Zero-curvature formulation for novel 2d field theories.

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Origin of the novel theories [Costello–Yamazaki, "Gauge Theory & Integrability III"]:

Let C be a Riemann surface, equipped with ω a holomorphic 1-form with simple zeroes and double poles. Consider the 4d Chern–Simons Lagrangian

$$S_{cs}[A] = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2\times C} \omega \wedge CS(A)$$

gauge field

(connection)

$$\frac{\delta S_{cs}}{\delta A} = 0$$

crit. pts of functional

where

$$CS(A) = \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right).$$

$$\boxed{\text{Equations of motion for theory: } F(A) = 0}$$

$$F = dA + \frac{1}{2}[A\wedge A]$$

Equip $\mathbb{R}^2$ with complex coordinates $w, \bar{w}$, and write

$$A = A_w \, dw + A_{\bar{w}} \, d\bar{w} + A_{\bar{z}}$$

Require:
- A has no $(1,0)$-component in the C-direction.
- At each simple zero of ω, either A_w or $A_{\bar{w}}$ has a simple pole.
- At each pole of ω, A vanishes.
- Only allow gauge transformations that vanish at the poles of ω.

To compactify on C:
- Consider $A_{\bar{z}}(w, \bar{w})$ as a $\mathbb{R}^2$-family deforming the bundle P $\qquad \bar{\partial}_P + A_{\bar{z}}(w, \bar{w})$
- Can solve uniquely for $A_w, A_{\bar{w}}$ given (some) $A_{\bar{z}}$.

$\Rightarrow$ Fields of 2d theory are maps

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] \arrow[dr, "\text{(really)}"'] & Bun_G(C, Q) \\ & \mathcal{M} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$Bun_G(C, Q)$: G-bundles on C trivialised on Q

Q: $\frac{1}{2}$-divisor of poles of ω

(Will see Lagrangian for 2d theory later.)

Mathematical Setup:

zeroes & poles

- C a proper curve over $\mathbb{C}$, equipped with 1-form ω
- effective divisors consisting of distinct points: P_1, P_2, Q satisfying for $D_i = P_i - Q$,

zeroes of ω $= P_1 + P_2$

$$\deg(D_i) = g-1, \quad \mathcal{O}(D_1 + D_2) = K_C.$$

- $\mathcal{M} = \{ P \mid H^0(C; \mathfrak{g}_P(D_1)) = 0 \} \subset \mathcal{B}un_G(C, Q)$
- $\mathcal{P} \longrightarrow C \times \mathcal{M}$ universal G-bundle
- $C_i = C \setminus D_i$, $\quad C_0 = C_1 \cap C_2$.

Will construct:

- An (algebraic) metric on $\mathcal{M}$
- A closed 3-form on $\mathcal{M}$
- Two families of flat connections on $P_z \to \{z\} \times \mathcal{M}$, $z \in C_0$

$$P \to C_0 \times \mathcal{M}$$
$$\downarrow$$
$$C_0$$

From this: Build

- an action $S[\sigma]$
- for each map σ to $\mathcal{M}$ a connection $D(\sigma)$ on spacetime

such that

$$\underbrace{\begin{array}{c}\sigma \text{ satisfies} \\ \text{EOM for} \\ \text{action } S\end{array} \iff \begin{array}{c} D(\sigma) \\ \text{is flat for all } z \in C_0\end{array}}_{\text{Main theorem (zero-curvature formulation)}}$$

> **For the moment, assume $Q = \emptyset$.**
> – simplifies notation
> – arguments generalise to $Q \neq \emptyset$

The metric:

Note that $\quad T_p\mathcal{M} = H^1(C; \mathfrak{g}_p)$.

Consider the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \Omega_C^{0,0}(\mathfrak{g}_p) \arrow[r] \arrow[d, "\bar\partial"'] & \Omega_C^{0,0}(\mathfrak{g}_p(D_i)) \arrow[d, "\bar\partial"'] \\
0 \arrow[r] & \Omega_C^{0,1}(\mathfrak{g}_p) \arrow[r] & \Omega_C^{0,1}(\mathfrak{g}_p(D_i)) \arrow[u, bend right, "\bar\partial^{-1}_{i,p}"']
\end{tikzcd}
$$

exists by cohomology vanishing assumption

$\bar\partial^{-1}_{i,p}$ can be represented by an integral kernel ("Szegö kernel"), which varies algebraically in p

see e.g. [Ben-Zvi – Biswas]

Let $\langle\, , \rangle$ be a nondegenerate invt pairing on $\mathfrak{g}$.

Define the metric at $P \in \mathcal{M}$ by

$$\boxed{g_p(A_1, A_2) = \int_C \omega \wedge \left\langle \bar{\partial}_1^{-1} A_1 \otimes A_2 + \bar{\partial}_1^{-1} A_2 \otimes A_1 \right\rangle, \quad A_1, A_2 \in \Omega_c^{0,1}(\mathfrak{g}_p).}$$

Proposition: This defines a metric.

Proof:

- smooth in P: follows from algebraic variation of Szegö kernel
- ~~gauge invariant~~: if $\phi \in \Omega_c^{0,0}(\mathfrak{g}_P)$, $A \in \Omega_c^{0,1}(\mathfrak{g}_P)$,

descends to cohomology then $\omega \langle \bar\partial_1^{-1} A, \phi \rangle$ is smooth (poles & zeroes cancel)

$$\Rightarrow g_P(A, \bar\partial\phi) = \int_C \omega \wedge \langle \bar\partial_1^{-1} A, \bar\partial \phi \rangle + \int_C \omega \wedge \langle A, \phi \rangle = \int_C \bar\partial\left(\omega \langle \bar\partial_1^{-1} A, \phi \rangle\right) = 0$$

- nondegeneracy: let $P_1 = p_1 + \cdots + p_n$

let $\mathbb{D}_i$ be a coord. disc around p_i with coord z_i s.t. $\omega = z_i \, dz_i$ — determines z_i up to ± 1

let $\{t_a\}$ be a basis of $\mathfrak{g}$

let $\delta_{|z_i|=\varepsilon}$ be the distributional $(0,1)$-form — or radially symmetric C^∞ mollification

$$\int g(z,\bar z)\, dz \wedge \delta_{|z|=\varepsilon} = \oint_{|z|=\varepsilon} g(z,\bar z)\, dz$$

Trivialise $\mathfrak{g}_P$ on the discs $\mathbb{D}_i$ and define

$$A_{ia} := \begin{cases} \frac{t_a}{z_i}\,\delta_{|z_i|=\varepsilon} & \text{on } \mathbb{D}_i \\ 0 & \text{on } C \setminus \mathbb{D}_i \end{cases}$$

gives a basis for tangent space

$$\rightsquigarrow \bar{\partial}_1^{-1} A_{ia} = \frac{t_a}{z_i}\,\delta_{|z_i|\le\varepsilon}$$

$$\Rightarrow g_P(A_{ia}, A_{jb}) = \int_C \omega \wedge \left\langle \frac{t_a}{z_i}\,\delta_{|z_i|\le\varepsilon}, \frac{t_b}{z_j}\,\delta_{|z_j|=\varepsilon} \right\rangle + \cdots$$

$\langle t_a, t_b \rangle$

$$= \delta_{ij} \oint_{|z_i|=\varepsilon} \frac{dz_i}{z_i} \langle t_a, t_b \rangle = 2\pi i\, \underbrace{\delta_{ij}\, K_{ab}}_{\text{nondegenerate}}$$

$\square$

The 3-form:

$$\Omega(A_1,A_2,A_3) = \sum_{s\in S_3} (-1)^s \int_C \omega \wedge \left\langle \left[A_{s(1)}, \bar\partial_1^{-1} A_{s(2)}\right], \bar\partial_2^{-1} A_{s(3)} \right\rangle , \quad A_1, A_2, A_3 \in \Omega^{0,1}_C(\mathfrak{g}_p)$$

Proof that Ω is a closed 3-form on $\mathcal{M}$ involves calculations similar to those we just did for g.

The connections:

Let $\mathcal{U} = \mathrm{Spec}(R) \subset \mathcal{M}$ be an affine patch, and consider the problem

"define a connection on $\mathcal{P}|_{C_0 \times \mathcal{U}}$ relative to C_0".

I will use the definition of a connection as an identification of fibres which lie in the same first-order nbhd. So, consider a square-zero extension

$$0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0, \quad \mathcal{U}' = \mathrm{Spec}(R')$$

Lifts of $\mathcal{P}|_{C \times \mathcal{U}}$ to $C \times \mathcal{U}'$ are parametrised by

$$R'\pi_{\mathcal{U}*}\left(\mathrm{ad}\left(\mathcal{P}|_{C \times \mathcal{U}}\right)\right) \underset{R}{\otimes} J$$

we'll calculate using a Čech complex

Take the cover of C

$$\mathcal{U} = \Big\{ \overset{C \setminus D_1}{C_1}, \coprod_{j=1}^{g-1} \mathbb{D}_j \Big\}$$

The exact sequence $0 \to \mathcal{O}_C \to \mathcal{O}_C(D_1) \to \mathcal{O}_{D_1}(D_1) \to 0$ on C yields

$$0 \to \mathrm{ad}(P|_{C\times U}) \longrightarrow \mathrm{ad}(P|_{C\times U}) \otimes \pi_u^* \mathcal{O}_C(D_1) \longrightarrow \bigoplus_{j=1}^{g-1} \mathrm{ad}(P|_{p_j \times U}) \otimes T_{p_j} C \to 0$$

on $C \times \mathcal{U}$.

Looking at Čech complexes we get a map

$$\begin{tikzcd}
\check{C}^0(\mathcal{U}; \mathrm{ad}(P)) \arrow[r, hook] \arrow[d] & \check{C}^0(\mathcal{U}; \mathrm{ad}(P) \otimes \pi_u^* \mathcal{O}_C(D_1)) \arrow[d, "\simeq"] \\
\check{C}^1(\mathcal{U}; \mathrm{ad}(P)) \arrow[r, hook] \arrow[ur, dashed] & \check{C}^1(\mathcal{U}; \mathrm{ad}(P) \otimes \pi_u^* \mathcal{O}_C(D_1))
\end{tikzcd}$$

lifts of P

trivialisation of lifts with first order poles on D_1

Given two lifts of P to $C \times U'$, $\dashrightarrow$ defines a unique isomorphism between the lifts after restricting to C_0.

Can patch this together on an open cover to define a cxn ∇^+.

Same argument for D_2 $\Longrightarrow$ cxn ∇^-

Can give a more explicit formula by choosing local coords and translating the above to the Dolbeault setting:

- Let $\{A_{ia}\}$ be Dolbeault reps of a basis of $T_p M$
- Let (λ^{ia}) be coordinates defined by $\bar{\partial}_{p+\vec{\lambda}} = \bar{\partial}_p + \lambda^{ia}[A_{ia}, -]$
- Write

$$\nabla^+ = d + \alpha_{ia} d\lambda^{ia}, \quad \nabla^- = d + \beta_{ia} d\lambda^{ia}$$

The connection components are given by the singular gauge transformations that trivialise the basis reps

$$\alpha_{ia}(\vec{\lambda}) = \bar{\partial}^{-1}_{p+\vec{\lambda},1} A_{ia} \qquad \beta_{ia}(\vec{\lambda}) = \bar{\partial}^{-1}_{p+\vec{\lambda},2} A_{ia}$$

Action of the 2d theory:

$$S[\sigma] = \int_{\mathbb{D}^2} \|d\sigma\|^2 \, dvol_{\mathbb{D}^2} + \frac{1}{3}\int_{\mathbb{D}^2\times\mathbb{R}_{\geq 0}} \tilde{\sigma}^* \Omega$$

$\tilde{\sigma}$ extension of σ to $\mathbb{D}^2\times\mathbb{R}_{\geq 0}$

$\mathbb{D}^2$ — 2-disc, coords t_1, t_2, metric η

$\sigma : \mathbb{D}^2 \longrightarrow M$

$\tilde{\sigma} : \mathbb{D}^2\times\mathbb{R}_{\geq 0} \longrightarrow M$

Let Σ be a 1-parameter family (in parameter τ) such that $\Sigma|_{\tau=0} = \sigma$.

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_{\mathbb{D}^2} \|d\Sigma\|^2 dvol$$

$$= \int_{\mathbb{D}^2} \frac{\partial \Sigma^{kc}}{\partial \tau}\Big|_{\tau=0} \eta^{\alpha\beta}\Bigg(-2\sigma^*(g_{ia,kc})\underbrace{\frac{\partial^2 \sigma^{ia}}{\partial t_\alpha \partial t_\beta}}_{\text{2nd order deriv. in }\sigma} - \sigma^*\underbrace{\left(\frac{\partial g_{kc,jb}}{\partial \lambda^{ia}} + \frac{\partial g_{ia,kc}}{\partial \lambda^{jb}} - \frac{\partial g_{ia,jb}}{\partial \lambda^{kc}}\right)}_{\text{Christoffel symbols}}\underbrace{\frac{\partial \sigma^{ia}}{\partial t_\alpha}\frac{\partial \sigma^{jb}}{\partial t_\beta}}_{\text{symmetric in } ia \leftrightarrow jb}\Bigg) dt_1 \wedge dt_2$$

$$\frac{d}{d\tau}\Big|_{\tau=0} \frac{1}{3}\int_{\mathbb{D}^2\times\mathbb{R}_{\geq 0}} \tilde{\Sigma}^*\Omega = \int_{\mathbb{D}^2} \frac{\partial \Sigma^{kc}}{\partial \tau}\Big|_{\tau=0} \underbrace{\sigma^*(\Omega_{kc,ia,jb})\epsilon^{\alpha\beta}}_{\text{anti-symmetric in indices } ia \leftrightarrow jb}\frac{\partial \sigma^{ia}}{\partial t_\alpha}\frac{\partial \sigma^{jb}}{\partial t_\beta} dt_1 \wedge dt_2$$

Add to get EOM

Induced connection:

Given a field $\sigma : \mathbb{D}^2 \longrightarrow \mathcal{M}$ define a cxn on $\mathbb{D}^2$ by

$$D(\sigma)_{\partial_1} := \left(\sigma^* \nabla^+\right)_{\partial_1} \; , \; D(\sigma)_{\partial_2} := \left(\sigma^* \nabla^-\right)_{\partial_2}$$

$\partial / \partial t_1$

In local coords:

$$D(\sigma) = d + \sigma^* \alpha\left(\frac{\partial}{\partial t_1}\right) dt_1 + \sigma^* \beta\left(\frac{\partial}{\partial t_2}\right) dt_2$$

This has curvature

$$\frac{F(\sigma)}{dt_1 \wedge dt_2} = \frac{\partial}{\partial t_1}\left(\sigma^*\beta\left(\frac{\partial}{\partial t_2}\right)\right) - \frac{\partial}{\partial t_2}\left(\sigma^*\alpha\left(\frac{\partial}{\partial t_1}\right)\right) + \left[\sigma^*\alpha\left(\frac{\partial}{\partial t_1}\right), \sigma^*\beta\left(\frac{\partial}{\partial t_2}\right)\right]$$

$$= \sigma^*\left(\beta_{ia} - \alpha_{ia}\right)\frac{\partial^2 \sigma^{ia}}{\partial t_1 \partial t_2} + \underbrace{\sigma^*\left(\frac{\partial \beta_{jb}}{\partial \lambda^{ia}} - \frac{\partial \alpha_{ia}}{\partial \lambda^{jb}} + [\alpha_{ia}, \beta_{jb}]\right)\frac{\partial \sigma^{ia}}{\partial t_1}\frac{\partial \sigma^{jb}}{\partial t_2}}_{\text{for later}}$$

Write as

$$\frac{1}{2}\sigma^*(\cdots)\left(\frac{\partial \sigma^{ia}}{\partial t_1}\frac{\partial \sigma^{jb}}{\partial t_2} + \frac{\partial \sigma^{ia}}{\partial t_2}\frac{\partial \sigma^{jb}}{\partial t_1}\right) + \frac{1}{2}\sigma^*(\cdots)\left(\frac{\partial \sigma^{ia}}{\partial t_1}\frac{\partial \sigma^{jb}}{\partial t_2} - \frac{\partial \sigma^{ia}}{\partial t_2}\frac{\partial \sigma^{jb}}{\partial t_1}\right)$$

THEN

symmetrise in $ia \leftrightarrow jb$

antisymmetrise in $ia \leftrightarrow jb$

Main Theorem:

$\sigma : \mathbb{D}^2 \longrightarrow \mathcal{M}$ is a solution to the EOM for S if and only if $D(\sigma)$ is flat.

Proof: Direct calculation – check that the eq^{ns} are proportional to eachother.

In particular, consider coefficients in each equation of

$$\frac{\partial^2 \sigma^{ia}}{\partial t_1 \partial t_2}, \quad \frac{\partial \sigma^{ia}}{\partial t_1}\frac{\partial \sigma^{jb}}{\partial t_2} \pm \frac{\partial \sigma^{ia}}{\partial t_2}\frac{\partial \sigma^{jb}}{\partial t_1}$$

and see that they are proportional by the same constant.

E.g. the coeffs of the second order term are

EOM	Flatness
$2g_{ia,kc}$	$(\beta_{ia} - \alpha_{ia})(z),\ z \in C$

For each basis vector A_{kc},

$$\int_C \omega \wedge \langle \beta_{ia} - \alpha_{ia}, A_{kc} \rangle = \int_C \omega \wedge \langle \bar\partial_2^{-1} A_{ia}, A_{kc} \rangle - \int_C \omega \wedge \langle \bar\partial_1^{-1} A_{ia}, A_{kc} \rangle$$

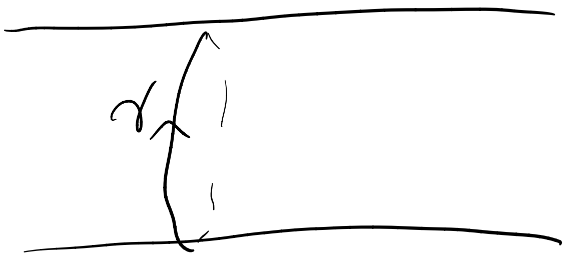
$\bar\partial_1^{-1}$ & $-\bar\partial_2^{-1}$ are adjoint

$$= -\int_C \omega \wedge \langle \bar\partial_1^{-1} A_{ia}, A_{kc} \rangle - \int_C \omega \wedge \langle A_{ia}, \bar\partial_1^{-1} A_{kc} \rangle$$

$$= -\frac{1}{2}\left(2g_{ia,kc}\right)$$

same constant appears in other calcs.

$S^1 \times \mathbb{R}$



- Have $D(\sigma)_z$ associated to each field, $z \in C_0$
- $\mathrm{Hol}_\gamma(D(\sigma)_z)$

Given $f \in \mathbb{C}[G]^G$,

$$F_{f,z}(\sigma) = f\left(\mathrm{Hol}_\gamma(D(\sigma)_z)\right)$$

Poisson structure

$\{F_{f,z} : EOM \longrightarrow \mathbb{C}\}$ is (supposed to be) an ∞-collection of Poisson commuting conserved quantities

Guess: in our case

$EOM(\mathbb{C}^\times \times \hat{\mathbb{D}}) = T^*LM$

$\Rightarrow$ Studying QM on LM

Example: $\mathbb{CP}^1$ with two marked points.

$$C = \mathbb{CP}^1 \qquad \omega = \frac{(z-p_1)(z-p_2)}{z^2}\,dz, \qquad \begin{array}{l} p_1 \neq p_2 \\ p_i \neq 0 \end{array}$$

2nd order poles at 0 & ∞

simple zeroes at p_1 & p_2

$$Q = 0 + \infty \qquad P_1 = p_1 \qquad D_1 = p_1 - 0 - \infty$$

Looking at G-bundles on $\mathbb{CP}^1$ with trivialisations at 0 & ∞.

Assume G is simple simply-connected.

<u>Claim</u>: cohomology vanishing condition $\Rightarrow$ G-bundles are trivial

$p_1 - 0 - \infty$

<u>Proof</u>: $H^0(\mathbb{CP}^1; \mathfrak{g}_P(-1)) = 0$. Suppose

$$E = \mathcal{O}(i_1) \oplus \cdots \oplus \mathcal{O}(i_n) \quad \text{with} \quad i_1 + \cdots + i_n = 0$$

Then $H^0(E \otimes \mathcal{O}(-1)) = 0 \iff i_k < 1 \ \forall k.$

So $i_1 = \cdots = i_n = 0.$ $\square$

Fix $P = \mathbb{CP}^1 \times G$, fix trivialisation at ∞ to agree with triv. of P.

Remaining degree of freedom is the trivialisation of P at 0.

$$\Rightarrow \mathcal{M} \cong G$$

(Sanity check: $T_p\mathcal{M} = H^1(\mathbb{CP}^1; \mathfrak{g} \otimes \mathcal{O}(-0-\infty)) \cong \mathfrak{g} \otimes H^1(\mathbb{CP}^1; \mathcal{O}(-2)) \cong \mathfrak{g}$.)

Szegö kernel:

$\mathfrak{g}\otimes\mathfrak{g}$-valued mero. f$^{\text{nc}}$ $\mathcal{S}(z,z')$ on $\mathbb{CP}^1\times\mathbb{CP}^1$ with

- simple pole along $z=z'$ with residue $c\in\mathfrak{g}\otimes\mathfrak{g}$ (Casimir)
- simple poles at $z=p_1$, $z'=p_2$
- simple zeroes at $z=0,\infty$, $z'=0,\infty$

$$\Rightarrow \quad \mathcal{S}(z,z') = \frac{zz'}{(z-z')(z-p_1)(z'-p_2)}\,c$$

Let $\{t_a\}$ be a basis for $\mathfrak{g}$. What is the corresponding basis for the tangent space to $\mathcal{M}$?

Tangent vector to $\mathcal{M}$ can be rep^d by

$$A \in \Omega^{0,1}(\mathbb{CP}^1; \mathcal{O}(-0-\infty)) \otimes \mathfrak{g}$$

Nonzero tangent vector $\Rightarrow$ A has no antideriv. in $\Omega^{0,0}(\mathbb{CP}^1; \mathcal{O}(-0-\infty))\otimes\mathfrak{g}$.

But $A = \bar\partial \chi$ for some $\chi \in \Omega^{0,0}(\mathbb{CP}^1; \mathcal{O}(-\infty)) \otimes \mathfrak{g}$ i.e. allow χ to be nonzero at 0

Then $\chi(0) \in \mathfrak{g}$ is the infinitesimal variation of the framing at 0.

So, take

$$A_a = (p_1 - p_2)\frac{z}{(z-p_1)(z-p_2)}\delta_{|z-p_1|=\varepsilon}\, t_a$$

Then $A_a = \bar{\partial}\chi_a$ for

$$\chi_a = -\left(\frac{p_1}{z-p_1}\delta_{|z-p_1|\geq\varepsilon} + \frac{p_2}{z-p_2}\delta_{|z-p_1|\leq\varepsilon}\right)t_a$$

and

$$\chi_a(0) = t_a$$

Then

$$g_{ab} = \int_{\mathbb{CP}^1} \omega \wedge \langle \bar{\partial}_1^{-1} A_a, A_b \rangle + \int_{\mathbb{CP}^1} \omega \wedge \langle \bar{\partial}_1^{-1} A_b, A_a \rangle$$

$$= K_{ab} \oint_{|z-p_1|=\varepsilon} \frac{dz}{z-p_1} \frac{(p_1-p_2)^2}{z-p_2} = 2\pi i (p_1-p_2) K_{ab}$$

and

$$\int_{\mathbb{CP}^1} \omega \wedge \langle [A_a, \bar{\partial}_1^{-1} A_b], \bar{\partial}_2^{-1} A_c \rangle$$

$$= \frac{1}{4} \oint_{|z-p_1|=\varepsilon} \frac{dz}{(z-p_1)^2} \frac{z}{(z-p_2)^2} \langle [t_a, t_b], t_c \rangle (p_1-p_2)^3$$

antisym. to get Ω_{abc}

$$= \frac{\pi i}{2} K_{cd} f^d_{ab} \frac{d}{dz}\left(\frac{z}{(z-p_2)^2} \right)_{z=p_1} \cdot (p_1-p_2)^3 = -\frac{\pi i}{2} K_{cd} f^d_{ab} (p_1+p_2)$$

$$g_{ab} = 2\pi i\, (p_1 - p_2)\, K_{ab}$$

$$\Omega_{abc} = -3\pi i\, (p_1 + p_2)\, K_{cd}\, f^{d}_{ab}$$

agree with calculations of Costello - Yamazaki (different method of calc.)

⚠ disagrees with C-Y

Note that for $p_1 = -p_2$, $\Omega \equiv 0$. So the main theorem in this case is a generalisation of the well-known result

"The harmonic map eq^{n} for surfaces mapping to Lie groups has a zero-curv. formulation."

[Pohlmeyer '76]

Some fun: the metric on real slices.

Physical theory requires target be a real pseudo-Riemannian mfld.

So let's take a real slice of $\mathcal{M}$, see what metric we get.

Take: • $\rho_C : C \to C$ antihol. involution satisfying $\rho_C^*(\omega) = \overline{\omega}$

• $\rho_G : G \to G$ —— " —— defining a real form of G

Defines a real structure on $\mathcal{M}$ by taking

$$(\varphi_{ij}) \xmapsto{\rho} (\rho_G \circ \varphi_{ij} \circ \rho_C)$$

$\leftarrow$ cocycle

Take real points $\mathcal{M}(\mathbb{R})$ with respect to this structure, and define a pseudo-Riemannian metric

$$g_{\mathbb{R}} := \frac{1}{2\pi i} g$$

Recall the basis $A_{ia} := \frac{t_a}{z_i} \delta_{|z_i|=\varepsilon}$. Assume $\{t_a\}$ is a basis for the real form of G.

$$\Rightarrow \rho^* A_{ia} = \frac{t_a}{\rho_c(\bar{z}_i)} \delta_{|z_i|=\varepsilon}$$

Since $\rho_c^* \omega = \overline{\omega}$, $\rho_c(z_i) = \lambda_i \bar{z}_i$ for $\lambda_i = \pm 1$.

- If $\lambda_i = +1$, A_{ia} is a tangent vector to $\mathcal{M}(\mathbb{R})$.
- If $\lambda_i = -1$, iA_{ia} is a tangent vector to $\mathcal{M}(\mathbb{R})$.

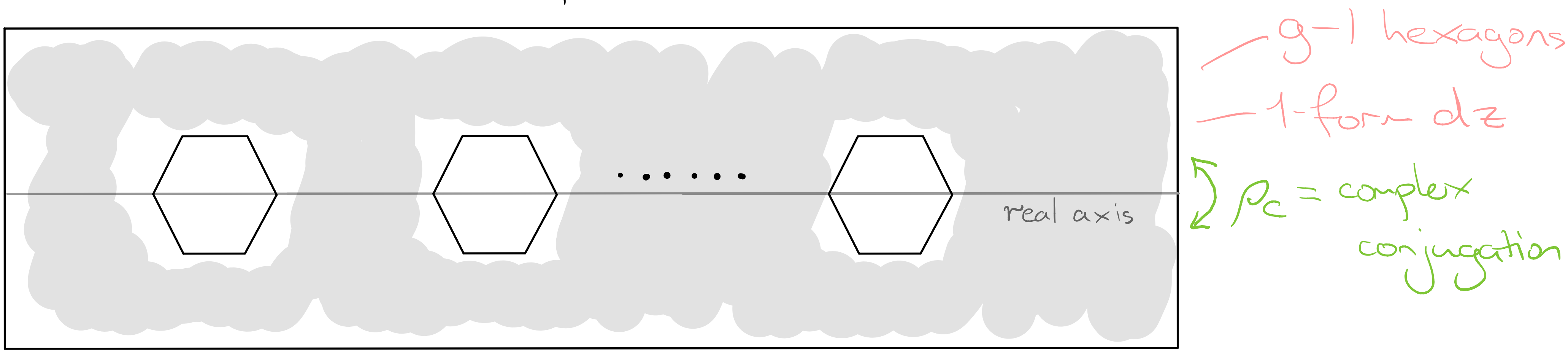
Since $\frac{1}{2\pi i} g(A_{ia}, A_{jb}) = \delta_{ij} \kappa_{ab}$ we obtain the following:

Prop^{n}: Let $\Sigma(\kappa)$ denote the signature of the form $\langle -, - \rangle$ restricted to the real form of $\mathfrak{g}$ defined by ρ_G.

Then the signature of $\mathfrak{g}_{\mathbb{R}}$ is $(\lambda_1 \Sigma(\kappa), \dots, \lambda_n \Sigma(\kappa))$.

What values can the tuple $(\lambda_1, \dots, \lambda_n)$ take?

Consider the following genus g surface with 1-form, rep^d as a translation surface:



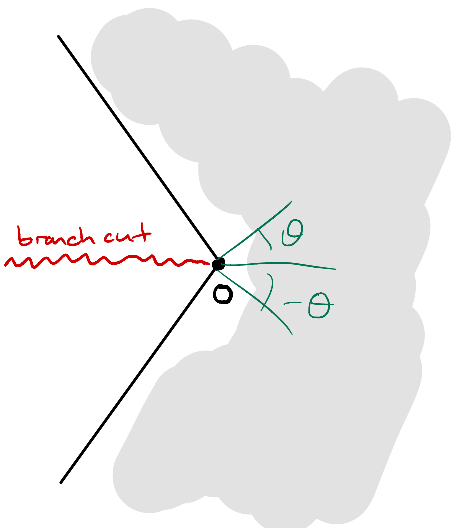
To define metric g, choose g−1 zeroes of ω (the corners of the hexagons). WLOG take zeroes on the real axis.

How does ρ_c act on a coordinate w centred at a zero?

$$w\,dw = dz \quad \Rightarrow \quad \text{“} w = 2\sqrt{z} \text{”}$$

Need to be careful with branch cuts! Let $z = re^{i\theta}$

"Rightmost" corners:



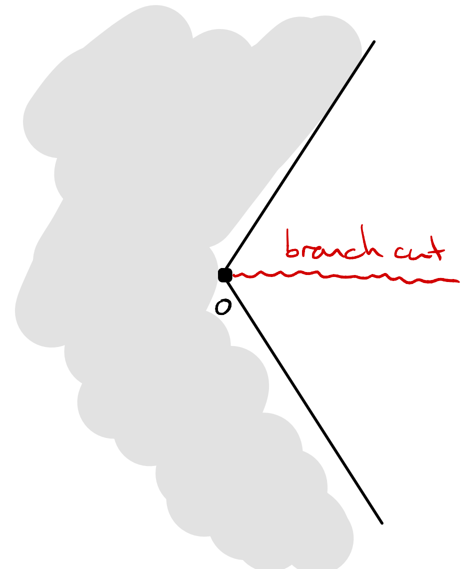
$w = 2\sqrt{r}\, e^{\frac{i\theta}{2}}, \quad -\pi < \theta < \pi$

$\rho_c : \theta \mapsto -\theta$

$\Rightarrow \rho_c(w) = \overline{w}$

$\Rightarrow \lambda_i = +1$

"Leftmost" corners:



$w = 2\sqrt{r}\, e^{\frac{i\theta}{2}}, \quad 0 < \theta < 2\pi$

$\rho_c : \theta \mapsto 2\pi - \theta$

$\Rightarrow \rho_c(w) = -\overline{w}$

$\Rightarrow \lambda_i = -1$

So by choosing left/rightmost points appropriately, can get any sequence of ± 1's!

Idea for example: $C = E_\tau$, $\tau \notin SL_2\mathbb{Z} \cdot i$

$$\omega = \wp(z)\,dz$$

1× double pole, 2× simple zeroes

For $\mathbb{CP}^1$: was able to write $A_a = \begin{pmatrix}\text{some nice meromorphic fn}\end{pmatrix} \cdot \delta$

$$H^\bullet(C; \mathfrak{g}_p(D_1)) = 0$$

SES:

$$0 \to \mathfrak{g}_P \longrightarrow \mathfrak{g}_P(D_1) \longrightarrow \bigoplus_{i=1}^{g-1} (\mathfrak{g}_P)_{p_i} \otimes T_{p_i}C \longrightarrow 0$$

on C

Cohomology LES:

$$0 \to \underbrace{\bigoplus_{i=1}^{g-1} (\mathfrak{g}_P)_{p_i} \otimes T_{p_i}C}_{\mathcal{O}_{p_i}(p_i)} \xrightarrow{\sim} \underset{T_P\mathcal{M}}{\overset{\|}{H^1(C;\mathfrak{g}_P)}} \longrightarrow 0$$

$\leftarrow$ "things near p_i with simple pole at p_i)

$$\frac{t_a}{z_i}\delta_{|z_i|=\varepsilon} \in \Omega^{0,1}_C(\mathfrak{g}_P)$$

$$\Omega^{0,0}_C(\mathfrak{g}_P(D_1)) \xrightarrow[\text{map"}]{\text{"residue}} \bigoplus_{i=1}^{g-1} (\mathfrak{g}_P)_{p_i} \otimes T_{p_i}C$$

$$\Omega^{0,1}_C(\mathfrak{g}_P) \longrightarrow \Omega^{0,1}_C(\mathfrak{g}_P(D_1)) \xrightarrow{\bar\partial^{-1}} \Omega^{0,0}_C(\mathfrak{g}_P(D_1))$$

$\{t_a\}$ basis for $\mathfrak{g}$

z_i coord at p_i

Trivialising $\mathfrak{g}_p$ near pts p_i, defining a $(0,1)$-form valued in $\mathfrak{g}_p$ by using this triv. to write

$$A_{ia} = \begin{cases} \dfrac{t_a}{z_i} \underbrace{\delta_{|z_i|=\varepsilon}}, \\ 0, \end{cases}$$

defined by prop.

$$\int g(z,\bar{z})\, dz \wedge \underbrace{\delta_{|z|=\varepsilon}} = \oint_{|z|=\varepsilon} g(z,\bar{z})\, dz$$

includes a $d\bar{z}$

<u>Idea</u>: sol^{ns} to eq^{ns} of motion on $\mathbb{C}^\times \times \hat{\mathbb{D}}$

$\overset{?}{\simeq} T^*L\mathcal{M}$ $\qquad\qquad\qquad\qquad \cup$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad S^1 \times$ inf. int.

<u>Vague</u>

Given $z \in \mathbb{C}$, $f \in \mathbb{C}[G]^G$

(Still a curve $\mathbb{C}$ floating around).

construct f^{ns} $\mathcal{F}_{z,f} : T^*L\mathcal{M} \longrightarrow \mathbb{C}$

$$\mathcal{F}_{z,f}(\sigma) = f\left(\mathrm{Hol}_{\mathbb{C}^\times} D(\sigma)\right)$$

Dream: this defines a bunch of commuting conserved quantities on $T^*L\mathcal{M}$.

$\nabla^+(z)$, $z \in C_0$ <u>family</u> of connections

Solve for A_w, $A_{\bar w}$ in $A_{\bar z}$:

- $A_w dw$ is the ∇^+ cxn form
- $A_{\bar w} d\bar w$ is the ∇^- cxn form

$$A_w(w, \bar w, z) \overset{?}{=} \bar\partial^{-1}_{1, p + A_{\bar z}} (A_{\bar z}) \quad \text{something like this}$$