the last equality is because $H^q_{(h)}(J^{\bullet,\bullet})$ is an injective resolution of $R^qF(A)$. Putting this together we obtain the Grothendieck Spectral Sequence

$$E_2^{p,q} = R^pG(R^qF(A)) \Rightarrow R^{p+q}(G \circ F)(A).$$

3 Application: Leray-Serre Spectral Sequences.

Ernie presented a version of this sequence on Wednesday – the version I give will look a little more general, as I will consider more general maps and will allow coeffiecients in any sheaf. I will finish by applying Leray-Serre to a particular setup in algebraic geometry, and will derive some geometric consequences.

3.1 Rediscovering Leray-Serre.

Consider a map of topological spaces

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ & \searrow \quad \swarrow & \\ & * & \end{array}$$

with the unique maps to the one point space shown explicitly. Pushforward to a point is exactly the functor of global sections, so we obtain a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Sh}(X) & \xrightarrow{f_*} & \mathrm{Sh}(B) \\ & {\scriptstyle\Gamma}\searrow \quad \swarrow{\scriptstyle\Gamma} & \\ & \mathrm{Ab} & \end{array}$$

where $\mathrm{Sh}(X)$ is the category of sheaves of abelian groups on X. These functors are all left exact, because they are right adjoints. Moreover, since f^{-1} (the left adjoint to f_*) is exact, one can show that f_* sends injectives to injectives. So from the GSS we obtain a spectral sequence for any $\mathcal{F} \in \mathrm{Sh}(X)$,

$$E_2^{p,q} = H^p(B; R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X; \mathcal{F})$$

We can recover the version of Leray-Serre described by Ernie as follows: suppose f is a fibration with fibre F. $R^q f_* \mathcal{F}$ is the sheafification of

$$U \mapsto H^q(f^{-1}(U); \mathcal{F}(f^{-1}(U)))$$

and $f^{-1}(U) \simeq F$ for small enough U. Assuming further that $\mathcal{F} = G_X$ is a constant sheaf, that F is connected, and that the monodromy of the pushforward is trivial, we may identify $R^q f_* \mathcal{F}$ as the constant sheaf $H^q(F; G)_B$, so that

$$E_2^{p,q} = H^p(B; H^q(F; G)) \Rightarrow H^{p+q}(X; G)$$

as in Ernie's talk.

3.2 Leray-Serre in algebraic geometry.

To finish up, let us now extract some *geometric* information from this spectral sequence. Let $f : X \to B$ be a map of noetherian schemes over a field k (or of complex analytic manifolds)[3]. Suppose further that $\mathrm{char}(k) = 0$, that B is normal, and that f is a faithfully flat proper morphism with geometrically connected fibres: for instance X could be a flat family of projective varieties over a smooth base B. Then $f_* \mathcal{O}_X = \mathcal{O}_B$.

Consider the sheaf $\mathcal{O}_X^\times \in \mathrm{Sh}(X)$. The first right derived functor $R^1 f_* \mathcal{O}_X^\times$ is the sheafification of

$$U \mapsto H^1(f^{-1}(U); \mathcal{O}_{f^{-1}(U)}^\times) = \mathrm{Pic}(f^{-1}(U))$$

and so carries information about line bundles along the fibres of f: it is called the *relative Picard sheaf.*

Then looking at the terms which converge to H^1, the Leray-Serre spectral sequence says, roughly, that "$H^1(X; \mathcal{O}_X^\times) = \mathrm{Pic}(X)$ can be approximated by $H^1(B; \mathcal{O}_B^\times) = \mathrm{Pic}(B)$ and $H^0(B; R^1 f_* \mathcal{O}_X^\times)$", or in plain

[3]Here I will be agnostic in notation as to whether we are working in the étale or analytic topologies.

language, that "a line bundles on X is approximated by a line bundle on the base B together with a line bundle along the fibres".

More than this, however, the spectral sequence tells us precisely *how* these pieces fit together. The *five-term exact sequence* (which we did not discuss, but whose derivation should be considered an exercise) is

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2$$

which in the case we are considering becomes

$$0 \to \mathrm{Pic}(B) \to \mathrm{Pic}(X) \to H^0(B; R^1 f_* \mathcal{O}_X^\times) \to H^2(B; \mathcal{O}_B^\times) \to H^2(X; \mathcal{O}_X^\times)$$

Note that there are potential obstructions to gluing "fibrewise defined" line bundles into line bundles on X, which lie in $H^2(B; \mathcal{O}_B^\times)$. These obstructions have a geometric incarnation as "$\mathcal{O}^\times$-gerbes" (which is a story for another time).

References

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[2] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, 2nd ed., ser. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. [Online]. Available: http://dx.doi.org.ezproxy.lib.utexas.edu/10.1007/978-3-662-12492-5