APM 346 Lecture 24.

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We've reached the last lecture of the semester – so let's have some fun!

References being used: [IvrXX, Ch.10] (§10.3) and [Str08, Ch.14.3].

1 Point particles and the strings they love

So far, we've studied functionals of the form

$$S[u^1, \dots, u^m] = \int_\Omega L(\vec{x}, u^i, \nabla u^i)\, d\vec{x} + \int_{\Sigma'} M(\vec{x}, u^i)\, d\mathrm{vol}_{\Sigma'}, \tag{1}$$

where M has been a fairly simple boundary term on $\Sigma' \subset \partial\Omega$, usually leading to Robin or Neumann BCs. Recall that the stationary points of S are given by functions

$$u = (u^1, \dots, u^m) : \Omega \to \mathbb{R}^m$$

which satisfy the Euler-Lagrange equations on Ω

$$\frac{\partial L}{\partial u_k} - \sum_{j=1}^{n} \partial_j \left(\frac{\partial L}{\partial(\partial_j u_k)} \right) = 0, \qquad k = 1, \dots, m, \tag{2}$$

and the boundary conditions on $\partial\Omega$

$$\left(- \sum_{j=1}^{n} \frac{\partial L}{\partial(\partial_j u_k)} \nu_j + \frac{\partial M}{\partial u_k} \right) \Bigg|_{\Sigma'} = 0, \qquad k = 1, \dots, m. \tag{3}$$

However, there was no particular reason why the boundary terms M had to be so simple. Indeed, suppose that we forget about the Lagrangian L and the domain Ω altogether! Then we would simply be studying the variation of a functional that takes as argument functions defined on the space Σ'.

Thinking about this physically, we come to the following realisation: we can define a physical theory in the "bulk" Ω via the Lagrangian L, and an entire *other* physical theory on the "boundary" $\partial\Omega$ via the boundary Lagrangian M. Provided that (at least some of) the functions that M takes in as input are required to be the restriction of functions that L takes in as input these two physical theories will now be coupled to each other, and we can study the (potentially very interesting!) physics of the entire bulk-boundary system.

Remark 1.1. The physics of bulk-boundary systems can be extremely intricate – for instance, one can study systems where the physical behaviour of the system on the boundary exhibits features that would be impossible absent effects from the bulk.

As a visualisation, imagine that you have a box, and that you can only perform measurements on the surface of this box. If you observe phenomena that could not occur in the physics of a two-dimensional system, then you know that *something* interesting is happening inside the box! Needless to say, this is well beyond the scope of this course...

1.1 Point particles

Let's begin by thinking about the motion of a particle in $\mathbb{R}^m$. In fact this is nothing but the σ-model we discussed last lecture, in the case where $\Omega \subset \mathbb{R}_t$:

$$S[q] = \int \left(\frac{M}{2} \left\| \frac{dq}{dt} \right\|^2 - V(q(t)) \right) dt, \tag{4}$$

for some potential energy function $V : \mathbb{R}^m \to \mathbb{R}$. The Euler-Lagrange equations for the σ-model become in this case

$$m \frac{d^2 q}{dt^2} = -\nabla V. \tag{5}$$

We can study the motion of a particle for a finite or infinite time interval, depending on our choice of domain Ω, and with Dirichlet or Neumann BCs depending on which variations δq we allow.

Example 1 (Free particle)**.** When $V \equiv 0$, we are studying the physics of a *free particle.*In this case the Euler-Lagrange equations take the simple form $\ddot{q} = 0$, which we can immediately solve to find

$$q(t) = t\vec{v} + \vec{q}_0, \tag{6}$$

where $\vec{q}_0$ and $\vec{v}$ are constant vectors. I.e. free particles move in straight lines.

Example 2 (Simple harmonic oscillator)**.** Another important but relatively simple example is given by the simple harmonic oscillator (which models, e.g., the motion of a mass on a spring). The potential energy of a simple harmonic oscillator is

$$V(q) = \frac{1}{2} k \|q\|^2, \tag{7}$$

and so the Euler-Lagrange equations become

$$M \frac{d^2 q}{dt^2} = -kq. \tag{8}$$

Note that this decouples into m separate equations $M(\ddot{q^i}) = -kq^i$, for $i = 1, \ldots, m$. Setting $\omega := \sqrt{\frac{k}{M}}$, each of these one dimensional equations has solution

$$q^i(t) = A_i \cos(\omega t + \varphi_i), \tag{9}$$

and so the general solution for the simple harmonic oscillator is

$$q(t) = \left(A_1 \cos(\omega t + \varphi_1), \ldots, A_m \cos(\omega t + \varphi_m)\right), \tag{10}$$

for some collection of amplitudes A_i and initial phases φ_i.

1.2 Strings

Now, let's up the dimension of the object whose motion we wish to study, and consider – instead of a point particle – a finite length piece of string. The dynamics of a string in $\mathbb{R}^m$ will be described by a map

$$\sigma : [0,1]_s \times \Omega_t \to \mathbb{R}^m, \tag{11}$$

and by the σ-model action[1]

$$S[\sigma] = \frac{\kappa}{2} \int_{\Omega_t} \int_0^1 \left(\left\| \frac{\partial \sigma}{\partial t} \right\|^2 - \left\| \frac{\partial \sigma}{\partial s} \right\|^2 \right) ds\, dt, \tag{12}$$

[1] This differs from the σ-model action describes last lecture in the *signature* of the metric on the space $[0,1]_s \times \Omega_t$ – instead of the Euclidean dot product $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$, we use the Lorentz signature inner product $\langle \vec{v}, \vec{w} \rangle = -v_1 w_1 + v_2 w_2$.

where κ is a constant. Note that we are assuming a "free" σ-model action: the potential term is zero. From last lecture, the corresponding Euler-Lagrange equations are

$$\frac{\partial^2 \sigma}{\partial t^2} - \frac{\partial^2 \sigma}{\partial s^2} = 0; \tag{13}$$

i.e. the dynamics of the string are governed by (a vector-valued function version of) the wave equation.

1.3 Point particles *attached* to strings

Now is the part where we really get to have fun. So far, we have studied point particles and strings. Let's now *attach* two point particles to the ends of our string segment, and study the resulting system.

What do we mean by this? Simple: setting

$$q_0(t) := \sigma(0,t),$$
$$q_1(t) := \sigma(1,t),$$

we take as our action functional

$$S[\sigma] = \int_{\Omega_t} \int_0^1 \underbrace{\frac{\kappa}{2}\left(\left\| \frac{\partial \sigma}{\partial t} \right\|^2 - \left\| \frac{\partial \sigma}{\partial s} \right\|^2 \right)}_{L(s,t,\sigma,\partial_s\sigma,\partial_t\sigma)} ds\, dt + \int_{\Omega_t} \underbrace{\left(\frac{m_0}{2} \left\| \frac{dq_0}{dt} \right\|^2 - V_0(q_0(t)) \right)}_{M_0(t,q_0,\dot{q}_0)} dt + \int_{\Omega_t} \underbrace{\left(\frac{m_1}{2} \left\| \frac{dq_1}{dt} \right\|^2 - V_1(q_1(t)) \right)}_{M_1(t,q_1,\dot{q}_1)} dt. \tag{14}$$

Remark 1.2. Note that this is form our action takes if we do not constrain our variations at all. If we wish we can constrain our variations, in which case the boundary terms may be altered or absent. For instance, if we require our variations to vanish both terms would be absent and we would have to specify Dirichlet boundary conditions at both ends of the string.

The Euler-Lagrange equations for the string are unchanged by the addition of boundary terms – i.e. the bulk dynamics of the string are still governed by the equation (13). On the boundary, however, we now have two terms: the usual Euler-Lagrange equations for the boundary term, and the terms

$$\frac{\partial L}{\partial(\partial_t \sigma^k)} + \frac{\partial L}{\partial(\partial_s \sigma^k)} = \kappa \left(\frac{\partial \sigma^k}{\partial t} - \frac{\partial \sigma^k}{\partial s} \right),$$

where $\sigma = (\sigma^1, \ldots, \sigma^m)$, and we subtract/add this term depending on whether we are at the left/right end of the interval $[0,1]_s$. Explicitly then, the boundary conditions are

$$-m_0 \ddot{q}_0 - \nabla V_0(q_0) - \kappa \left(\frac{\partial \sigma}{\partial t} - \frac{\partial \sigma}{\partial s} \right)\Bigg|_{s=0} = 0, \tag{15}$$

$$-m_1 \ddot{q}_1 - \nabla V_1(q_1) + \kappa \left(\frac{\partial \sigma}{\partial t} - \frac{\partial \sigma}{\partial s} \right)\Bigg|_{s=1} = 0. \tag{16}$$

Rearranging these boundary conditions, we arrive at the following system of PDEs in the bulk ($[0,1]_s \times \Omega_t$) and on the boundary (Ω_t):

$$\begin{aligned}
&\frac{\partial^2 \sigma}{\partial t^2} - \frac{\partial^2 \sigma}{\partial s^2} = 0, && \text{in } [0,1]_s \times \Omega_t, \\
&m_0 \ddot{q}_0 + \kappa \dot{q}_0 = -\nabla V_0(q_0) + \kappa \frac{\partial \sigma}{\partial s}\bigg|_{s=0}, && \text{on } \{s=0\} \times \Omega_t, \\
&m_1 \ddot{q}_1 - \kappa \dot{q}_1 = -\nabla V_1(q_1) - \kappa \frac{\partial \sigma}{\partial s}\bigg|_{s=1}, && \text{on } \{s=1\} \times \Omega_t.
\end{aligned} \tag{17}$$

1.3.1 Interpretation of the coupled system

We are not going to attempt to explicitly solve the system of equations (17). Instead, let's try to qualitatively understand how attaching our particles to the end of a string has affected the behaviour of the particles.

We'll focus on the particle at the right end of the string and drop all of the subscript 1's. To understand how the behaviour of the particle has changed, let's see what happens when we evolve a solution forward over a short time interval, $t \to t + \epsilon$.

First, let's see what happens for particles that are *not* attached to a string:

- For the free particle (Example 1) $\ddot{q}(t) = 0$; taking a Taylor expansion of $q(t+\epsilon)$ around $\epsilon = 0$ gives
$$q(t+\epsilon) = q(t) + \epsilon v(t),$$
where we write $v(t) := \dot{q}(t)$ for the velocity of the particle. (Of course in this situation, this is the exact solution – free particles move in straight lines.)
- For the simple harmonic oscillator (Example 2) $m\ddot{q} = -kq$. As an exercise you should show that the Taylor series of $q(t+\epsilon)$ can be summed explicitly to give
$$q(t+\epsilon) = \cos\left(\epsilon\sqrt{\frac{k}{m}}\right) q(t) + \sqrt{\frac{m}{k}} \sin\left(\epsilon\sqrt{\frac{k}{m}}\right) v(t).$$
This is explicit, but it is not immediately apparent what this tells us qualitatively about the short time evolution of a solution. So instead, assuming that ϵ is small, we can take a quadratic approximation to the exact answer,
$$q(t+\epsilon) \simeq q(t) + \epsilon v(t) + \frac{1}{2}\ddot{q}(t)\epsilon^2 = q(t) + \epsilon\left(v(t) - \frac{k\epsilon}{2m} q(t)\right).$$
There are now two terms contributing to the short-time evolution of the particle. The velocity of the particle (which drove the free particle onwards in a straight line) is now compensated by a "restoring" force proportional to the displacement of the particle from the origin. When the particle is very close[2] to the origin
$$q(t+\epsilon) \simeq q(t) + \epsilon v(t),$$
and the particle behaves like a free particle. Provided the velocity doesn't blow up, at large displacements the restorative term becomes dominant and
$$q(t+\epsilon) \simeq q(t) - \frac{k\epsilon^2}{2m} q(t),$$
i.e. the particle starts to move closer to the origin. Since
$$v(t+\epsilon) \simeq v(t) - \frac{k\epsilon}{m} q(t),$$
the velocity also experiences a restorative effect, preventing it from blowing up.
- In general, the quadratic approximation is
$$q(t+\epsilon) \simeq q(t) + \epsilon v(t) - \frac{\epsilon^2}{2m}\nabla V.$$

Remark 1.3. It is of course reasonable to worry that the long time evolution of a solution will be quite different to the short time behaviour, e.g. because terms of higher order in the Taylor expansion will dominate the linear and quadratic terms. We just aren't going to worry about this *now*.[3]

[2] To quantify "very close" one considers the ratio $\frac{k}{m}$. E.g. for a fixed value of k, heavy particles can travel further than light particles before they feel the effect of the restorative force.

[3] You can worry about it later, in the privacy of your home.

Now, let's return to the case of the particle attached to a string – for simplicitly, we'll assume that $V = 0$. Then the boundary condition is

$$m\ddot{q} = -\kappa\dot{q} - \kappa\frac{\partial\sigma}{\partial s}\bigg|_{s=1}.$$

To simplify the notation we'll again write $v(t) := \dot{q}(t)$, and we'll also introduce the notation

$$\alpha(t) := \frac{\partial\sigma}{\partial s}\bigg|_{s=1}(t).$$

We can interpret α as a measure of how "stretched" the string is at the endpoint (**Exercise:** Draw a picture to convince yourself of this.) Then the equation becomes

$$m\ddot{q}(t) = \kappa v(t) - \kappa\alpha(t).$$

The quadratic approximation of $q(t+\epsilon)$ is

$$q(t+\epsilon) \simeq q(t) + \epsilon v(t) + \frac{\epsilon^2}{2}\ddot{q}(t) = q(t) + \epsilon\left(1 + \frac{\kappa\epsilon}{2m}\right)v(t) - \frac{\kappa\epsilon^2}{2m}\alpha(t).$$

Hence the short-time evolution of a free particle that has been attached to the end of a string is as follows:

- The standard displacement of the particle due to its velocity at time t is slightly magnified by the factor $1 + \frac{\kappa\epsilon}{2m}$.
- The particle will experience some displacement in the *opposite* direction to which the string is pointing, and proportional to the magnitude of α at time t.

Remark 1.4. Even though a picture might have convinced you α is *a* measure of how stretched the string is, you may have noticed that the string itself is not stretched *proportionally* to α.

Last Exercise: Come up with a sensible physical interpretation of $\alpha(t)$.[4]

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.

[4] If you are having trouble with this exercise, you could start by thinking about the following two observations:

- If $\alpha(t) = 0$ then σ is roughly constant in some neighbourhood $(1-\epsilon, 1]$ – but this means that every point on the string in that neighbourhood is mapped to the *same* point of $\mathbb{R}^m$! So $\alpha(t) = 0$ represents **maximal compression** of the string at the endpoint.
- If the string were neither compressed nor stretched at the endpoint, then the segment $(1-\epsilon, 1]$ would be mapped to a short segment of length ϵ in $\mathbb{R}^m$. In this case, what can we say about $\alpha(t)$?