APM 346 Lecture 22.

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Today we begin studying our last topic for the semester: the calculus of variations!

References being used: [IvrXX, Ch.10] (§10.1) and [Str08, Ch.14.3].

1 Motivation: Dirichlet's Principle

To motivate the study of calculus of variations, we consider the following problem. Let Ω be a bounded domain, and consider the collection of functions w on Ω which satisfy the Dirichlet boundary condition

$$w|_{\partial\Omega}(\vec{x}) = h(\vec{x}). \tag{1}$$

Define the energy of such a function by

$$E[w] := \frac{1}{2}\int_{\Omega} \|\nabla w\|^2 d\vec{x}. \tag{2}$$

Physical systems tend to "prefer" to be in the state of least possible energy, so by minimizing $E[w]$ we can discover what our system will look like at equilibrium. The claim is the following:

Theorem 1.1 (Dirichlet's Principle)**.** *The energy* (2) *is minimized by the unique harmonic function that satisfies the boundary condition* (1)*.*

Idea of proof. Suppose that $u(\vec{x})$ is a function that minimises the energy, and let $v(\vec{x})$ be an arbitrary function that vanishes on $\partial\Omega$. Then $u + \epsilon v$ still satisfies the boundary condition (1), and so by hypothesis

$$E[u] \leq E[u + \epsilon v] = E[u] - \epsilon \int_{\Omega} \Delta u v \, d\vec{x} + \epsilon^2 E[v].$$

Think of the right hand side as a function of ϵ, $e(\epsilon) = E[u + \epsilon v]$. It is minimised at $\epsilon = 0$, and so by first year calculus,

$$\left.\frac{de}{d\epsilon}\right|_{\epsilon=0} = 0.$$

But this is just

$$\int_{\Omega} \Delta u v d\vec{x} = 0,$$

and since this hold for arbitrary v we must have $\Delta u = 0$ on Ω. □

2 Variation of functionals

This leads us naturally to the study of *functionals*.

Definition 2.1. A *functional* is a map from some space of functions $\mathcal{F}$ to the real or complex numbers,

$$\Phi : \mathcal{F} \to \mathbb{R}/\mathbb{C}. \tag{3}$$

If $\mathcal{F}$ is a vector space and Φ is a linear map, we say that Φ is a *linear functional*.

Example 1. Let's consider some examples of functionals.

(i) We have already seen an example: the *energy functional* (2),

$$E[w] = \frac{1}{2}\int_\Omega \|\nabla w\|^2 d\vec{x}.$$

Exercise: Think about what the domain of the energy functional could be; i.e. what space of functions could w belong to?

(ii) Let $\mathcal{F} = C_c^\infty(\Omega)$, the space of infinitely differentiable functions which vanish outside a closed and bounded subset of Ω, and let f be a function satisfying $\int_\Omega |f| < \infty$. Then we have a functional $\Phi_f : C_c^\infty(\Omega) \to \mathbb{R}/\mathbb{C}$ defined by

$$\Phi_f[u] := \int_\Omega f(\vec{x})u(\vec{x})\, d\vec{x}. \tag{4}$$

(iii) Let $\mathcal{F} = C(\Omega)$, the space of continuous functions on Ω, and choose a point $\vec{x} \in \Omega$. Then there is a functional

$$\delta_{\vec{x}}[u] := u(\vec{x}), \tag{5}$$

which just evaluates each function at the chosen point. Note that we have encountered this before, where we called it the "Dirac-δ function" and wrote

$$\delta_{\vec{x}}[u] = \int_\Omega \delta(\vec{y} - \vec{x})u(\vec{y})\, d\vec{y}.$$

(iv) Let $I \subset \mathbb{R}$ be a closed interval, and let $\mathcal{F} = C(I, \mathbb{R})$ (continuous *real-valued* functions). Then there are functionals

$$M[u] := \max_{x\in I} u(x), \tag{6}$$

$$m[u] := \min_{x\in I} u(x). \tag{7}$$

We could also replace $u(x)$ by $|u(x)|$ in the above definitions – this then immediately generalises to continuous maps to any space with a notion of "magnitude".

Much as in the example of the energy functional in Section 1, the core problem we will be considering in what follows is the question of how to maximise or minimise a given functional.

2.1 Warm-up: functions of one variable

To start with, let's begin by considering a space of real valued functions of a single variable, $q(t)$ on an interval $[t_0, t_1]$,

$$q : [t_0, t_1] \to \mathbb{R}.$$

For the time derivative of q, we adopt the "dot notation" sometimes used in physics, $\frac{dq}{dt}(t) = \dot{q}(t)$. To make life easier, let's assume that $t_0 = 0$ and $t_1 = 1$.

2.1.1 Calculating the variation

Consider a functional of the form

$$S[q] = \int_0^1 L(q, \dot{q}, t)\, dt, \tag{8}$$

where $L(q, \dot{q}, t)$, the *Lagrangian*, is an integrable expression in the variables $t, q, \dot{q}$.

Consider making a "small" variation of the function q,

$$q \to q + \delta q, \tag{9}$$

where for the purposes of this class you should think of $\delta q = \epsilon\phi$ for some fixed function ϕ and ϵ a small parameter. Under this variation, our functional changes as

$$\begin{aligned} S[q + \delta q] &= \int_0^1 L(q + \delta q, \dot{q} + \delta\dot{q}, t)\, dt \\ &= \int_0^1 \left(L(q, \dot{q}, t) + \frac{\partial L}{\partial q}(q, \dot{q}, t)\delta q + \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, t)\delta\dot{q} + \cdots \right) dt \\ &= S[q] + \int_0^1 \left(\frac{\partial L}{\partial q}(q, \dot{q}, t)\delta q + \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, t)\delta\dot{q} \right) dt + \cdots \end{aligned}$$

where we have taken the Taylor expansion of L and the "$\cdots$" represent terms which are of quadratic or higher order in the variations δq, $\delta\dot{q}$. The term which is linear is δq and $\delta\dot{q}$ is something like the "first derivative of S with respect to q".

Definition 2.2. The *variation of the functional* S is the the linear functional with respect to δq given by

$$\delta S_q := \int_0^1 \left(\frac{\partial L}{\partial q}(q, \dot{q}, t)\delta q + \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, t)\delta\dot{q} \right) dt. \tag{10}$$

Now, we make the assumption that all of the functions we are considering are "sufficiently smooth" – i.e. that whenever we come across an operation that would require some measure of differentiability to perform, we will assume that our functions satisfy that requirement. Having made this assumption, we integrate by parts to obtain

$$\delta S_q = \int_0^1 \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q\, dt + \frac{\partial L}{\partial \dot{q}}\delta q \bigg|_0^1. \tag{11}$$

2.1.2 Stationary points

Now, consider placing restrictions on the variations δq that we are allowed to consider – call this a choice of *admissible variations.*

Definition 2.3. If $\delta S_q = 0$ for all admissible variations δq, we call q a *stationary point* of the functional S.

For now, let's take as admissible those variations satisfying

$$\delta q(0) = \delta q(1) = 0.$$

Then

$$\delta S = \int_0^1 \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q\, dt.$$

Lemma 2.1. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. If*

$$\int_0^1 f(t)\phi(t)\,dt = 0$$

for all ϕ satisfying $\phi(0) = \phi(1) = 0$, then $f \equiv 0$ on $[0,1]$.

Proof. It suffices to check that f is nowhere strictly positive. Suppose that $f(t_*) > 0$ for some $t_* \in [0,1]$. Then $f(t) > 0$ for all t on a small interval (a,b) containing t_*.

Now, consider a function $\phi(x)$ satisfying

$$\phi(t) \geq 0, t \in (a,b), \qquad \phi(t_*) > 0,$$

and $\phi = 0$ outside (a,b).[1] Then we also have $f(t)\phi(t) > 0$ on (a,b), and we derive a contradiction. □

We have now shown:

Theorem 2.2. *The function q is a stationary point of the action* (8) *with respect to variations satisfying $\delta q(0) = \delta q(1) = 0$ if and only if it satisfies the* Euler-Lagrange equation

$$\frac{\delta S}{\delta q} := \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{12}$$

Example 2. As a warm-up example, consider the Lagrangian

$$L = \frac{1}{2}m\dot{q}^2 - V(q),$$

where V is some "potential energy" function that depends only on q. Then

$$\frac{\partial L}{\partial q} = -\frac{dV}{dq},$$
$$\frac{\partial L}{\partial \dot{q}} = m\dot{q},$$

and so the Euler Lagrange equation reads

$$-\frac{dV}{dq} = m\ddot{q}.$$

If we interpret q as the position of a particle, suggestively relabel

$$q(t) \to x(t),$$
$$\dot{q}(t) \to v(t),$$
$$\ddot{q}(t) \to a(t),$$

and define $F := -\frac{dV}{dx}$, we obtain

$$F = ma,$$

i.e. Newton's second law of motion!

From (12) we can immediately obtain a "conservation law" – if L does not explicitly depend on q, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0,$$

i.e.

$$\frac{\partial L}{\partial \dot{q}} \equiv C$$

is constant with time. The can be regarded as a "conservation of momentum" statement – indeed, in Example 2 above if we take $V \equiv 0$ (i.e. a "free particle", moving in the absence of any force), we find that $p = mv$ is the conserved quantity, which is precisely the classical definition of the linear momentum of a particle.

[1] We take the existence of an appropriately smooth such function for granted, but one can be explicitly constructed.

2.1.3 Extremums

Recall that the "core problem" we were considering was the minimisation or maximisation of particular functionals.

Definition 2.4. Let S be a functional, and suppose that we have specfied a class of admissible variations.

(a) If $S[q] \geq S[q+\delta q]$ for all small admissible variations δq, we call q a *local maximum* of S.

(b) If $S[q] \leq S[q+\delta q]$ for all small admissible variations δq, we call q a *local minimum* of S.

In either of the above cases, we call q a *local extremum* of S.

Theorem 2.3. *If q is a local extrumum of S, and the variation with respect to all small admissible variations exists, then q is a stationary point of S.*

Proof. Suppose that q is a local minimum, and let $\delta q = \epsilon\phi$ for some fixed ϕ and small ϵ. Then

$$S[q+\delta q] - S[q] = \epsilon \cdot \delta S(\phi) + O(\epsilon^2).$$

Suppose that $\delta S(\phi) > 0$ (< 0 can be dealt with similarly). Take $\epsilon < 0$, so that

$$\epsilon \delta S(\phi) = -|\epsilon| \delta S(\phi) < 0.$$

By choose $|\epsilon|$ small enough, we can arrange the terms of $O(\epsilon^2)$ to have smaller magnitude that $|\epsilon\delta S(\phi)|$ – but then

$$S[q+\delta q] - S[q] < 0,$$

and we arrive at a contradiction. □

2.1.4 Boundary terms

Suppose now that we change our class of admissible variations δq – rather than placing a vanishing condition at *both* endpoints, we only require

$$\delta q(0) = 0.$$

If q is to be a stationary point of S, it is still necessary for q to satisfy the Euler-Lagrange equation – however it is no longer *sufficient*. From integration by parts, we are left with the term

$$\delta S = \left.\frac{\partial L}{\partial \dot{q}} \delta q \right|_{t=1},$$

and since $\delta q(1)$ is arbitrary we arrive at the condition

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{t=1} = 0.$$

Example 3. For instance, consider the Lagrangian of Example 2,

$$L = \frac{1}{2} m \dot{q}^2 - V(q)$$

from which we derived Newton's second law $F = ma$ in the form

$$-\frac{\partial V}{\partial q} = m\ddot{q}.$$

Relaxing the class of admissible variations as above, we find that

$$m\dot{q}(1) = 0,$$

i.e. since $m \neq 0$ we have the Neumann BC $\left.\frac{dq}{dt}\right|_{t=1} = 0$.

Since we are now allowing variations of our function at $t = 1$, we can consider a more general functional

$$S[q] = \int_0^1 L(q, \dot{q}, t)dt + M_1(q(1)), \tag{13}$$

where M_1 is called a *boundary term.* The linear term in a Taylor approximation to M_1 is

$$\frac{\partial M_1}{\partial q}\delta q,$$

and so

$$\delta S = \left(\frac{\partial L}{\partial \dot{q}} + \frac{\partial M_1}{\partial q}\right)\delta q(1).$$

The same arguments can be applied to the left boundary point $t = 0$ also, and we find:

Theorem 2.4. *The function q is a stationary point of the action*

$$S[q] = \int_0^1 L(q, \dot{q}, t)\,dt + M_0(q(0)) + M_1(q(1)) \tag{14}$$

with respect to all variations δq if and only if it satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0,$$

together with the boundary conditions

$$\left(\frac{\partial L}{\partial \dot{q}} - \frac{\partial M_0}{\partial q}\right)\bigg|_{t=0} = \left(\frac{\partial L}{\partial \dot{q}} + \frac{\partial M_1}{\partial q}\right)\bigg|_{t=1} = 0. \tag{15}$$

Exercise: Suppose that our functional includes higher-order derivatives,

$$S[q] = \int_0^1 L(q, \dot{q}, \ddot{q}, t)\,dt + M_0(q(0), \dot{q}(0)) + M_1(q(1), \dot{q}(1)).$$

Show that q is a stationary point iff it satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0,$$

together with the boundary conditions

$$\left(\frac{\partial M_i}{\partial \dot{q}(i)} - (-1)^i\,\frac{\partial L}{\partial \ddot{q}}\bigg|_{t=i}\right) = 0,$$

$$\left(\frac{\partial M_i}{\partial q(i)} - (-1)^i\,\frac{\partial L}{\partial \dot{q}}\bigg|_{t=i} + (-1)^i\,\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\bigg|_{t=i}\right) = 0,$$

where $i = 0, 1$. What are the admissible variations I have taken in order to obtain these boundary conditions?

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.