APM 346 Lecture 23.

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Having warmed up by considering calculus of variations for functions of a single variable, we now move on to the multidimensional case.

References being used: [IvrXX, Ch.10] (§10.3) and [Str08, Ch.14.3].

1 Functionals in higher dimensions

Consider a domain $\Omega \subset \mathbb{R}^n$ and a functional

$$\Phi[u] = \int_\Omega L(\vec{x}, u, \nabla u)\, d\vec{x}, \tag{1}$$

where the Lagrangian L is now (roughly) a function of $n+2$ variables.

Remark 1.1. A discussion of what precisely the domain of the Lagrangian L is would take us quite far afield,[1] however it is worth commenting on a question that may have occured to you: Why do we say it is roughly a function of $n+2$ variables when ∇u is itself an n-component vector?

The answer to this question is the same, regardless of whether one wishes to consider it from a physical or a mathematical perspective: we do not want our Lagrangian to depend upon a particular choice of coordinate system.

- Physically, this is the idea that "the laws of physics should not be different in different coordinate systems" (although a careful discussion would note that depending on what theory you are studying,[2] not all coordinate systems may be considered kosher!).
- Mathematically, this is the claim that the Lagrangian makes sense on more general looking spaces than domains of $\mathbb{R}^n$ – spaces that could be curved, or that might not have a global system of coordinates.[3]

The claim is then that while the individual partial derivatives $\frac{\partial}{\partial x_i}$ are not independent of the coordinate system (indeed they depend explicitly on the choice of coordinates $\{x_i\}$), provided one is dealing with a space in which one can do calculus and measure lengths and angles the gradient ∇u *is* a coordinate independent function.

Of course, rules were made to be broken, and we'll probably break this rule later on by singling our preferred directions, breaking the symmetry of our problem (e.g. by having a spherically symmetric potential term that singles out an origin – thus breaking translational symmetry, but retaining rotational symmetry).

[1] If you are interested in this question, you ought to consult the marvellous notes on *Classical Field Theory* by Pierre Deligne and Dan Freed.

[2] E.g. for the physicists: consider Newtonian mechanics in a non-inertial reference frame.

[3] This is the case even on the sphere – in the spherical coordinates we have been using the poles $\theta = 0, \pi$ are coordinate singularities.

1.1 Variation of functionals

As in the 1d case, we now consider a *variation* of u, $u \to u + \delta u$, where for our purposes we may consider taking $\delta u = \epsilon\phi$ for a fixed function ϕ and "small" ϵ. Then

$$\Phi[u+\delta u] - \Phi[u] = \int_\Omega \left(L(x, u+\delta u, \nabla u + \nabla(\delta u)) - L(x, u, \nabla u)\right) dx \simeq \int_\Omega \left(\frac{\partial L}{\partial u}\delta u + \sum_{j=1}^{n} \frac{\partial L}{\partial u_{x_j}} \delta u_{x_j}\right) dx \quad (2)$$

up to linear terms in a Taylor approximation.

Definition 1.1. We call the RHS of (2) the *variation of* Φ, and denote it by $\delta\Phi$.

We now make the same assumption that we made in our treatment of the 1d case: that all functions are "sufficiently smooth" (i.e. we assume that our functions have whatever degreee of differentiability is necessary for our calculations to work nicely – we may differentiate with impunity). Then we may integrate (2) by parts to obtain the expression

$$\delta\Phi = \int_\Omega \left(\frac{\partial L}{\partial u} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{x_j}}\right)\right) \delta u \, dx - \int_{\partial\Omega} \left(\sum_{j=1}^{n} \frac{\partial L}{\partial u_{x_j}} \nu_j\right) \delta u \, d\mathrm{vol}_{\partial\Omega} \quad (3)$$

where ν is the inward pointing unit normal vector field along $\partial\Omega$.

1.2 Stationary points

As in Lecture 22, we now choose a class of *admissible variations* and make the following definition:

Definition 1.2. If $\delta\Phi = 0$ for all admissible variations δu of u, we call u a *stationary point* of the functional Φ.

1.2.1 Vanishing boundary variation

To begin with, let us take as admissible those variations which satisfy

$$\delta u|_{\partial\Omega} = 0. \quad (4)$$

If Ω is unbounded, we will interpret (4) as a vanishing condition at infinity. With this as our class of admissible variations,

$$\delta\Phi = \int_\Omega \left(\frac{\partial L}{\partial u} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{x_j}}\right)\right) \delta u \, dx. \quad (5)$$

Lemma 1.1. *Let f be a continuous function in Ω. If*

$$\int_\Omega f(x)\phi(x)dx = 0$$

for all ϕ such that $\phi|_{\partial\Omega} = 0$, then $f \equiv 0$ in Ω.

Proof. Same as the 1d case – if f is positive at x_* it is positive on some open set U; construct a function that is also positive at x_*, is non-negative on U, and vanishes outside of U. Then the integral must be positive, and we derive a contradiction. □

Hence we arrive at the following result:

Theorem 1.2. *A function u is a stationary point of the functional* (1) *with respect to admissible variations* (4) *if and only if it satisfies the* Euler-Lagrange equation

$$\frac{\partial \Phi}{\partial u} := \frac{\partial L}{\partial u} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{x_j}}\right) = 0. \tag{6}$$

Example 1. Suppose we work in 2-dimensions, and we take as our Lagrangian

$$L(u, \nabla u) = \frac{1}{2}(u^2 - \|\nabla u\|^2) = \frac{1}{2}(u^2 - u_x^2 - u_y^2).$$

Then the derivatives of L we need are

$$\frac{\partial L}{\partial u} = u,$$
$$\frac{\partial L}{\partial u_x} = -u_x,$$
$$\frac{\partial L}{\partial u_y} = -u_y,$$

and the Euler-Lagrange equation is

$$0 = u + u_{xx} + u_{yy} = u + \Delta u.$$

1.2.2 Extrema of functionals

Recall that the "core problem" in calculus of variations in often the maximisation or minimisation of a particular quantity.

Definition 1.3. If $\Phi[u] \geq \Phi[u + \delta u]$ (resp. $\leq$) for all small admissible variations δu, we call u a *local maximum* (resp. *local minimum*) of the functional Φ. In either situation we sat that u is a *local extremum* of Φ.

Theorem 1.3. *If u is a local extremum of Φ, and the variation of u with respect to all small admissible variations is defined, then u is a stationary point of Φ.*

Proof. Again, the proof is identical to the 1d case: if for $\delta u = \epsilon\phi$ we have $\delta\Phi \neq 0$, one can show that for small enough ϵ we have $\Phi[u + \delta u] < \Phi[u]$ (for u a prospective minimum) or $\Phi[u + \delta u] > \Phi[u]$ (for u a prospective maximum), giving a contradiction. □

Example 2 (Minimal surface problem)**.** Suppose that $\Omega \subset \mathbb{R}^2_{x,y}$ has a nice boundary (e.g. smooth), and consider a surface in $\Sigma \subset \mathbb{R}^3$ defined by the graph of a function $u : \Omega \to \mathbb{R}$,

$$\Sigma = \{(x, y, z) \,|\, (x, y) \in \Omega,\, z = u(x, y)\}.$$

Σ has surface area

$$A(\Sigma) = \iint_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy.$$

The *minimal surface problem* seeks to find the surface Σ with minimal area, subject to the restriction $u|_{\partial\Omega} = g$. Physically, this corresponds to determining the shape of a soap film on a wire[4] – the function g determines the shape of the wire in 3d, and the minimal surface is the corresponding soap film.

The Euler-Lagrange equation for this problem is

$$-\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0.$$

[4]Minimising surface area and minimising potential energy usually coincide for this problem.

Example 3. Assume that $u_x, u_y \ll 1$. Then taking a Taylor expansion of square-root function, we see that we can approximate

$$A(\Sigma) - A(\Omega) \simeq \frac{1}{2} \iint_\Omega (u_x^2 + u_y^2)\, dx\, dy.$$

This functional has Euler-Lagrange equation

$$-\Delta u = 0,$$

so we see that harmonic functions provide approximations to minimal surfaces which are not particularly "steep".

2 Functionals with boundary terms

Suppose that we now consider variations with the restrictive constraint

$$\delta u|_\Sigma = 0 \tag{7}$$

for some (possibly empty) subset $\Sigma \subset \partial\Omega$. Set

$$\Sigma' = \partial\Omega \setminus \Sigma.$$

Then from our previous calculations we see that in order for u to be a stationary point of a the functional Φ of (1) it must not only satisfy the Euler-Lagrange equation (6), but also the boundary condition

$$-\sum_{j=1}^n \frac{\partial L}{\partial u_{x_j}} \nu_j \bigg|_{\Sigma'} = 0.$$

Much as in the 1d case, however, once we have relaxed the constraint to (7) we can consider a more general class of functionals – those which contain *boundary terms*:

$$\Phi[u] = \int_\Omega L(x, u, \nabla u)\, dx + \int_{\Sigma'} M(x, u)\, \mathrm{dvol}_{\Sigma'}. \tag{8}$$

The variation of the boundary term is

$$\int_{\Sigma'} \frac{\partial M}{\partial u} \delta u\, \mathrm{dvol}_{\Sigma'},$$

and so if u satisfies the Euler-Lagrange equation the remaining boundary terms in the variation are

$$\delta\Phi = \int_{\Sigma'} \left(-\sum_{j=1}^n \frac{\partial L}{\partial u_{x_j}} \nu_j + \frac{\partial M}{\partial u} \right) \delta u\, \mathrm{dvol}_{\Sigma'}.$$

So, we have:

Theorem 2.1. *A function u is a stationary point of the functional* (8) *with respect to admissible variations* (7) *if and only if it satisfies the Euler-Lagrange equation*

$$\frac{\partial \Phi}{\partial u} := \frac{\partial L}{\partial u} - \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\partial L}{\partial u_{x_j}} \right) = 0,$$

and also the boundary condition

$$\left(-\sum_{j=1}^n \frac{\partial L}{\partial u_{x_j}} \nu_j + \frac{\partial M}{\partial u} \right) \bigg|_{\Sigma'} = 0. \tag{9}$$

Example 4 (Poisson Equation)**.** Consider the functional

$$\Phi[u] = \int_\Omega \left(\frac{1}{2}\|\nabla u\|^2 - f(\vec{x})u\right) d^n\vec{x} + \int_{\Sigma'} \left(\frac{1}{2}\alpha(\vec{x})|u|^2 - h(\vec{x})u\right) d\mathrm{vol}_{\Sigma'}$$

together with the partial Dirichlet boundary condition

$$u|_\Sigma = g.$$

The Euler-Lagrange equation is the Poisson equation

$$\Delta u = -f,$$

and the boundary conditions we obtain by finding stationary points are Robin (or Neumann for $\alpha \equiv 0$) BCs

$$\left.\left(\frac{\partial u}{\partial \nu} - \alpha u\right)\right|_{\Sigma'} = -h.$$

The calculations are left as a relatively straightforward exercise in calculus.

3 Functionals of vector valued functions

In the above we have only considered functionals where elements of the function space are of the form

$$u : \Omega \to \mathbb{R}.$$

However, the machinery of calculus of variations can be applied far more generally than this. For just a taste of what can be done, we will consider variations of *vector valued functions*.

So, let's study functions on a domain $\Omega \subset \mathbb{R}^n$ which are valued in a vector space,

$$u : \Omega \to \mathbb{R}^m;$$

by choosing a basis for $\mathbb{R}^m$ we can write this as an m-tuple of functions

$$u = (u_1, \dots, u_m).$$

We now have more degrees of freedom in which we can vary our functions; specifically, when we take $u \to u + \delta u$, δu is itself a vector valued function. With our choice of basis we can write

$$\delta u = (\delta u_1, \dots, \delta u_m),$$

and so we can reduce our problem to that of varying each of the component functions u_k separately.

The calculations for each u_k are identical to those for a single function u that we have considered previously. As such, we obtain a system of differential equations, each one corresponding to one of the component functions. By calculations we have done previously, these *Euler-Lagrange equations* are given by

$$\frac{\partial L}{\partial u_k} - \sum_{j=1}^n \partial_j \left(\frac{\partial L}{\partial(\partial_j u_k)}\right) = 0, \qquad k = 1, \dots, m. \tag{10}$$

Similarly, if we have a boundary functional on some $\Sigma' \subset \partial\Omega$ we obtain a system of boundary conditions

$$\left.\left(-\sum_{j=1}^n \frac{\partial L}{\partial(\partial_j u_k)}\nu_j + \frac{\partial M}{\partial u_k}\right)\right|_{\Sigma'} = 0, \qquad k = 1, \dots, m. \tag{11}$$

Here we have chosen coordinates $x^1, \dots, x^n$ on Ω, and written $\partial_j = \frac{\partial}{\partial x^j}$.

Example 5. Consider the functional

$$\Phi[u] = \frac{1}{2}\int_\Omega \left(\alpha|\nabla\otimes u|^2 + \beta|\nabla\cdot u|^2\right)\, d\vec{x}$$

where

$$\begin{aligned} u &= (u_1, \dots, u_n), \\ |\nabla\otimes u|^2 &= \sum_{j,k} |\partial_j u_k|^2, \\ \nabla\cdot u &= \sum_j \partial_j u_j, \end{aligned}$$

and α, β are constant. If we consider variations δu which vanish on $\partial\Omega$, we obtain

$$\delta\Phi = \int_\Omega \left(-\alpha\Delta u - \beta\nabla\left(\nabla\cdot u\right)\right)\cdot\delta u\, d\vec{x},$$

and so the Euler-Lagrange equations are

$$-\alpha\Delta u - \beta\nabla(\nabla\cdot u) = 0.$$

Example 6. If we take the functional of Example 5 but do not fix $\delta u = 0$ on some segment $\Sigma' \subset \Omega$, we find that even if u satisfies the Euler-Lagrange equations there is a remaining boundary term

$$\delta\Phi = -\int_{\Sigma'}\left(\alpha\frac{\partial u}{\partial\nu} + \beta(\nabla\cdot u)\nu\right)\cdot\delta u\, d\mathrm{vol}_{\Sigma'} = 0.$$

If we assume that $\delta u|_{\Sigma'}$ may be arbitrary, then we must supplement the Euler-Lagrange equations with the boundary condition

$$\alpha\frac{\partial u}{\partial\nu} + \beta(\nabla\cdot u)\nu = 0.$$

Example 7. Suppose now that we take the functional of Example 5 and we do not assume vanishing of δu on Σ' as in Example 6, but we do not allow our variations to be *arbitrary.* There are other natural constraints we could allow on our variations: for instance,

(a) If we assume that δu is parallel to ν on Σ', we obtain that boundary condition

$$\left(\alpha\frac{\partial u}{\partial\nu} + \beta(\nabla\cdot u)\nu\right)\cdot\nu = 0.$$

Geometrically, this says that we are prescribing the component of $\alpha\frac{\partial u}{\partial\nu}$ orthogonal to the boundary to be given precisely by $-\beta(\nabla\cdot u)$.

(b) If we assume that δu is tangent to Σ', i.e. that $\delta u\cdot\nu = 0$, and we assume $\alpha \neq 0$, we find that

$$\frac{\partial u}{\partial\nu}\cdot\delta u = 0.$$

This says that $\frac{\partial u}{\partial\nu}$ has no components in any of the directions tangent to the boundary; i.e. that it is itself orthogonal to Σ' (or equivalently that it is parallel to ν).

Example 8 (σ-models)**.** Let's finish up by considering an interesting example from physics, known as a "σ-model".[5] We consider functions

$$\vec{u} = (u^1, \dots u^m) : \Omega \to \mathbb{R}^m$$

[5] In the original paper [GML60], "σ" was the name of a particle.

where $\Omega \subset \mathbb{R}^n$ with coordinates $x^1, \ldots, x^n$, and define the functional

$$S[\vec{u}] := \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{\partial u^i}{\partial x^j} \right)^2 - V(u^1(\vec{x}), \ldots, u^m(\vec{x})) \right) d\vec{x}, \tag{12}$$

where V is a "potential" function

$$V : \mathbb{R}^m \to \mathbb{R}. \tag{13}$$

As an exercise, you should calculate the variation of S: under the variation $\vec{u} \to \vec{u} + \delta\vec{u}$ you should find

$$\delta S = -\int_\Omega (\Delta\vec{u} + \mathrm{grad}(V)) \cdot \delta\vec{u}\, d\vec{x} - \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial \nu} \cdot \delta\vec{u}\, d\mathrm{vol}_{\partial\Omega},$$

where ν is the inward pointing unit normal vector field along $\partial\Omega$, and $\mathrm{grad}(V)$ is the gradient of V (we avoid the notation "∇V" to prevent confusion between the gradient of a function defined on Ω, and the gradient of a function defined on $\mathbb{R}^m$).

Hence the Euler-Lagrange equations for the σ-model are

$$\Delta\vec{u} = -\mathrm{grad}(V) \quad \text{on } \Omega,$$

and these may be supplemented by Dirichlet BCs (if we require $\delta\vec{u}|_{\partial\Omega} = 0$) or Neumann BCs (if we do not restrict our variations at all). (Note that this is not an exhaustive list of possible BCs!)

References

[GML60] M. Gell-Mann and M. Lévy. The axial vector current in beta decay. *Nuovo Cimento (10)*, 16:705–726, 1960.

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.