APM 346 – Final Exam Practice Problems.

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(Problems are mostly taken from or variants of problems from [IvrXX] or [Str08].)

1 Introductory explicitly solvable problems

Problem 1. *Solve the equation* $5u_y + u_{xy} = 0$.

Problem 2. *Solve the equation* $u_{xy} - 4u_x = e^{x+5y}$.

Problem 3. *Solve the equation* $u_{xy} = u_x u_y$.

Problem 4. *Solve the system of equations*

$$\begin{aligned} u_{xy} &= 0, \\ u_{yz} &= 0, \\ u_{zx} &= 1. \end{aligned}$$

2 Method of characteristics

Problem 5. *Solve the problem*

$$\begin{aligned} 2u_t + 3u_x &= 0, \\ u(x,0) &= \sin(x), \end{aligned}$$

and sketch the characteristic curves.

Problem 6. *Solve the problem*

$$\begin{aligned} u_x + u_y + u &= e^{x+2y}, \\ u(x,0) &= 0, \end{aligned}$$

and sketch the characteristic curves.

Problem 7. *Find the general solution to the equation*

$$(1+t^2)u_t + u_x = 0,$$

and sketch the characteristic curves.

Problem 8. *Solve the problem*

$$\begin{aligned} u_t + txu_x &= 0, \\ u(x,0) &= \frac{1}{1+x^2}, \end{aligned}$$

and sketch the characteristic curves.

Problem 9. *Solve the problem*

$$\begin{aligned} u_t + t^2 u_x &= 0, \\ u(x,0) &= e^x, \end{aligned}$$

and sketch the characteristic curves.

Problem 10. *Find the general solution to the equation*

$$xu_x + yu_y = 0,$$

and sketch the characteristic curves.

Problem 11. *Solve the problem*

$$\begin{aligned} \sqrt{1-x^2}\,u_x + u_y &= 0, \\ u(0,y) &= y, \end{aligned}$$

and sketch the characteristic curves.

Problem 12. *Solve the problem*

$$\begin{aligned} u_t + xu_x &= x, \\ u(x,0) &= -x, \end{aligned}$$

and sketch the characteristic curves.

3 The wave equation

Problem 13. *Solve the IVP*

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \\ u|_{t=0} &= \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases} \\ u_t|_{t=0} &= \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \end{aligned}$$

Problem 14. *Solve the IVP*

$$\begin{aligned} u_{tt} - 3u_{xx} &= 0, \\ u|_{t=0} &= e^x, \\ u_t|_{t=0} &= \sin(x). \end{aligned}$$

Problem 15. *Solve the IVP*

$$\begin{aligned} u_{tt} - u_{xx} &= xt, \\ u(x,0) &= 0, \\ u_t(x,0) &= 0. \end{aligned}$$

Problem 16. *Solve the IBVP ($x, t > 0$)*

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \\ u(x,0) &= \sin(x), \\ u_t(x,0) &= 0, \\ u_x(0,t) &= 0. \end{aligned}$$

Problem 17. *Determine* $u|_{(x,t)=(50.1,12)}$ *when* u *is a solution to the problem*

$$\begin{aligned} u_{tt} - \pi^2 u_{xx} &= 0, \\ u|_{t=0} &= \begin{cases} e^{-\frac{x^2}{7}}, & x < 3, \\ 0, & x > 3, \end{cases} \\ u_t|_{t=0} &= 0. \end{aligned}$$

Problem 18. *Suppose that* $u(x,y,z,t)$ *solves the wave equation* $u_{tt} = c^2 \Delta u$ *on the bounded domain* Ω*, with homogeneous Dirichlet boundary conditions on* $\partial\Omega$*. Prove that the energy of* u

$$E_\Omega(t) := \frac{1}{2} \iiint_\Omega (u_t^2 + c^2 |\nabla u|^2)\, dx\, dy\, dz$$

is conserved.

Problem 19. *Suppose that* $u(x,y,z,t)$ *solves the wave equation* $u_{tt} = c^2 \Delta u$ *on the bounded domain* Ω*, with homogeneous Neumann boundary conditions on* $\partial\Omega$*. Prove that the energy of* u

$$E_\Omega(t) := \frac{1}{2} \iiint_\Omega (u_t^2 + c^2 |\nabla u|^2)\, dx\, dy\, dz$$

is conserved.

Problem 20. *Suppose that* $u(x,y,z,t)$ *solves the wave equation* $u_{tt} = c^2 \Delta u$ *on the bounded domain* Ω*, with boundary conditions* $\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial t}$ *on* $\partial\Omega$ *(where* ν *is the outward pointing normal vector field on* $\partial\Omega$*). Is the energy of* u

$$E_\Omega(t) := \frac{1}{2} \iiint_\Omega (u_t^2 + c^2 |\nabla u|^2)\, dx\, dy\, dz$$

increasing, decreasing, or constant?

Problem 21. *Where does a solution* $u(x,y,z,t)$ *to the homogeneous wave equation have to vanish if its initial data vanishes outside of the unit ball* $\{\vec{x} \in \mathbb{R}^3 \mid \|x\| \le 1\}$*?*

4 The heat equation

Problem 22. *Solve the heat equation IVP*

$$\begin{aligned} u_t - u_{xx} &= 0, & -\infty < x, t < \infty, \\ u(x,0) &= \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases} \end{aligned}$$

Express your answer in terms of the error function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz.$$

Problem 23. *Solve the heat equation IVP*

$$\begin{aligned} 4u_t - u_{xx} &= 0, & -\infty < x, t < \infty, \\ u(x,0) &= e^{-x}. \end{aligned}$$

Problem 24. *Suppose that* u *is a solution to the 1d heat equation on* $(0,1)$*, satisfying the boundary conditions*

$$\begin{aligned} u_x(0,t) - u(0,t) &= 0, \\ u_x(1,t) &= 0. \end{aligned}$$

Show that the function

$$E(t) = \int_0^1 u(x,t)^2 \, dx$$

is nonincreasing, and that it decreases unless $u(x,t)$ is identically zero.

Problem 25. *Suppose that u is a solution to the 1d heat equation $u_t = u_{xx}$ on $\{0 < x < 1,\, 0 < t < \infty\}$, with homogeneous Dirichlet boundary conditions and initial condition*

$$u(x,0) = 4x(1-x).$$

Prove that $0 < u(x,t) < 1$ for all $t > 0$ and all $0 < x < 1$.

Problem 26. *Suppose that u is a solution to the 1d heat equation $u_t = u_{xx}$ on $\{0 < x < 1,\, 0 < t < \infty\}$, with homogeneous Dirichlet boundary conditions and initial condition*

$$u(x,0) = 1 - x^2.$$

(a) Prove that $u(x,t)$ is strictly positive for all $t > 0$ and $0 < x < 1$.

(b) Prove that

$$\mu(t) := \max_{0 \le x \le 1} u(x,t)$$

is a decreasing function of t.

5 Fourier series

Problem 27. *Determine the real Fourier series representation of $\sin\left(\frac{x}{2}\right)$ on the interval $(-\pi,\pi)$.*

Problem 28. *Determine the real Fourier series representation of $\sinh(x)$ on the interval $(-\pi,\pi)$.*

Problem 29. *Determine the complex Fourier series representation of $e^{\alpha x}$ on the interval $(-\pi,\pi)$, for $\alpha \in \mathbb{C}$. Which values of α are "exceptional"?*

Problem 30. *Determine the real Fourier series representation of $|x|$ on the interval $(-1,1)$.*

Problem 31. *Determine the sine Fourier series representation of $x(\pi - x)$ on the interval $(0,\pi)$.*

Problem 32. *Determine the sine Fourier series representation of x^2 on the interval $(0,1)$.*

Problem 33. *Determine the sine Fourier series representation of 1 on the interval $(0,\pi)$.*

Problem 34. *Determine the cosine Fourier series representation of 1 on the interval $(0,\pi)$.*

Problem 35. *Determine the cosine Fourier series representation of x on the interval $(0,1)$.*

Problem 36. *Determine the cosine Fourier series representation of x^2 on the interval $(0,1)$.*

6 Separation of variables

Problem 37. *Using the method of separation of variables, solve the following problem:*

$$\begin{aligned} u_{tt} - u_{xx} &= 0, && -\pi < x < \pi, \\ u(-\pi,t) &= 0, \\ u(\pi,t) &= 0, \\ u(x,0) &= \sinh(x), \\ u_t(x,0) &= 0. \end{aligned}$$

Problem 38. *Using the method of separation of variables, solve the following problem:*

$$\begin{aligned} u_{tt} - 8u_{xx} &= 0, && 0 < x < \pi, \\ u(0,t) &= u(\pi,t), \\ u_x(0,t) &= u_x(\pi,t), \\ u(x,0) &= x(\pi - x), \\ u_t(x,0) &= 0. \end{aligned}$$

Problem 39. *Using the method of separation of variables, solve the following problem:*

$$\begin{aligned} u_t - 7u_{xx} &= 0, && 0 < x < 1, \\ u(0,t) &= 0, \\ u_x(1,t) &= 0, \\ u(x,0) &= 1. \end{aligned}$$

Problem 40. *Using the method of separation of variables, solve the following problem:*

$$\begin{aligned} u_t - u_{xx} &= 10u, && -1 < x < 1, \\ u_x(-1,t) &= 0, \\ u_x(1,t) &= 0, \\ u(x,0) &= |x|. \end{aligned}$$

Problem 41. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\begin{aligned} \Delta u &= 0, && 0 \le r < 2,\ -\pi \le \theta \le \pi, \\ u(2,\theta) &= \pi^2 - \theta^2. \end{aligned}$$

Here (r,θ) *are the standard polar coordinates on* $\mathbb{R}^2$*:*

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Problem 42. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\begin{aligned} \Delta u &= 0, && 1 < r < 2,\ -\pi \le \theta \le \pi, \\ u(1,\theta) &= \sin(2\theta), \\ u(2,\theta) &= |\theta|. \end{aligned}$$

Here (r,θ) *are the standard polar coordinates on* $\mathbb{R}^2$*:*

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Problem 43. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\begin{aligned} \Delta u &= 0, && 1 < r,\ -\pi \le \theta \le \pi, \\ u(1,\theta) &= \theta^4. \end{aligned}$$

Here (r,θ) *are the standard polar coordinates on* $\mathbb{R}^2$*:*

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Problem 44. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\begin{aligned} \Delta u &= 0, && 1 < r < 2,\ -\pi \le \theta \le \pi, \\ u(1,\theta) &= 1 + \theta^2, \\ u_r(2,\theta) &= 0. \end{aligned}$$

Here (r,θ) are the standard polar coordinates on $\mathbb{R}^2$:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Problem 45. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\begin{aligned} \Delta u &= 0, && 0 \le r < 3,\ 0 \le \theta \le \pi, \\ u(3,\theta) &= e^{\theta}, \\ u(r,0) = u(r,\pi) &= 0. \end{aligned}$$

Here (r,θ) are the standard polar coordinates on $\mathbb{R}^2$:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Problem 46. *Using the method of separation of variables solve the following problem for the 2d Laplace equation:*

$$\begin{aligned} \Delta u &= 0, && 0 \le r < 2,\ 0 \le \theta \le \frac{\pi}{2}, \\ u(2,\theta) &= \theta, \\ u(r,0) = u_\theta\left(r,\frac{\pi}{2}\right) &= 0. \end{aligned}$$

Here (r,θ) are the standard polar coordinates on $\mathbb{R}^2$:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Problem 47. *Consider the 2d Helmholtz equation*

$$(\Delta + \omega^2)u = 0,$$

where ω is a constant. Separate variables in cartesian coordinates $u(x,y) = X(x)Y(y)$, and write down the ODEs that X and Y must satisfy.

Problem 48. *Consider the 2d Helmholtz equation*

$$(\Delta + \omega^2)u = 0,$$

where ω is a constant. Separate variables in polar coordinates $u(r,\theta) = R(r)\Theta(\theta)$, and write down the ODEs that R and Θ must satisfy.

Problem 49. *Consider the 3d Helmholtz equation*

$$(\Delta + \omega^2)u = 0,$$

where ω is a constant. Separate variables in cartesian coordinates $u(x,y,z) = X(x)Y(y)Z(z)$, and write down the ODEs that X, Y and Z must satisfy.

Problem 50. *Consider the 3d Helmholtz equation*

$$(\Delta + \omega^2)u = 0,$$

where ω is a constant. Separate variables in spherical coordinates $u(\rho,\theta,\phi) = R(\rho)\Theta(\theta)\Phi(\phi)$, and write down the ODEs that R, Θ and Φ must satisfy.

7 Fourier transforms

Problem 51. *Calculate the Fourier transform of*

$$f(x) = \begin{cases} 1, & |x| < 5, \\ 0, & |x| > 5. \end{cases}$$

Problem 52. *Calculate the Fourier transform of*

$$f(x) = \begin{cases} x, & |x| < 5, \\ 0, & |x| > 5. \end{cases}$$

Problem 53. *Calculate the Fourier transform of* e^{-4x^2}.

Problem 54. *Calculate the Fourier transform of* $e^{-3|x|}$.

Problem 55. *Calculate the Fourier transform of* $x^2e^{-|x|}$.

Problem 56. *Calculate the Fourier transform of* $x^4e^{-4x^2}$.

Problem 57. *Calculate the Fourier transform of*

$$f(x) = \begin{cases} 1-|x|, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

Problem 58. *Use the Fourier transform to solve the heat equation with convection problem*

$$\begin{aligned} u_t &= \kappa u_{xx} + \mu u_x, && -\infty < x < \infty, \\ u(x,0) &= \phi(x), \\ \max|u| &< \infty, \end{aligned}$$

where $\kappa > 0$.

Problem 59. *Use the Fourier transform to solve*

$$\begin{aligned} \Delta u &= 0, && -\infty < x < +\infty,\ y > 0, \\ u(x,0) &= x^4 e^{-4x^2}, \\ \max|u| &< \infty. \end{aligned}$$

Problem 60. *Use the Fourier transform to solve*

$$\begin{aligned} \Delta u &= 0, && -\infty < x < +\infty,\ 0 < y < 1, \\ u(x,0) &= \begin{cases} x, & |x| < 5, \\ 0, & |x| > 5, \end{cases} \\ u(x,1) &= \begin{cases} 1, & |x| < 5, \\ 0, & |x| > 5. \end{cases} \end{aligned}$$

Problem 61. *Use the Fourier transform to solve the 2d heat equation*

$$\begin{aligned} 4u_t &= \Delta u, && -\infty < x,y < +\infty,\ t > 0, \\ u(x,y,0) &= \begin{cases} e^{-\frac{y^2}{2}}, & |x| < 5, \\ 0, & |x| > 5. \end{cases} \end{aligned}$$

8 Harmonic functions

Problem 62. *Find all the harmonic functions on* $\mathbb{R}^2_{x,y}$ *which depend only on the radial coordinate* $r = \sqrt{x^2+y^2}$.

Problem 63. *Suppose that* u *is a harmonic function on the open unit disc* $\{x^2+y^2<1\}$ *which is continuous on the closed unit disc* $\{x^2+y^2\le 1\}$ *and has boundary value*

$$u|_{x^2+y^2=1} = |\theta|^3, \qquad -\pi \le \theta \le \pi.$$

(a) Determine the maximum value that u *takes on the closed unit disc.*

(b) Determine $u(0)$.

Problem 64. *Suppose that* u *is a harmonic function on the open unit disc* $\{x^2+y^2<1\}$ *which is continuous on the closed unit disc* $\{x^2+y^2\le 1\}$ *and has boundary value*

$$u|_{x^2+y^2=1} = \theta^2 - \theta^4, \qquad -\pi \le \theta \le \pi.$$

(a) Determine the maximum value that u *takes on the closed unit disc.*

(b) Determine $u(0)$.

Problem 65. *Suppose that* u *is a harmonic function on the open unit disc* $\{x^2+y^2<1\}$ *which is continuous on the closed unit disc* $\{x^2+y^2\le 1\}$ *and has boundary value*

$$u|_{x^2+y^2=1} = |\theta| + \sin(\theta), \qquad -\pi \le \theta \le \pi.$$

(a) Determine the minimum value that u *takes on the closed unit disc.*

(b) Determine $u(0)$.

Problem 66. *Suppose that* u *is a harmonic function on the open unit disc* $\{x^2+y^2<1\}$ *which is continuous on the closed unit disc* $\{x^2+y^2\le 1\}$ *and has boundary value*

$$u|_{x^2+y^2=1} = \left|\sin\left(\frac{\theta}{2}\right)\right| \qquad -\pi \le \theta \le \pi.$$

(a) Determine the maximum value that u *takes on the closed unit disc.*

(b) Determine $u(0)$.

Problem 67. *Suppose that* u *is a harmonic function on the open disc* $\{x^2+y^2<4\}$ *which is continuous on the closed disc* $\{x^2+y^2\le 4\}$ *and has boundary value*

$$u|_{x^2+y^2=4} = \frac{3}{2}xy + 1.$$

(a) Determine the maximum value that u *takes on the closed unit disc.*

(b) Determine $u(0)$.

9 Calculus of variations

Problem 68. *Find the curve* $y = u(x)$ *that makes the integral*

$$\int_0^1 \left[\left(\frac{du}{dx} \right)^2 + xu \right] dx$$

stationary, subject to the constraints $u(0) = 0$, $u(1) = 1$.

Problem 69. *Find the Euler-Lagrange equation for the action*

$$S[u] = \iint \left(\frac{1}{2} u_x u_t + u_x^3 - \frac{1}{2} u_{xx}^2 \right) dx\, dt.$$

Problem 70. *Find the Euler-Lagrange equation for the functional*

$$T[y] = \int_0^a \sqrt{\frac{1 + (y')^2}{2gy}}\, dx.$$

Problem 71. *Find the Euler-Lagrange equations and boundary conditions for the functional*

$$S[u] = \int_0^1 \int_0^1 \left(\frac{1}{2} \|\nabla u\|^2 + \frac{x}{1+y^2} u \right) dx\, dy + \int_{\partial([0,1]\times[0,1])} \left(\frac{x}{2} u^2 - u \right) dvol.$$

Problem 72. *Find the Euler-Lagrange equation for the functional*

$$S[u] = \int_{-2}^2 \frac{u^2 \sqrt{1 + \left(\frac{du}{dx}\right)^2}}{2}\, dx.$$

Problem 73. *Let* $\Omega \subset \mathbb{R}^2$ *be an open domain with smooth boundary. The area of a surface in* $\mathbb{R}^3$ *defined as the graph of a function* $z : \Omega \to \mathbb{R}$ *is*

$$A[z] = \iint_\Omega \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy.$$

Find the Euler-Lagrange equation for the functional A.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.