APM 346 Lecture 21.

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Last lecture we used separation of variables to solve the 3d wave equation on the solid ball. Today we'll try to understand more generally the propagation of waves in 3 spatial dimensions.

References being used: [IvrXX, Ch.9] (§8.1) and [Str08, Ch.9].

1 Waves in three dimensions

Our goal today is to understand the propagation of waves in three spatial dimensions – i.e. solutions to the differential equation

$$u_{tt} - c^2 \Delta u = u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0.$$

1.1 Conservation of energy and causality

Recall that in the 1d wave equation we saw that the value of a solution at (x, t) depended only on the source function and initial conditions within the domain of causality. In particular, we saw that a wave in 1d propagates with speed no greater than c. We would like to derive a similar conclusion for the 3d wave equation.

1.1.1 Conservation of energy

Assuming that the partial derivatives of the wave $u(\vec{x}, t)$ decay sufficiently quickly at ∞, define the *total energy* of u to be

$$E(t) := \frac{1}{2} \iiint_{\mathbb{R}^3} (u_t^2 + c^2 |\nabla u|^2)\, d\vec{x}. \tag{1}$$

Proposition 1.1. *The total energy of a solution to the wave equation on $\mathbb{R}^3$ is conserved, i.e.*

$$\frac{dE}{dt} = 0.$$

Proof. Since u solves the wave equation, we have $(u_{tt} - c^2 \Delta u) = 0$. So,

$$0 = (u_{tt} - c^2 \Delta u) u_t = \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 \right) - c^2 \nabla \cdot (u_t \nabla u). \tag{2}$$

When this is integrated over all of $\mathbb{R}^3$ the divergence term disappears, and we are left with

$$0 = \iiint \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 \right) d\vec{x} = \frac{d}{dt} \iiint \frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 \, d\vec{x} = \frac{dE}{dt}.$$

□

From now on we will simply write E instead of $E(t)$.

Remark 1.1. What do the terms in (1) actually mean? For simplicity, let's interpret the terms in E for the 1d case of a vibrating string.

- The first term, $\frac{1}{2}u_t^2$, is the kinetic energy of the string – this is the energy that the string has because it is vibrating, with faster vibrations corresponding to higher energy.
- The second term is the potential energy – this is the energy that the string gets from the particular setup of our system (so it depends on the displacement, mass density, and tension of the string).

Remark 1.2. What would happen if we restricted our domain to some subset $\Omega \subset \mathbb{R}^3$? In our argument above we claimed that the divergence term would disappear in our integration; if $\Omega \neq \mathbb{R}^3$ then this claim is no longer valid, as there will be contributions from the boundary.

Say for concreteness that our domain is the upper-half space $\mathbb{H}^+ = \{(x,y,z) \in \mathbb{R}^3 \,|\, z \geq 0\}$. The inward-pointing unit normal vector field along $\partial\mathbb{H}^+ = \mathbb{R}^2_{x,y,z=0}$ is the constant vector field $\nu = (0,0,1) = \frac{\partial}{\partial z}$;[1] hence by the divergence theorem,

$$\iiint_{\mathbb{H}^+} \nabla \cdot (u_t \nabla u)\, d^3\vec{x} = -\iint_{\mathbb{R}^2} \frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\, dx\, dy.$$

Integrating (2) we obtain

$$\frac{dE_{\mathbb{H}^+}}{dt} + c^2 \iint_{\mathbb{R}^2} \frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\, dx\, dy = 0 \tag{3}$$

where $E_{\mathbb{H}^+}$ is defined as in (1), but with integral over $\mathbb{H}^+$ instead of all of $\mathbb{R}^3$. (3) is telling us something reasonable: instead of conservation of energy, we now have the statement that the only way energy can be gained or lost is through fluctuations on the boundary.

1.1.2 The light cone

Let's now introduce the geometric object in $\mathbb{R}^3_{\vec{x}} \times \mathbb{R}^1_t$ analogous to the causal triangle of $\mathbb{R}_x \times \mathbb{R}_t$. Write $\|\vec{x}\| = \sqrt{x^2+y^2+z^2}$ for the usual vector norm in $\mathbb{R}^3$, and define the *light cone* and *solid light cone*[2] at $(\vec{x}_0, t_0)$ respectively by

$$\Lambda(\vec{x}_0,t_0) = \{(\vec{x},t) \,|\, \|\vec{x}-\vec{x}_0\| = c|t-t_0|\}, \tag{4}$$
$$C(\vec{x}_0,t_0) = \{(\vec{x},t) \,|\, \|\vec{x}-\vec{x}_0\| < c|t-t_0|\}. \tag{5}$$

Later we are going to need the unit normal vector field $\hat{n}$ along $\Lambda(\vec{x}_0,t_0)$ for some calculations, so let's calculate it now. $\Lambda(\vec{x}_0,t_0)$ is the zero level set of the function

$$\phi(x,y,z,t) := (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 - c^2(t-t_0)^2,$$

so we can calculate the unit normal vector fields using the gradient of ϕ. We have

$$\nabla\phi = 2\left(x-x_0, y-y_0, z-z_0, -c^2(t-t_0)\right), \tag{6}$$
$$|\nabla\phi|^2 = 4\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 + c^4(t-t_0)^2\right), \tag{7}$$

so

$$\hat{n} = \pm\frac{(x-x_0, y-y_0, z-z_0, -c^2(t-t_0))}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 + c^4(t-t_0)^2}}. \tag{8}$$

[1]In this notation I have identified the vector field $(0,0,1)$ with the directional derivative it induces.
[2]Nomenclature borrowed from the physics of relativity.

Define r by the equation $r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$. Then the equation for the cone is

$$r^2 = c^2(t - t_0)^2$$

and so we can rewrite the norm of $\nabla\phi$ as

$$\frac{1}{4}|\nabla\phi|^2 = (1 + c^2)r^2 = c^2(1 + c^2)(t - t_0)^2.$$

Then

$$\begin{aligned}\hat{n} &= \pm\left(\frac{x - x_0}{r\sqrt{c^2+1}}, \frac{y - y_0}{r\sqrt{c^2+1}}, \frac{z - z_0}{r\sqrt{c^2+1}}, \frac{-c^2(t - t - 0)}{\sqrt{c^2(c^2+1)(t - t_0)^2}}\right) \\ &= \pm\frac{c}{\sqrt{c^2+1}}\left(\frac{x - x_0}{cr}, \frac{y - y_0}{cr}, \frac{z - z_0}{cr}, \frac{t - t_0}{|t - t_0|}\right)\end{aligned} \tag{9}$$

1.1.3 The principle of causality

Now, suppose that $u(\vec{x}, t)$ is a solution to the wave equation with initial conditions

$$u(\vec{x}, 0) = \phi(\vec{x}), \qquad u_t(\vec{x}, 0) = \psi(\vec{x}). \tag{10}$$

Let $\vec{x}_0$ be any point in $\mathbb{R}^3$, and let $t_0 > 0$ be any time. We wish to prove the following:

Theorem 1.2. *The value of u at $(\vec{x}_0, t_0)$ depends only on the values of $\phi(\vec{x})$ and $\psi(\vec{x})$ in the ball*

$$B := \{\|\vec{x} - \vec{x}_0\| \le ct_0\} = \overline{C(\vec{x}_0, t_0)} \cap \{t = 0\}.$$

To prove this, we'll consider the spacetime integral of (2) over a frustum F of the solid light cone with bottom B. Call the top face of the frustum T and mantle of the frustum M, so $\partial F = B \cup M \cup T$. Without loss of generality we may assume that $\vec{x}_0 = 0$ is the origin of our coordinate system. (Exercise: Think about why this is allowed.)

Lemma 1.3. *We have an inequality*

$$\iiint_T \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) d\vec{x} \le \iiint_B \left(\frac{1}{2}\psi^2 + \frac{1}{2}c^2|\nabla\phi|^2\right) d\vec{x}. \tag{11}$$

Proof. Begin by rewriting the "local conservation of energy" equality (2) explicitly as

$$\begin{aligned}0 &= \frac{\partial}{\partial x}(-c^2u_tu_x) + \frac{\partial}{\partial y}(-c^2u_tu_y) + \frac{\partial}{\partial z}(-c^2u_tu_z) + \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) \\ &= (\partial_x, \partial_y, \partial_z, \partial_t) \cdot \left(-c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z, \frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right).\end{aligned} \tag{12}$$

We now integrate this expression over F. (12) makes explicit that the integrand is the divergence of a vector field, and so we may apply the divergence theorem to obtain

$$\iiint_{\partial F} \vec{n} \cdot \left(-c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z, \frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) d\mathrm{vol}_{\partial F} = 0, \tag{13}$$

where $\vec{n}$ is the outward pointing unit normal vector field along ∂F. Since ∂F decomposes into three pieces we can write this as

$$\iiint_T + \iiint_B + \iiint_M = 0,$$

and consider each piece separately.

Along T the outward pointing normal vector is $\vec{n} = (0, 0, 0, 1)$, so this contributes

$$\iiint_T \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) d\vec{x}$$

to the integral. Along B, $\vec{n} = (0, 0, 0, -1)$, so that contributes

$$\iint_B (-1)\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) d\vec{x} = -\iiint_B \left(\frac{1}{2}\psi(\vec{x}) + \frac{1}{2}c^2|\nabla \phi|^2\right) d\vec{x}.$$

Finally, using the formula (9) with $\vec{x} = 0$, the integral over M is

$$\frac{c}{\sqrt{c^2+1}} \iiint_M \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2\|\nabla u\|^2 + \frac{x}{cr}(-c^2 u_t u_x) + \frac{y}{cr}(-c^2 u_t u_y) + \frac{z}{cr}(-c^2 u_t u_z)\right) \mathrm{dvol}_M.$$

Let consider the radial unit vector $\hat{r} := \frac{(x,y,z)}{r}$, and the radial derivative

$$u_r = \frac{\partial u}{\partial r} = \hat{r} \cdot \nabla u = \frac{x u_x + y u_y + z u_z}{r}.$$

Using this we can rewrite the M contribution as

$$\frac{c}{2\sqrt{c^2+1}} \iiint_M \left(u_t^2 + c^2\|\nabla u\|^2 - 2c u_t u_r\right) \mathrm{dvol}_M.$$

Now, complete the square of the integrand:

$$\begin{aligned} u_t^2 - 2cu_t u_r + c^2 u_r^2 - c^2 u_r^2 + c^2\|\nabla u\|^2 &= (u_t - cu_r)^2 + c^2(\|\nabla u\|^2 - u_r^2) \\ &= (u_t - cu_r)^2 + c^2(\|\nabla u\|^2 - 2u_r^2 + u_r^2) \\ &= (u_t - cu_r)^2 + c^2(\|\nabla u\|^2 - 2\nabla u \cdot (u_r \hat{r}) + u_r^2) \\ &= (u_t - cu_r)^2 + c^2\|\nabla u - u_r \hat{r}\|^2 \geq 0. \end{aligned}$$

So the contribution from M is nonnegative; using this we have that

$$0 = \iiint_T + \iiint_B + \iiint_M \geq \iint_T + \iint_B = \iiint_T \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) d\vec{x} - \iiint_B \left(\frac{1}{2}\psi(\vec{x}) + \frac{1}{2}c^2|\nabla \phi|^2\right) d\vec{x},$$

which is the desired inequality. □

Proof of Theorem 1.2. First, suppose that ψ and ϕ vanish on B. Then our inequality (11) tells us that

$$\iiint_T \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) d\vec{x} \leq 0,$$

which, since each term in the integrand is nonnegative, implies that

$$u_t|_T = 0, \qquad \nabla u|_T = 0.$$

Since the height of the frustum is arbitrary, this tells us that u_t and ∇u vanish everywhere on the solid cone between B and the vertex $(0, t_0)$. Therefore u is constant on this solid cone, and since $u|_B \equiv 0$, u must vanish on the solid cone. In particular, $u(0, t_0) = 0$.

Now, suppose that we have two solutions u and v to the wave equation which both satisfy (10) on B. Then $u - v$ is a solution to the wave equation with vanishing initial conditions on B, and so u and v agree on the solid cone with vertex $(0, t_0)$. In particular, $u(0, t_0) = v(0, t_0)$, which is the equality we wanted to show. □

The solid light cone decomposes into "forward" and "backwards" components,

$$C(\vec{x}_0, t_0) = S^+(\vec{x}_0, t_0) \cup S^-(\vec{x}_0, t_0),$$

where

$$S^+(\vec{x}_0, t_0) = \{(\vec{x}, t) \in C(\vec{x}_0, t_0) \,|\, t > t_0\}, \tag{14}$$
$$S^-(\vec{x}_0, t_0) = \{(\vec{x}, t) \in C(\vec{x}_0, t_0) \,|\, t < t_0\}. \tag{15}$$

Rephrasing Theorem 1.2 in terms of a solution propagating forward in time (rather than asking what a solution depended on in the past) gives the following:

Corollary 1.4. *The initial data ϕ, ψ at the spatial point $\vec{x}_0$ can influence the solution only in the closure of the forward solid light cone $\overline{S^+(\vec{x}_0, 0)}$.*

Exercise: Convince yourself that this argument works in dimensions other than 3 (i.e. that waves in any dimension propagate at speed less than or equal to c).

1.2 Kirchhoff's Formula

In one dimension, we solved the wave equation on the real line explicitly – the result was the *D'Alembert formula.* We would like a similarly explicit solution to the 3d problem

$$\begin{aligned} u_{tt} &= c^2 \Delta u, \\ u(\vec{x}, 0) &= \phi(\vec{x}), \\ u_t(\vec{x}, 0) &= \psi(\vec{x}). \end{aligned} \tag{16}$$

Recall that we denote the sphere and ball of radius R centred at a point $\vec{y}$ by

$$S_R(\vec{y}) = \{\vec{x} \in \mathbb{R}^3 \,|\, \|\vec{y} - \vec{x}\| = R\}, \tag{17}$$
$$B_R(\vec{y}) = \{\vec{x} \in \mathbb{R}^3 \,|\, \|\vec{y} - \vec{x}\| < R\}, \tag{18}$$

Theorem 1.5 (Kirchhoff's Formula)**.** *The solution to* (16) *is given by* Kirchhoff's formula*,*

$$u(\vec{x}, t) = \frac{1}{4\pi c^2 t} \iint_{S_{ct}(\vec{x})} \psi \, dvol + \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \iint_{S_{ct}(\vec{x})} \phi \, dvol \right). \tag{19}$$

Proof. Our plan to derive this formula is follows:

(i) Solve the problem with $\phi \equiv 0$, $\psi \not\equiv 0$ via the *method of spherical means.* (This method reduces the problem to a 1d wave IBVP in the radial coordinate.)

(ii) Solve the problem with $\phi \not\equiv 0$, $\psi \equiv 0$ by reducing to the case dealt with in (i).

(iii) Solve the general problem via the principle of superposition.

Step (iii) in our plan is immediate – the wave equation is linear, and so the general solution can be decomposed into the sum of a solution with $\phi \equiv 0$ and a solution with $\psi \equiv 0$.

Step (i): First, observe that it suffices to solve the problem at $\vec{x}_0 = 0$ – this solution may then be translated to any other point $\vec{x}_0$. So we will assume that $\vec{x}_0 = 0$.

Now, denote the average of $u(\vec{x},t)$ on the sphere $S_r(0)$ by

$$\bar{u}(r,t) := \frac{1}{4\pi r^2}\iint_{S_r(0)} u(\vec{x},t)\,\mathrm{dvol} = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(\vec{x},t)\sin(\theta)\,d\theta\,d\phi. \tag{20}$$

We claim that $\bar{u}$ satisfies the PDE

$$\bar{u}_{tt} = c^2\bar{u}_{rr} + c^2\frac{2}{r}\bar{u}_r. \tag{21}$$

To prove this, we use the following fact (proof as an exercise): the mean of the Laplacian is the Laplacian of the mean, i.e.

$$\Delta(\bar{u}) = \overline{\Delta u}.$$

Since $\overline{\Delta u} = \overline{(u_{tt})} = \bar{u}_{tt}$, this gives

$$\bar{u}_{tt} = \Delta\bar{u}.$$

Recall that the Laplacian in spherical coordinates is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}.$$

Since $\bar{u}$ is independent of the angular variables, $\Delta_{S^2}\bar{u} = 0$, which proves (21).

Now, let $v(r,t) := r\bar{u}(r,t)$, so that

$$\begin{aligned} v_r &= r\bar{u}_r + \bar{u},\\ v_{rr} &= r\bar{u}_{rr} + 2\bar{u}_r. \end{aligned}$$

In terms of v, (21) simplifies to

$$v_{tt} = c^2 v_{rr}, \qquad 0 \le r < \infty, \tag{22}$$

with BC

$$v(0,t) = 0 \tag{23}$$

and ICs (recall we are assuming $\phi \equiv 0$)

$$v(r,0) = 0, \quad v_r(r,0) = r\bar{\psi}(r). \tag{24}$$

This is just the wave equation IBVP on the half-line, with a homogeneous Dirichlet boundary condition. We solved this (long ago):

$$v(r,t) = \frac{1}{2c}\int_{ct-r}^{ct+r} s\bar{\psi}(s)\,ds, \qquad\qquad 0 \le r \le ct. \tag{25}$$

We have ignored the range $r > ct$ since we are about to take the $r \to 0$ limit – we are interested in $u(0,t)$, and this is given by

$$\begin{aligned} u(0,t) = \bar{u}(0,t) &= \lim_{r\to 0}\frac{v(r,t)}{r}\\ &= \lim_{r\to 0}\frac{v(r,t) - v(0,t)}{r} = \frac{\partial v}{\partial r}(0,t). \end{aligned}$$

We calculate this as

$$\begin{aligned} \left.\frac{\partial v}{\partial r}\right|_{r=0} &= \left.\frac{(ct+r)\bar{\psi}(ct+r) + (ct-r)\bar{\psi}(ct-r)}{2c}\right|_{r=0}\\ &= \frac{ct\bar{\psi}(ct) + ct\bar{\psi}(ct)}{2c} = t\bar{\psi}(ct) = \frac{t}{4\pi(ct)^2}\iint_{S_{ct}(0)}\psi(\vec{x})\,\mathrm{dvol}, \end{aligned}$$

i.e.

$$u(0,t) = \frac{1}{4\pi c^2 t} \iint_{S_{ct}(0)} \psi(\vec{x})\, d\text{vol}. \tag{26}$$

This complete Step (i).

Step (ii): Consider the auxilliary problem

$$\begin{aligned} w_{tt} &= c^2 \Delta w, \\ w(\vec{x}, 0) &= 0, \\ w_t(\vec{x}, 0) &= \phi(\vec{x}). \end{aligned} \tag{27}$$

By Step (i), this has solution

$$w(\vec{x},t) = \frac{1}{4\pi c^2 t} \iint_{S_{ct}(\vec{x})} \phi\, d\text{vol}. \tag{28}$$

Since w vanishes at $t = 0$, we also have $\Delta w|_{t=0} = 0$, and since w solves the wave equation this implies that $w_{tt}|_{t=0} = 0$. Setting $u := w_t$, we have that u solves the original problem (16) with $\psi \equiv 0$,

$$\begin{aligned} u_{tt} &= c^2 \Delta u, \\ u(\vec{x}, 0) &= \phi(\vec{x}), \\ u_t(\vec{x}, 0) &= 0, \end{aligned}$$

and since

$$u(\vec{x},t) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \iint_{S_{ct}(\vec{x})} \phi\, d\text{vol} \right), \tag{29}$$

this completes the proof of Step (ii), and thus of the theorem. □

1.2.1 Huygen's principle

It would be nice if we could read off from (19) the principle of causality we discussed in Section 1.1.3. Looking at the formula, we see that we have indeed expressed the solution u in a way that makes its dependence on ψ and ϕ explicit – the integrals $u(\vec{x}, t)$ depends only on the values of ϕ and ψ on the sphere $S_{ct}(\vec{x})$ (and importantly, *not* on the ball $B_{ct}(\vec{x})$!)

Inverting this statement yields *Huygen's principle*: The values of ϕ and ψ at a spatial point $\vec{x} \in \mathbb{R}^3$ can only influence the solution on the spherical surface $S_{ct}(\vec{x})$.

I.e. a solution to the 3d wave equation propagates at speed *exactly* c. This is a stronger statement than the principle of causality!

1.2.2 Inhomogeneous wave equation

Suppose now that we wanted to solve the problem

$$\begin{aligned} u_{tt} - c^2 \Delta u &= f, \\ u(\vec{x}, 0) &= 0, \\ u_t(\vec{x}, 0) &= 0. \end{aligned} \tag{30}$$

We can construct a solution using Duhamel's principle – recall that this gave us a method of constructing a solution to the inhomogeneous problem by solving a family of homogeneous problems. Specifically, given the solution to the auxilliary problem

$$\begin{aligned} \tilde{u}_{tt} - c^2 \Delta \tilde{u} &= 0, \\ \tilde{u}|_{t=\tau} &= 0, \\ \tilde{u}_t|_{t=\tau} &= f(\vec{x}, \tau), \end{aligned} \tag{31}$$

the solution to (30) is

$$u(\vec{x}, t) = \int_0^t \tilde{u}(\vec{x}, t - \tau)\, d\tau. \tag{32}$$

From our solution to the homogeneous equation, we know that $\tilde{u}$ is given by

$$\tilde{u}(\vec{x}, t, \tau) = \frac{1}{4\pi c^2 (t - \tau)} \iint_{S_{c|t-\tau|}(\vec{x})} f\, d\text{vol}, \tag{33}$$

so we find:

Theorem 1.6 (Kirchhoff's Formula – inhomogeneous version)**.** *The solution to* (30) *is given by*

$$u(\vec{x}, t) = \int_0^t \left(\frac{1}{4\pi c^2 (t - \tau)} \iint_{S_{c|t-\tau|}(\vec{x})} f\, dvol \right) d\tau. \tag{34}$$

2 Waves in two dimensions

Let's now leverage our understanding of the 3d wave equation to study solutions to the 2d wave equation,

$$\begin{aligned} u_{tt} &= c^2 (u_{xx} + u_{yy}), \\ u(x, y, 0) &= \phi(x, y), \\ u_t(x, y, 0) &= \psi(x, y). \end{aligned} \tag{35}$$

The trick we will use is the *method of descent*: we will introduce a third spatial variable z, and regard a solution to (35) as a solution to (16) which is independent of the coordinate z.

We will derive the value of u at $(x, y) = (0, 0)$, assuming that $\phi \equiv 0$. The formula at other points will follow by translation, and the formula for nonvanishing ϕ will be $\frac{\partial}{\partial t}$ of the expression we derive (with $\psi \to \phi$), just as before.

Thinking of u as a solution to the 3d wave equation, we know that its solution must be given by Kirchhoff's formula (19), so that

$$u(0, 0, t) = \frac{1}{4\pi c^2 t} \iint_{x^2+y^2+z^2=c^2t^2} \psi(x, y)\, d\text{vol}. \tag{36}$$

We simplify this expression as follows. First, since the integrand is independent of z, and the domain is unchanged by the map $z \mapsto -z$, this is twice the integral of the same integrand over the upper hemisphere,

$$\begin{aligned} u(0, 0, t) &= \frac{1}{4\pi c^2 t} \iint_{x^2+y^2+z^2=c^2t^2} \psi(x, y)\, d\text{vol} \\ &= \frac{1}{2\pi c^2 t} \iint_{z=+\sqrt{c^2t^2-x^2-y^2}} \psi(x, y)\, d\text{vol}. \end{aligned}$$

Next, on this hemisphere we can write the volume form in terms of x and y as

$$\begin{aligned} d\text{vol} &= \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy \\ &= \sqrt{1+\left(\frac{-x}{z}\right)^2+\left(\frac{-y}{z}\right)^2}\,dx\,dy \\ &= \sqrt{\frac{z^2+x^2+y^2}{z^2}}\,dx\,dy \\ &= \frac{ct}{\sqrt{c^2t^2-x^2-y^2}}\,dx\,dy \end{aligned}$$

Using this expression for the volume form gives

$$u(0,0,t)=\frac{1}{2\pi c}\iint_{x^2+y^2\le c^2t^2}\frac{\psi(x,y)}{\sqrt{c^2t^2-x^2-y^2}}\,dx\,dy$$

Let $D_r(\vec{x})=\{\vec{x}'\in\mathbb{R}^2\mid \|\vec{x}-\vec{x}'\|\le r\}$ be the disc of radius r centred on the point $\vec{x}$. We have shown:

Theorem 2.1. *The solution to* (35) *is given by*

$$u(\vec{x},t)=\iint_{D_{ct}(\vec{x})}\frac{\psi(\vec{x}')}{\sqrt{c^2t^2-\|\vec{x}-\vec{x}'\|^2}}\frac{d\vec{x}'}{2\pi c}+\frac{\partial}{\partial t}\left(\iint_{D_{ct}(\vec{x})}\frac{\phi(\vec{x}')}{\sqrt{c^2t^2-\|\vec{x}-\vec{x}'\|^2}}\frac{d\vec{x}'}{2\pi c}\right). \tag{37}$$

2.1 Failure of Huygen's principle

Note an immediate consequence of (37) is that Hugyen's principle fails in two dimensions! More precisely, by looking at the domain of integration, we see that the values of ϕ and ψ at point $\vec{x}\in\mathbb{R}^2$ can influence the solution u at any point in the disc $D_{ct}(\vec{x})$, and *not* just on the bounding circle $\{\|\vec{x}'-\vec{x}\|=ct\}$.

To understand this difference in behaviour, let's consider the solution to the wave equation in 2d and 3d where we start from the zero solution and hit our wave with an instantaneous impulse at $t=0$; i.e. our initial conditions are[3]

$$\begin{aligned} u(\vec{x},0)&=\phi(\vec{x})=0,\\ u_t(\vec{x},0)&=\psi(\vec{x})=2\pi c\delta(\vec{x}). \end{aligned}$$

We have mentioned (but not explicitly studied) the Dirac-δ previously: all that we need to know is that it's defining property is that

$$\int_\Omega \delta(\vec{x}-\vec{y})f(\vec{y})d\vec{y}=\begin{cases} f(\vec{x}), & \vec{x}\in\Omega,\\ 0, & \vec{x}\notin\Omega.\end{cases} \tag{38}$$

You can picture $\delta(\vec{x})$ as an infinitely tall spike, concentrated at the origin, with integral 1 (this picture may help you understand the claim this $\delta(\vec{x})$ represents an instantaneous concentrated impluse).

By Kirchhoff's formula (19), the corresponding 3d wave is given by

$$u(\vec{x},t)=\begin{cases} 0\,, & \|\vec{x}\|\ne ct,\\ \dfrac{1}{2ct}\,, & \|\vec{x}\|=ct.\end{cases} \tag{39}$$

So the wave propagates outwards, supported on the spherical shell $\|\vec{x}\|=ct$, and with amplituding decaying proportionally to $\frac{1}{ct}=\frac{1}{\|\vec{x}\|}$.

[3]The factor of $2\pi c$ is simply a convenient choice of magnitude for our impulse.

We can constrast this with the behaviour of the corresponding 2d wave, which from (37) is given by

$$u(\vec{x},t) = \begin{cases} 0\,, & \|x\| > ct, \\ \dfrac{1}{\sqrt{c^2t^2 - \|\vec{x}\|^2}}\,, & \|x\| \leq ct. \end{cases} \tag{40}$$

This wave also propagates outwards, however now it is supported on the entire disc $\|\vec{x}\| < ct$ – one the wavefront hits a point $\vec{x} \in \mathbb{R}^2$, the solution at that point is never zero again (although it does decay to zero as $t \to \infty$). Moreover, the amplitude of the wavefront in 2d does not decay over time – indeed, at the actual wavefront $\|x\| = ct$ the solution always blows up to $+\infty$! For a visualisation of this solution, see the animation on the visualisation page corresponding to this lecture.

Exercise: Check that (40) is a solution to the 2d wave equation.

Remark 2.1. You can use the same sorts of techniques to prove that Huygen's principle holds in every odd dimension ≥ 3; using the method of descent, you can then also show that it fails in every even dimension. See [IvrXX, §9.1] for the explicit formulae.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.