APM 346 – Final Exam Practice Problems.

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(Problems are mostly taken from or variants of problems from [IvrXX] or [Str08].)

1 Introductory explicitly solvable problems

Problem 1. *Solve the equation* $5u_y + u_{xy} = 0$.

Problem 2. *Solve the equation* $u_{xy} - 4u_x = e^{x+5y}$.

Problem 3. *Solve the equation* $u_{xy} = u_x u_y$.

Problem 4. *Solve the system of equations*

$$\begin{aligned} u_{xy} &= 0, \\ u_{yz} &= 0, \\ u_{zx} &= 1. \end{aligned}$$

Solution. *Integrate* $u_{xy} = 0$ *to obtain* $u_y = f(y,z)$. *The second equation gives* $0 = f_z$, *so in fact* $u_y = f(y)$. *Hence* $u(x,y,z) = F(y) + G(x,z)$. *Now,*

$$1 = u_{zx} = G_{xz}.$$

So $G_x(x,z) = z + h(x)$, *and* $G(x,z) = xz + H(x) + A(z)$. *Putting it all together (and renaming the arbitrary functions in the solution) we have*

$$u(x,y,z) = xz + f(x) + g(y) + h(z).$$

2 Method of characteristics

Problem 5. *Solve the problem*

$$\begin{aligned} 2u_t + 3u_x &= 0, \\ u(x,0) &= \sin(x), \end{aligned}$$

and sketch the characteristic curves.

Problem 6. *Solve the problem*

$$\begin{aligned} u_x + u_y + u &= e^{x+2y}, \\ u(x,0) &= 0, \end{aligned}$$

and sketch the characteristic curves.

Solution. *First, let's convert this into a homogeneous linear problem. Let* $p(x, y) = Ae^{x+2y}$*, so that*

$$\begin{aligned} p_x &= p, \\ p_y &= 2p, \\ p_x + p_y + p &= 4Ae^{x+2y}. \end{aligned}$$

Then p *is a particular solution to our equation if* $A = \frac{1}{4}$*. Now, let's find the general solution to the homogeneous problem*

$$v_x + v_y = -v.$$

The characteristic curves are given by

$$x - y = C$$

for C *constant (I believe y'all can sketch these particular characteristic curves). These can be parametrised by* $\gamma(s) = (x(s), y(s)) = (s + C, s)$*, and the corresponding ODE to solve along the characteristic curves is*

$$\frac{dv}{ds} = -v \qquad \Rightarrow \qquad v(\gamma(s)) = Ae^{-s}$$

where A *is constant along* $\gamma(s)$*. I.e. the general solution to the homogeneous problem is* $v(x, y) = \phi(x-y)e^{-y}$ *for an arbitrary function* ϕ*.*

So the general solution to the inhomogeneous problem is

$$u(x, y) = \phi(x - y)e^{-y} + \frac{1}{4}e^{x+2y},$$

and applying the BC at $y = 0$ *gives*

$$0 = \phi(x) + \frac{1}{4}e^x$$

so that $\phi(x) = -\frac{1}{4}e^x$*. Putting this together gives*

$$u(x, y) = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-y}e^{-y} = \frac{e^x}{2}\sinh(2y).$$

Problem 7. *Find the general solution to the equation*

$$(1 + t^2)u_t + u_x = 0,$$

and sketch the characteristic curves.

Problem 8. *Solve the problem*

$$\begin{aligned} u_t + txu_x &= 0, \\ u(x, 0) &= \frac{1}{1 + x^2}, \end{aligned}$$

and sketch the characteristic curves.

Problem 9. *Solve the problem*

$$\begin{aligned} u_t + t^2u_x &= 0, \\ u(x, 0) &= e^x, \end{aligned}$$

and sketch the characteristic curves.

Problem 10. *Find the general solution to the equation*

$$xu_x + yu_y = 0,$$

and sketch the characteristic curves.

Problem 11. *Solve the problem*

$$\sqrt{1-x^2}u_x + u_y = 0,$$
$$u(0,y) = y,$$

and sketch the characteristic curves.

Problem 12. *Solve the problem*

$$u_t + xu_x = x,$$
$$u(x,0) = -x,$$

and sketch the characteristic curves.

3 The wave equation

Problem 13. *Solve the IVP*

$$u_{tt} - u_{xx} = 0,$$
$$u|_{t=0} = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases}$$
$$u_t|_{t=0} = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Solution. *Let's assume $t \geq 0$ (if not we just have to care about a couple of extra regions). We can apply D'Alembert's formula*

$$u(x,t) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds$$

where $g(x) = u(x,0)$ and $h(x) = u_t(x,0)$. Then the solution is piecewise defined over three regions:

- $x < -t$*: I.e. $x + t < 0$. In this region the $h(s)$ integral does not contribute, and we have*

$$u(x,t) = \frac{1+1}{2} + 0 = 1.$$

- $|x| < t$*: Then $g(x-t) = 0$ and $g(x+t) = 1$, so we have*

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\int_0^{x+t} ds = \frac{1}{2} + \frac{1}{2}(x+t).$$

- $x > t$*: Then $g(x \pm t) = 0$ and the solution is*

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} ds = \frac{(x+t)-(x-t)}{2} = t.$$

Problem 14. *Solve the IVP*

$$u_{tt} - 3u_{xx} = 0,$$
$$u|_{t=0} = e^x,$$
$$u_t|_{t=0} = \sin(x).$$

Problem 15. *Solve the IVP*

$$u_{tt} - u_{xx} = xt,$$
$$u(x,0) = 0,$$
$$u_t(x,0) = 0.$$

Problem 16. *Solve the IBVP $(x,t>0)$*

$$u_{tt} - u_{xx} = 0,$$
$$u(x,0) = \sin(x),$$
$$u_t(x,0) = 0,$$
$$u_x(0,t) = 0.$$

Problem 17. *Determine $u|_{(x,t)=(50.1,12)}$ when u is a solution to the problem*

$$u_{tt} - \pi^2 u_{xx} = 0,$$
$$u|_{t=0} = \begin{cases} e^{-\frac{x^2}{7}}, & x<3, \\ 0, & x>3, \end{cases}$$
$$u_t|_{t=0} = 0.$$

Problem 18. *Suppose that $u(x,y,z,t)$ solves the wave equation $u_{tt} = c^2\Delta u$ on the bounded domain Ω, with homogeneous Dirichlet boundary conditions on $\partial\Omega$. Prove that the energy of u*

$$E_\Omega(t) := \frac{1}{2}\iiint_\Omega (u_t^2 + c^2|\nabla u|^2)\,dx\,dy\,dz$$

is conserved.

Problem 19. *Suppose that $u(x,y,z,t)$ solves the wave equation $u_{tt} = c^2\Delta u$ on the bounded domain Ω, with homogeneous Neumann boundary conditions on $\partial\Omega$. Prove that the energy of u*

$$E_\Omega(t) := \frac{1}{2}\iiint_\Omega (u_t^2 + c^2|\nabla u|^2)\,dx\,dy\,dz$$

is conserved.

Solution. *Homogeneous Neumann BCs means that the normal derivative $\frac{\partial u}{\partial \nu}$ along the boundary $\partial\Omega$ vanishes identically. So we calculate:*

$$\frac{dE_\Omega}{dt} = \frac{1}{2}\iiint_\Omega \left(2u_t u_{tt} + 2c^2 \nabla u_t \cdot \nabla u\right)\, d^3\vec{x} = c^2 \iiint_\Omega \left(u_t \Delta u + \nabla u_t \cdot \nabla u\right)\, d^3\vec{x}$$
$$= c^2 \iiint_\Omega \nabla\cdot(u_t\nabla u)\, d^3\vec{x} = c^2 \iint_{\partial\Omega} u_t \nabla u \cdot \nu \, dvol_{\partial\Omega} = c^2 \iint_{\partial\Omega} u_t \frac{\partial u}{\partial \nu}\, dvol_{\partial\Omega} = 0.$$

Problem 20. *Suppose that $u(x,y,z,t)$ solves the wave equation $u_{tt} = c^2\Delta u$ on the bounded domain Ω, with boundary conditions $\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial t}$ on $\partial\Omega$ (where ν is the outward pointing normal vector field on $\partial\Omega$). Is the energy of u*

$$E_\Omega(t) := \frac{1}{2}\iiint_\Omega (u_t^2 + c^2|\nabla u|^2)\,dx\,dy\,dz$$

increasing, decreasing, or constant?

Problem 21. *Where does a solution $u(x,y,z,t)$ to the homogeneous wave equation have to vanish if its initial data vanishes outside of the unit ball $\{\vec{x} \in \mathbb{R}^3 \mid \|x\| \leq 1\}$?*

4 The heat equation

Problem 22. *Solve the heat equation IVP*

$$u_t - u_{xx} = 0, \qquad -\infty < x, t < \infty,$$
$$u(x,0) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

Express your answer in terms of the error function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

Problem 23. *Solve the heat equation IVP*

$$4u_t - u_{xx} = 0, \qquad -\infty < x, t < \infty,$$
$$u(x,0) = e^{-x}.$$

Problem 24. *Suppose that u is a solution to the 1d heat equation on $(0,1)$, satisfying the boundary conditions*

$$u_x(0,t) - u(0,t) = 0,$$
$$u_x(1,t) = 0.$$

Show that the function

$$E(t) = \int_0^1 u(x,t)^2 \, dx$$

is nonincreasing, and that it decreases unless $u(x,t)$ is identically zero.

Problem 25. *Suppose that u is a solution to the 1d heat equation $u_t = u_{xx}$ on $\{0 < x < 1, 0 < t < \infty\}$, with homogeneous Dirichlet boundary conditions and initial condition*

$$u(x,0) = 4x(1-x).$$

Prove that $0 < u(x,t) < 1$ for all $t > 0$ and all $0 < x < 1$.

Solution. *The (strong) maximum/minimum principles tell us that the max/min of the solution u must occur either at the endpoints $x = 0, 1$ or at time $t = 0$, and moreover that if the max/min occurs anywhere in the interior $0 < x < 1$, $t > 0$, then the function must be constant. The non-constant IC tells us that our solution is not constant – hence it suffices to show that at the endpoints at at time zero, the function takes minimum 0 and maximum 1.*

The endpoints are held constant at $u(0,t) = u(1,t) = 0$, and the function $g(x) = u(x,0) = 4x(1-x)$ is ≥ 0, so $\min u = 0$. Further,

$$g'(x) = 4 - 8x = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$

so that $x = \frac{1}{2}$ is the only interior critical point; since $g'' = -8 < 0$ this critical point is a maximum, and $g(1/2) = 2(1 - \frac{1}{2}) = 1$.

Problem 26. *Suppose that u is a solution to the 1d heat equation $u_t = u_{xx}$ on $\{0 < x < 1, 0 < t < \infty\}$, with homogeneous Dirichlet boundary conditions and initial condition*

$$u(x,0) = 1 - x^2.$$

(a) Prove that $u(x,t)$ is strictly positive for all $t > 0$ and $0 < x < 1$.

(b) Prove that

$$\mu(t) := \max_{0 \leq x \leq 1} u(x,t)$$

is a decreasing function of t.

5 Fourier series

Problem 27. *Determine the real Fourier series representation of* $\sin\left(\frac{x}{2}\right)$ *on the interval* $(-\pi,\pi)$.

Problem 28. *Determine the real Fourier series representation of* $\sinh(x)$ *on the interval* $(-\pi,\pi)$.

Problem 29. *Determine the complex Fourier series representation of* $e^{\alpha x}$ *on the interval* $(-\pi,\pi)$, *for* $\alpha\in\mathbb{C}$. *Which values of* α *are "exceptional"?*

Solution. *The Fourier coefficients are given by*

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\alpha x}e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(\alpha-in)x}\,dx \\ &= \frac{(-1)^n}{2\pi(\alpha-in)}(e^{\alpha\pi}-e^{-\alpha\pi}) \end{aligned}$$

provided $\alpha\neq in$ *for any* $n\in\mathbb{Z}$ *(the "exceptional" values). So*

$$e^{\alpha x}=\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{2\pi(\alpha-in)}(e^{\alpha\pi}-e^{-\alpha\pi})e^{inx}.$$

Problem 30. *Determine the real Fourier series representation of* $|x|$ *on the interval* $(-1,1)$.

Problem 31. *Determine the sine Fourier series representation of* $x(\pi-x)$ *on the interval* $(0,\pi)$.

Problem 32. *Determine the sine Fourier series representation of* x^2 *on the interval* $(0,1)$.

Problem 33. *Determine the sine Fourier series representation of* 1 *on the interval* $(0,\pi)$.

Problem 34. *Determine the cosine Fourier series representation of* 1 *on the interval* $(0,\pi)$.

Problem 35. *Determine the cosine Fourier series representation of* x *on the interval* $(0,1)$.

Problem 36. *Determine the cosine Fourier series representation of* x^2 *on the interval* $(0,1)$.

6 Separation of variables

Problem 37. *Using the method of separation of variables, solve the following problem:*

$$\begin{aligned} u_{tt}-u_{xx}&=0, && -\pi<x<\pi, \\ u(-\pi,t)&=0, \\ u(\pi,t)&=0, \\ u(x,0)&=\sinh(x), \\ u_t(x,0)&=0. \end{aligned}$$

Solution. *Looking for a separated solution* $u(x,t)=X(x)T(t)$ *gives the system of equations*

$$\begin{aligned} X''+\lambda X&=0 \\ T''+\lambda T&=0 \\ X(-\pi)=X(\pi)&=0 \end{aligned}$$

We have homogeneous Dirichlet BCs on both ends, so there are no solutions for $\lambda < 0$ *or* $\lambda = 0$. *For* $\lambda = \omega^2 > 0$, $\omega > 0$, *we find*

$$\begin{aligned} X(x) &= A\cos(\omega x) + B\sin(\omega x), \\ X(\pi) &= A\cos(\omega\pi) + B\sin(\omega\pi) = 0, \\ X(-\pi) &= A\cos(\omega\pi) - B\sin(\omega\pi) = 0. \end{aligned}$$

The ICs are odd, so we may take $A = 0$ *and look for solutions to*

$$\sin(\omega\pi) = 0.$$

These are given by $\omega = n \in \mathbb{Z}_{>0}$, *i.e.* $n = 1, 2, 3, \dots$. *Using these eigenvalues, we obtain the solutions*

$$\begin{aligned} \lambda_n &= n^2 \\ X_n(x) &= \sin(nx) \\ T_n(t) &= A_n\cos(nt) + B_n\sin(nt) \end{aligned}$$

So the general solution looks like

$$u(x,t) = \sum_{n=1}^{\infty} (A_n\cos(nt) + B_n\sin(nt))\sin(nx).$$

$u_t(x,0) = 0$ *implies that all of the* $B_n = 0$, *so*

$$u(x,t) = \sum_{n=1}^{\infty} A_n\cos(nt)\sin(nx).$$

The other IC gives

$$u(x,0) = \sinh(x) = \sum_{n=1}^{\infty} A_n\sin(nx),$$

so we need to calculate the Fourier series for $\sinh(x)$ *on* $(-\pi,\pi)$. *We could find this using our solution to Problem 29, but instead let's calculate the Fourier coefficients directly:*

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sinh(x)\sin(nx)\,dx = Im\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\sinh(x)e^{inx}\,dx\right) \\ &= Im\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^x e^{inx} - e^{-x}e^{inx})\,dx\right) \\ &= \frac{1}{2\pi} Im\left(\int_{-\pi}^{\pi} e^{(1+in)x}\,dx - \int_{-\pi}^{\pi} e^{-(1-in)x}\,dx\right) \\ &= \frac{1}{2\pi} Im\left(\frac{e^{\pi}e^{in\pi}}{1+in} - \frac{e^{-\pi}e^{-in\pi}}{1+in} + \frac{e^{-\pi}e^{in\pi}}{1-in} - \frac{e^{\pi}e^{-in\pi}}{1-in}\right) \\ &= \frac{2\sinh(\pi)}{\pi}(-1)^{n+1}\frac{n}{n^2+1} \end{aligned}$$

So

$$u(x,t) = \frac{2}{\pi}\sinh(\pi)\sum_{n=1}^{\infty}\frac{n}{n^2+1}(-1)^{n+1}\cos(nt)\sin(nx).$$

Problem 38. *Using the method of separation of variables, solve the following problem:*

$$\begin{aligned} u_{tt} - 8u_{xx} &= 0, \qquad\qquad 0 < x < \pi, \\ u(0,t) &= u(\pi,t), \\ u_x(0,t) &= u_x(\pi,t), \\ u(x,0) &= x(\pi - x), \\ u_t(x,0) &= 0. \end{aligned}$$

Problem 39. *Using the method of separation of variables, solve the following problem:*

$$\begin{aligned} u_t - 7u_{xx} &= 0, && 0 < x < 1, \\ u(0,t) &= 0, \\ u_x(1,t) &= 0, \\ u(x,0) &= 1. \end{aligned}$$

Problem 40. *Using the method of separation of variables, solve the following problem:*

$$\begin{aligned} u_t - u_{xx} &= 10u, && -1 < x < 1, \\ u_x(-1,t) &= 0, \\ u_x(1,t) &= 0, \\ u(x,0) &= |x|. \end{aligned}$$

Problem 41. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\begin{aligned} \Delta u &= 0, && 0 \le r < 2, \ -\pi \le \theta \le \pi, \\ u(2,\theta) &= \pi^2 - \theta^2. \end{aligned}$$

Here (r,θ) are the standard polar coordinates on $\mathbb{R}^2$:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Problem 42. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\begin{aligned} \Delta u &= 0, && 1 < r < 2, \ -\pi \le \theta \le \pi, \\ u(1,\theta) &= \sin(2\theta), \\ u(2,\theta) &= |\theta|. \end{aligned}$$

Here (r,θ) are the standard polar coordinates on $\mathbb{R}^2$:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Solution. *In polar coordinates, the Laplace equation is*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

and separating variables $u(r,\theta) = R(r)\Theta(\theta)$ gives the system of equations

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ r^2R'' + rR' - \lambda R &= 0 \end{aligned}$$

with 2π-periodic BCs for Θ. The eigenvalues and Θ eigenfunctions are

$$\begin{aligned} \lambda_0 &= 0, && \Theta_0 = 0, \\ \lambda_n &= n^2, && \Theta_n = C_n\cos(n\theta) + D_n\sin(n\theta). \end{aligned}$$

Solving the Euler type equation for R gives

$$\begin{aligned} R_0(r) &= A_0 + B_0\log(r), \\ R_n(r) &= A_nr^n + B_nr^{-n}, \end{aligned}$$

and so

$$u(r,\theta) = \frac{1}{2}(A_0 + B_0\log(r)) + \sum_{n=1}^{\infty}(A_n r^n + B_n r^{-n})(C_n\cos(n\theta) + D_n\sin(n\theta)).$$

At $r = 1$*,*

$$\sin(2\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n + B_n)(C_n\cos(n\theta) + D_n\sin(n\theta)),$$

from which we obtain the equations

$$\begin{aligned} A_0 &= 0 = C_2, & (A_2 + B_2)D_2 &= 1, \\ (A_n + B_n)C_n &= 0, & n &\neq 2, \\ (A_n + B_n)D_n &= 0, & n &\neq 2. \end{aligned}$$

At $r = 2$ *we have*

$$u(2,\theta) = |\theta| = \frac{\log(2)}{2}B_0 + \sum_{n=1}^{\infty}(2^n A_n + 2^{-n}B_n)(C_n\cos(n\theta) + D_n\sin(n\theta)).$$

Comparing this with the Fourier expansion of $|\theta|$ *on* $(-\pi,\pi)$

$$|\theta| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)\theta)$$

we obtain the equations

$$\begin{aligned} \frac{\log(2)}{2}B_0 &= \frac{\pi}{2}, & & \\ (2^n A_n + 2^{-n}B_n)D_n &= 0, & &\text{for all } n, \\ (2^n A_n + 2^{-n}B_n)C_n &= 0, & &\text{for even } n, \\ (2^n A_n + 2^{-n}B_n)C_n &= -\frac{4}{\pi n^2}, & &\text{for odd } n. \end{aligned}$$

Let's take these two systems of equations and use them to simplify the series expression before we calculate the final answer. We have:

$$\begin{aligned} B_0 &= \frac{\pi}{\log(2)}, & 4A_2 + \frac{B_2}{4} &= 0, \\ D_2 &= \frac{1}{A_2 + B_2} = -\frac{1}{15A_2}, & D_n &= 0 \text{ for } n \neq 2, \\ C_n &= 0 \text{ for } n \text{ even}, & B_n &= -A_n \text{ for } n \text{ odd}. \end{aligned}$$

Rewriting the series solution for u *using this information, reindexing to sum over only odd integers, and collecting together various constants, we have*

$$u(r,\theta) = \frac{\pi}{2}\frac{\log(r)}{\log(2)} - \frac{r^2 - 16r^{-2}}{15}\sin(2\theta) + \sum_{n=1}^{\infty}A_{2n-1}(r^{2n-1} - r^{-2n+1})\cos((2n-1)\theta).$$

Comparing this again at $r = 2$ *with the Fourier series for* $|\theta|$ *gives*

$$(2^{2n-1} - 2^{-2n+1})A_{2n-1} = -\frac{4}{\pi(2n-1)^2}.$$

So the solution to the problem is

$$u(r,\theta) = \frac{\pi}{2}\frac{\log(r)}{\log(2)} - \frac{r^2 - 16r^{-2}}{15}\sin(2\theta) - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{r^{2n-1} - r^{-2n+1}}{2^{2n-1} - 2^{-2n+1}}\cdot\frac{\cos((2n-1)\theta)}{(2n-1)^2}.$$

Problem 43. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\Delta u = 0, \qquad 1 < r,\ -\pi \le \theta \le \pi,$$
$$u(1,\theta) = \theta^4.$$

Here (r,θ) *are the standard polar coordinates on* $\mathbb{R}^2$*:*

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Problem 44. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\Delta u = 0, \qquad 1 < r < 2,\ -\pi \le \theta \le \pi,$$
$$u(1,\theta) = 1 + \theta^2,$$
$$u_r(2,\theta) = 0.$$

Here (r,θ) *are the standard polar coordinates on* $\mathbb{R}^2$*:*

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Problem 45. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\Delta u = 0, \qquad 0 \le r < 3,\ 0 \le \theta \le \pi,$$
$$u(3,\theta) = e^{\theta},$$
$$u(r,0) = u(r,\pi) = 0.$$

Here (r,θ) *are the standard polar coordinates on* $\mathbb{R}^2$*:*

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Problem 46. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\Delta u = 0, \qquad 0 \le r < 2,\ 0 \le \theta \le \frac{\pi}{2},$$
$$u(2,\theta) = \theta,$$
$$u(r,0) = u_\theta\left(r,\frac{\pi}{2}\right) = 0.$$

Here (r,θ) *are the standard polar coordinates on* $\mathbb{R}^2$*:*

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Problem 47. *Consider the 2d Helmholtz equation*

$$(\Delta + \omega^2)u = 0,$$

where ω *is a constant. Separate variables in cartesian coordinates* $u(x,y) = X(x)Y(y)$*, and write down the ODEs that* X *and* Y *must satisfy.*

Problem 48. *Consider the 2d Helmholtz equation*

$$(\Delta + \omega^2)u = 0,$$

where ω *is a constant. Separate variables in polar coordinates* $u(r,\theta) = R(r)\Theta(\theta)$*, and write down the ODEs that* R *and* Θ *must satisfy.*

Problem 49. *Consider the 3d Helmholtz equation*

$$(\Delta + \omega^2)u = 0,$$

where ω *is a constant. Separate variables in cartesian coordinates* $u(x,y,z) = X(x)Y(y)Z(z)$*, and write down the ODEs that* X*,* Y *and* Z *must satisfy.*

Problem 50. *Consider the 3d Helmholtz equation*

$$(\Delta + \omega^2)u = 0,$$

where ω *is a constant. Separate variables in spherical coordinates* $u(\rho,\theta,\phi) = R(\rho)\Theta(\theta)\Phi(\phi)$*, and write down the ODEs that* R*,* Θ *and* Φ *must satisfy.*

7 Fourier transforms

Problem 51. *Calculate the Fourier transform of*

$$f(x) = \begin{cases} 1, & |x| < 5, \\ 0, & |x| > 5. \end{cases}$$

Solution.

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-5}^{5} e^{-ikx}\,dx = \frac{e^{-5ik} - e^{5ik}}{(-ik)\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\frac{\sin(5k)}{k}.$$

Problem 52. *Calculate the Fourier transform of*

$$f(x) = \begin{cases} x, & |x| < 5, \\ 0, & |x| > 5. \end{cases}$$

Solution. *This function is* x *times the function of* x *in Problem 51, so using properties of the Fourier transform,*

$$\hat{f}(k) = i\frac{d}{dk}\left(\sqrt{\frac{2}{\pi}}\frac{\sin(5k)}{k}\right) = i\sqrt{\frac{2}{\pi}}\frac{5k\cos(5k) - \sin(5k)}{k^2}.$$

Problem 53. *Calculate the Fourier transform of* e^{-4x^2}*.*

Problem 54. *Calculate the Fourier transform of* $e^{-3|x|}$*.*

Problem 55. *Calculate the Fourier transform of* $x^2e^{-|x|}$*.*

Problem 56. *Calculate the Fourier transform of* $x^4e^{-4x^2}$*.*

Problem 57. *Calculate the Fourier transform of*

$$f(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

Problem 58. *Use the Fourier transform to solve the heat equation with convection problem*

$$\begin{aligned} u_t &= \kappa u_{xx} + \mu u_x, && -\infty < x < \infty, \\ u(x,0) &= \phi(x), \\ \max|u| &< \infty, \end{aligned}$$

where $\kappa > 0$.

Problem 59. *Use the Fourier transform to solve*

$$\begin{aligned} \Delta u &= 0, && -\infty < x < +\infty,\ y > 0, \\ u(x,0) &= x^4 e^{-4x^2}, \\ \max|u| &< \infty. \end{aligned}$$

Problem 60. *Use the Fourier transform to solve*

$$\begin{aligned} \Delta u &= 0, && -\infty < x < +\infty,\ 0 < y < 1, \\ u(x,0) &= \begin{cases} x, & |x| < 5, \\ 0, & |x| > 5, \end{cases} \\ u(x,1) &= \begin{cases} 1, & |x| < 5, \\ 0, & |x| > 5. \end{cases} \end{aligned}$$

Problem 61. *Use the Fourier transform to solve the 2d heat equation*

$$\begin{aligned} 4u_t &= \Delta u, && -\infty < x, y < +\infty,\ t > 0, \\ u(x,y,0) &= \begin{cases} e^{-\frac{y^2}{2}}, & |x| < 5, \\ 0, & |x| > 5. \end{cases} \end{aligned}$$

Solution. *Take the Fourier transform in both* x *and* y, $(x,y) \to (k_x, k_y) =: \vec{k}$, *to transform the PDE into the differential equation*

$$\hat{u}_t = -\frac{\|\vec{k}\|^2}{4}\hat{u}.$$

This has solution

$$\hat{u}(\vec{k},t) = \hat{g}(\vec{k}) e^{-\frac{\|\vec{k}\|}{4}t},$$

where $\hat{g}$ *is the Fourier transform of the initial condition* $g(x,y) = u(x,y,0)$.

There are two possible ways you could be asked to "solve" the problem from this point:

(i) Write the final answer in terms of a convolution (I'll leave this method up to you).

(ii) Calculate $\hat{g}$ *and write the answer as an inverse Fourier transform. For this:*

$$\begin{aligned} \hat{g}(\vec{k}) &= \frac{1}{2\pi}\iint g(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}\,dx\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} e^{-ik_y y}\,dy \cdot \frac{1}{\sqrt{2\pi}}\int_{-5}^{5} e^{-ik_x x}\,dx \\ &= \sqrt{\frac{2}{\pi}}\frac{\sin(5k_x)}{k_x} e^{-\frac{k_y^2}{2}}. \end{aligned}$$

So,

$$\hat{u}(k_x, k_y, t) = \sqrt{\frac{2}{\pi}}\frac{\sin(5k_x)}{k_x} e^{-\frac{k_y^2}{2}} e^{-\frac{\|\vec{k}\|}{4}t},$$

and

$$u(x,y,t) = \frac{1}{\sqrt{2\pi^3}} \iint \frac{\sin(5k_x)}{k_x} e^{-\frac{k_y^2}{2}} e^{-\frac{\|\vec{k}\|}{4}t} e^{i\vec{k}\cdot\vec{x}}\, dk_x\, dk_y.$$

8 Harmonic functions

Problem 62. *Find all the harmonic functions on* $\mathbb{R}^2_{x,y}$ *which depend only on the radial coordinate* $r = \sqrt{x^2+y^2}$.

Problem 63. *Suppose that* u *is a harmonic function on the open unit disc* $\{x^2+y^2<1\}$ *which is continuous on the closed unit disc* $\{x^2+y^2\le 1\}$ *and has boundary value*

$$u|_{x^2+y^2=1} = |\theta|^3, \qquad -\pi \le \theta \le \pi.$$

(a) Determine the maximum value that u *takes on the closed unit disc.*

(b) Determine $u(0)$.

Problem 64. *Suppose that* u *is a harmonic function on the open unit disc* $\{x^2+y^2<1\}$ *which is continuous on the closed unit disc* $\{x^2+y^2\le 1\}$ *and has boundary value*

$$u|_{x^2+y^2=1} = \theta^2 - \theta^4, \qquad -\pi \le \theta \le \pi.$$

(a) Determine the maximum value that u *takes on the closed unit disc.*

(b) Determine $u(0)$.

Problem 65. *Suppose that* u *is a harmonic function on the open unit disc* $\{x^2+y^2<1\}$ *which is continuous on the closed unit disc* $\{x^2+y^2\le 1\}$ *and has boundary value*

$$u|_{x^2+y^2=1} = |\theta| + \sin(\theta), \qquad -\pi \le \theta \le \pi.$$

(a) Determine the minimum value that u *takes on the closed unit disc.*

(b) Determine $u(0)$.

Solution. *Write* $g(\theta) := |\theta| + \sin(\theta)$.

(a) By the minimum principle, the minimum of u *on the closed disc is the minimum of* u *on the boundary circle. So we need to find the minimum of* $g(\theta)$, $-\pi < \theta < \pi$. g *is differentiable away from* $\theta = 0$, *and*

$$g'(\theta) = \begin{cases} 1+\cos(\theta), & 0<\theta<\pi, \\ -1+\cos(\theta), & -\pi<\theta<0. \end{cases}$$

Since $|\cos(\theta)| < 1$ *on these domains,* $g'(\theta) \neq 0$ *for any of these values. So to find the minimum, it reamins to check the endpoints* $\theta = 0, \pi$:

$$g(0) = 0,$$
$$g(\pi) = \pi.$$

So $\min u = 0$.

(b) By the mean value formula,

$$u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (|\theta| + \sin(\theta))\, d\theta = \frac{1}{\pi}\int_0^{\pi} \theta\, d\theta + \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} \sin(\theta)\, d\theta}_{=0 \text{ (odd function)}} = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2}.$$

Problem 66. *Suppose that u is a harmonic function on the open unit disc $\{x^2+y^2<1\}$ which is continuous on the closed unit disc $\{x^2+y^2\le 1\}$ and has boundary value*

$$u|_{x^2+y^2=1} = \left|\sin\left(\frac{\theta}{2}\right)\right| \qquad -\pi \le \theta \le \pi.$$

(a) Determine the maximum value that u takes on the closed unit disc.

(b) Determine $u(0)$.

Problem 67. *Suppose that u is a harmonic function on the open disc $\{x^2+y^2<4\}$ which is continuous on the closed disc $\{x^2+y^2\le 4\}$ and has boundary value*

$$u|_{x^2+y^2=4} = \frac{3}{2}xy + 1.$$

(a) Determine the maximum value that u takes on the closed unit disc.

(b) Determine $u(0)$.

9 Calculus of variations

Problem 68. *Find the curve $y = u(x)$ that makes the integral*

$$\int_0^1 \left[\left(\frac{du}{dx}\right)^2 + xu\right] dx$$

stationary, subject to the constraints $u(0) = 0$, $u(1) = 1$.

Problem 69. *Find the Euler-Lagrange equation for the action*

$$S[u] = \iint \left(\frac{1}{2}u_x u_t + u_x^3 - \frac{1}{2}u_{xx}^2\right) dx\,dt.$$

Solution. *Explicitly expanding $S[u+\delta u]$ in powers of δu gives*

$$S[u+\delta u] - S[u] = \iint \left(\frac{1}{2}u_x\delta u_t + \frac{1}{2}u_t\delta u_x + 3u_x^2\delta u_x - u_{xx}\delta u_{xx}\right) dx\,dt + O(\delta u^2),$$

so that

$$\begin{aligned}
\delta S &= \iint \left(\frac{1}{2}u_x\delta u_t + \frac{1}{2}u_t\delta u_x + 3u_x^2\delta u_x - u_{xx}\delta u_{xx}\right) dx\,dt \\
&= \iint \left(-\frac{1}{2}u_{xt}\delta u - \frac{1}{2}u_{xt}\delta u - 3\frac{\partial}{\partial x}\left(u_x^2\right)\delta u - u_{xxxx}\delta u\right) dx\,dt + (\textit{bdy terms}) \\
&= \iint \left(-u_{xt} - 6u_x u_{xx} - u_{xxxx}\right)\delta u\, dx\,dt + (\textit{bdy terms}).
\end{aligned}$$

Setting $\delta S = 0$ we find the Euler-Lagrange equation

$$u_{xt} + 6u_x u_{xx} + u_{xxxx} = 0.$$

Problem 70. *Find the Euler-Lagrange equation for the functional*

$$T[y] = \int_0^a \sqrt{\frac{1+(y')^2}{2gy}}\,dx.$$

Problem 71. *Find the Euler-Lagrange equations and boundary conditions for the functional*

$$S[u] = \int_0^1 \int_0^1 \left(\frac{1}{2}\|\nabla u\|^2 + \frac{x}{1+y^2}u\right) dx\,dy + \int_{\partial([0,1]\times[0,1])} \left(\frac{x}{2}u^2 - u\right) dvol.$$

Problem 72. *Find the Euler-Lagrange equation for the functional*

$$S[u] = \int_{-2}^2 \frac{u^2\sqrt{1+\left(\frac{du}{dx}\right)^2}}{2}\,dx.$$

Problem 73. *Let* $\Omega \subset \mathbb{R}^2$ *be an open domain with smooth boundary. The area of a surface in* $\mathbb{R}^3$ *defined as the graph of a function* $z : \Omega \to \mathbb{R}$ *is*

$$A[z] = \iint_\Omega \sqrt{1+z_x^2+z_y^2}\,dx\,dy.$$

Find the Euler-Lagrange equation for the functional A.

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