APM 346 Lecture 20.

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Continuing from the end the last lecture, we will consider some applications of spherical harmonics.

References being used: [IvrXX, §8.1] (§8.1) and [Str08, Ch.10].

1 Applications of spherical harmonics

Recall from Lecture 19 that we found a complete orthogonal system for L^2-functions on the unit sphere,

$$Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)e^{im\theta}, \qquad 0 \le l < \infty,\ |m| \le l, \tag{1}$$

which were simultaneous eigenvectors for the Laplacian and the generator of rotations around the z-axis $\frac{\partial}{\partial\phi}$

$$\begin{aligned}\Delta_{S^2} Y_l^m &= -l(l+1)Y_l^m,\\ \frac{\partial}{\partial\phi}Y_l^m &= -m^2 Y_l^m.\end{aligned}$$

The functions P_l^m appearing in (1) are the *associated Legendre functions* (or *Legendre polynomials* for $m = 0$) given by

$$P_l(s) = \frac{1}{2^l l!}\frac{d^l}{ds^l}(s^2-1)^l, \tag{2}$$

$$P_l^m(s) = (-1)^m(1-s^2)^{\frac{m}{2}}\frac{d^m}{ds^m}P_l(s); \tag{3}$$

these solved the differential equations

$$\frac{d}{ds}\left((1-s^2)\frac{dP}{ds}\right) + \left(l(l+1) - \frac{m^2}{1-s^2}\right)P = 0. \tag{4}$$

1.1 Dirichlet problem inside and outside the ball

Let's start by considering the problem that we started thinking about last lecture (before we discovered spherical harmonics):

$$\Delta u = 0, \qquad \text{in or outside of } B_R(0), \tag{5}$$

$$u = g, \qquad \text{on } S_R(0) = \partial B_R(0). \tag{6}$$

By separating variables $u = R(\rho)Y(\theta,\phi)$ we obtained the system of DEs

$$\rho^2 R''(\rho) + 2R'(\rho) - \lambda R(\rho) = 0, \tag{7}$$

$$\Delta_{S^2} Y(\theta,\phi) + \lambda Y(\theta,\phi) = 0. \tag{8}$$

Solving (8), we found the spherical harmonics Y_l^m, $0 \le l < \infty$, $|m| \le l$, where the eigenvalues of the Laplace operator are given by

$$\lambda = l(l+1). \tag{9}$$

The radial equation (7) is of Euler type, and has solutions $R(\rho) = \rho^\alpha$ where $\alpha(\alpha+1) - l(l+1) = 0$. The roots of this equation are $\alpha = l, -l-1$, one of which is positive, the other of which is negative.

Inside the ball we require the solution be nonsingular at 0; outside the ball we require boundedness. Discarding either the positive or negative powers of ρ as is appropriate, we arrive at the separated solutions

$$u^m_{\mathrm{in},l}(\rho,\theta,\phi) = \rho^l Y_l^m(\theta,\phi), \tag{10}$$
$$u^m_{\mathrm{out},l}(\rho,\theta,\phi) = \rho^{-l-1} Y_l^m(\theta,\phi),, \tag{11}$$

and so the Dirichlet problem has series solutions given by

$$u_{\mathrm{in}} = \sum_{l=0}^{\infty} A_m^l \rho^l Y_l^m, \tag{12}$$
$$u_{\mathrm{out}} = \sum_{l=0}^{\infty} B_m^l \rho^{-l-1} Y_l^m, \tag{13}$$
$$\tag{14}$$

where the coefficients A_m^l and B_m^l are determined by the decomposition of the boundary data $g(\theta,\phi)$ into spherical harmonics.

Example 1. Suppose that we want to solve the Dirichlet problem in ball or radius R, $B_R(0)$ and with boundary conditions

$$g(\theta,\phi) = \sin(\theta)\cos(\theta)\cos(\phi).$$

Recalling the table of spherical harmonics from last time, we see that this is in fact the real spherical harmonic P_2^1 ($l = 2$). So the series solution collapses to a single term

$$u = \left(\frac{\rho}{R}\right)^2 \sin(\theta)\cos(\theta)\cos(\phi).$$

This can alternately be expressed in cartesian coordinates as

$$u = \frac{\rho^2}{R^2}\frac{xz}{\rho^2} = \frac{xz}{R^2}.$$

Remark 1.1. In fact all of the separated solutions for the Laplace equation in the solid ball $u^m_{\mathrm{in},l}$ are given by polynomials in the cartesian coordinates x, y, z (called – unsurprisingly – *harmonic polynomials*). As an exercise you should try to show this – alternately, consult [Str08, Ch.10.3].

1.2 Wave equation on the ball

Now, instead of the Laplace equation, let's consider the homogeneous Dirichlet problem for the wave equation

$$u_{tt} - c^2 \Delta u = 0, \qquad \text{in} \qquad B_R(0), \tag{15}$$
$$u|_{S_R(0)} = 0. \tag{16}$$

First let's separate out the time variable, $u = T(t)v(\vec{x})$. We get

$$\frac{T''(t)}{T(t)} - c^2 \underbrace{\frac{\Delta v}{v}}_{=-\lambda} = 0,$$

which gives the system of equations

$$\Delta v = -\lambda v, \tag{17}$$
$$T''(t) + c^2\lambda T(t) = 0. \tag{18}$$

Equation (17) is called the *Helmholtz equation*. Separating variables again as $v = R(\rho)Y(\theta, \phi)$, we obtain

$$\left(\frac{\rho^2 R'' + 2\rho R'}{R} + \lambda\rho^2\right) + \frac{\Delta_{S^2} Y}{Y} = 0.$$

We already know that the solutions to the Y-eigenvalue problem are spherical harmonics Y_l^m, with $0 \le l < \infty$ (since we are inside the ball). Substituting in $-l(l+1)$ for the Y-terms in the equation we arrive at the equation

$$R''(\rho) + \frac{2}{\rho}R'(\rho) + \left(\lambda - \frac{l(l+1)}{\rho^2}\right)R(\rho) = 0. \tag{19}$$

1.2.1 Bessel functions

To solve (19) we will put it into a standard form, whose solutions are known. First make the change of coordinates $w(\rho) := \sqrt{\rho}R(\rho)$. We now have

$$\frac{d^2 w}{d\rho^2}(\rho) + \frac{1}{\rho}\frac{dw}{d\rho}(\rho) + \left(\lambda - \frac{l(l+1) + \frac{1}{4}}{\rho^2}\right)w(\rho) = 0. \tag{20}$$

Next, rescale the radial variable as

$$r = \sqrt{\lambda}\rho, \qquad \frac{d}{d\rho} = \sqrt{\lambda}\frac{d}{dr}.$$

With this rescaling, the ODE becomes

$$\frac{d^2 w}{dr} + \frac{1}{r}\frac{dw}{dr} + \left(1 - \frac{l(l+1) + \frac{1}{4}}{r^2}\right)w = 0, \tag{21}$$

which is exactly *Bessel's differential equation*

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \left(1 - \frac{s^2}{r^2}\right)w = 0, \tag{22}$$

at $s = \sqrt{l(l+1) + \frac{1}{4}} = \sqrt{(l + \frac{1}{2})^2} = \pm\left(l + \frac{1}{2}\right)$.

How would one actually *solve* an ODE like (22)? One way would be to search for a series solution, of the form

$$w(r) = r^\alpha \sum_{n=0}^{\infty} a_n r^n, \tag{23}$$

and then use the Bessel equation (22) to obtain

- the possible values of α (solution: $\alpha = \pm s$),
- vanishing of the odd coefficients, and
- a recursion relation on the even coefficients $a_n = -\frac{a_{n-2}}{(\alpha+n)^2 - s^2}$.

For any $s \in \mathbb{R} \setminus \{-1, -2, -3, \ldots\}$ this logic can in fact be followed through to obtain an honest series solution to (22), which with its standard normalisation is given by

$$J_s(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+s+1)} \left(\frac{r}{2}\right)^{2n+s}. \tag{24}$$

Here Γ is the *gamma function* which extends the factorial function on the integers in the sense that

$$\Gamma(n+1) = n!$$

for $n \in \mathbb{Z}_{\geq 0}$.

For an introduction to some of the properties of Bessel functions, see [Str08, Ch.10.5].

1.2.2 Back to the wave equation

Let's now apply our new-found knowledge of Bessel functions to the wave equation. Our solution $w(\rho)$ is going to be a Bessel function, and since we require the solution to be nonsingular at zero we see from (24) that we need a solution with $s > 0$. Putting this together, we have that

$$w(\rho) = J_{l+\frac{1}{2}}(\sqrt{\lambda}\rho),$$

and so the radial function R is

$$R(\rho) = \frac{J_{l+\frac{1}{2}}(\sqrt{\lambda}\rho)}{\sqrt{\rho}}. \tag{25}$$

For an idea of what (25) looks like see Figure 1.

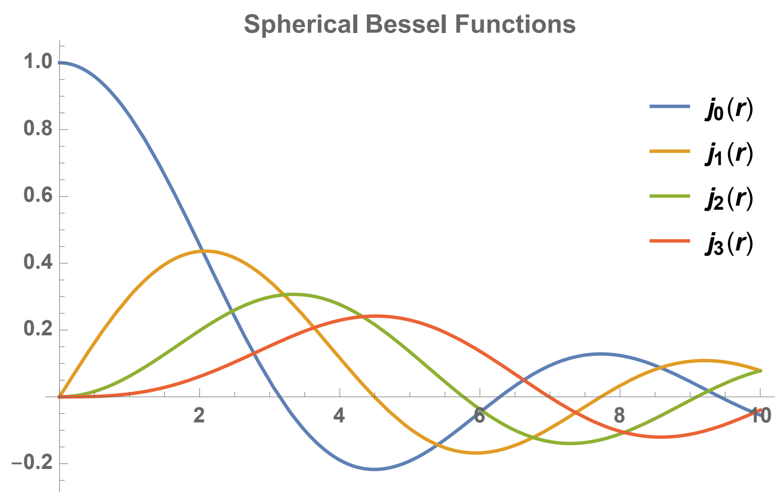

Figure 1: The first four spherical Bessel functions.

To determine the allowed values of λ, apply the Dirichlet boundary condition $R(R) = 0$ to find the equation

$$J_{l+\frac{1}{2}}(\sqrt{\lambda}R) = 0. \tag{26}$$

Remark 1.2. Compare this to the application of Dirichlet BCs to the equation $X'' + \lambda X = 0$ on $0 \leq x \leq l$ – there the equation to solve was $\sin(\sqrt{\lambda}l) = 0$, which we could do explicitly.

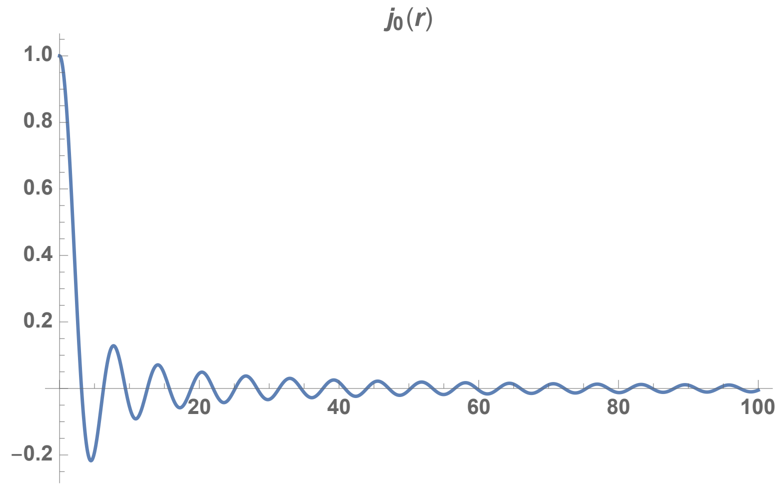

Figure 2: Zeroes of the first spherical Bessel function.

Although we cannot solve (26) explicitly, we are able to say that there are infinitely many zeros, tending towards infinity (see e.g. Figure 2). Denote by

$$\lambda_{l,1} < \lambda_{l,2} < \lambda_{l,3} < \cdots \tag{27}$$

the values of λ which solve (26). Then the ρ-components of our separated solution may be written as

$$R_{l,k}(\rho) = \frac{J_{l+\frac{1}{2}}(\sqrt{\lambda_{l,k}}\rho)}{\sqrt{\rho}}. \tag{28}$$

Finally, to finish solving the wave equation via separation of variables, we return to the equation

$$T'' + c^2\lambda T = 0,$$

which has solutions

$$T_{lk}(t) = A\cos(c\sqrt{\lambda_{l,k}}t) + B\sin(c\sqrt{\lambda_{l,k}}t). \tag{29}$$

So we find that

$$u(\rho,\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m=-l}^{l}(A_{lkm}\cos(c\sqrt{\lambda_{l,k}}t) + B_{lkm}\sin(c\sqrt{\lambda_{l,k}}t))\frac{J_{l+\frac{1}{2}}(\sqrt{\lambda_{l,k}}\rho)}{\sqrt{\rho}}Y_l^m(\theta,\phi). \tag{30}$$

As always, the coefficients A_{lkm} and B_{lkm} will be determined by two initial conditions.

1.2.3 Heat equation on the ball

Note that the same separation of variables arguments could have been run up until (28) to solve the heat equation[1]

$$u_t = D\Delta u$$

on the ball $B_R(0)$ with homogeneous Dirichlet boundary conditions. The final result would be the familiar looking expression

$$u(\rho,\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m=-l}^{l}A_{lkm}e^{-D\lambda_{l,k}}\frac{J_{l+\frac{1}{2}}(\sqrt{\lambda_{l,k}}\rho)}{\sqrt{\rho}}Y_l^m(\theta,\phi), \tag{31}$$

where the coefficients A_{lkm} are determined by the initial condition.

[1]We use the notation D instead of k for the coefficient in the heat equation here to avoid confusion with the summation index.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.