APM 346 Lecture 19.

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Today we introduce the notion of a Green's function, before moving on to the Laplace equation in spherical coordinates.

References being used: [IvrXX, §7.3,§8.1] (§7.3,§8.1) and [Str08, Ch.7,Ch.10].

1 Green's functions

Recall that last lecture we proved that for u a function on a bounded domain[1] Ω, and for $y \in \Omega$ (not on the boundary), we had the equality

$$u(y) = \int_{x\in\Omega} G_0(x,y)\Delta u(x)\, d\mathrm{vol}_\Omega + \int_{x\in\partial\Omega} \left(-u(x)\frac{\partial G_0}{\partial \nu_x}(x,y) + G_0(x,y)\frac{\partial u}{\partial \nu}(x)\right) d\mathrm{vol}_{\partial\Omega} \tag{1}$$

where

$$G_0(x,y) = \begin{cases} -\frac{|x-y|^{2-n}}{(n-2)\sigma_n}, & n \neq 2, \\ \frac{1}{2\pi}\log|x-y|, & n = 2. \end{cases} \tag{2}$$

We would like to be able to use (1) to actually *solve* the *Poisson equation*[2]

$$\Delta u = f, \tag{3}$$

with some prescribed boundary conditions.

Unfortunately, the data required to reconstruct the function u in (1) would specify an *overdetermined* problem – one cannot arbitrarily specify both $u|_{\partial\Omega}$ and $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$ – and so as written the equation is ill-suited to solving well-posed problems. Our goal now is to remedy this deficit.

1.1 Green's function for the Dirichlet problem

Ultimately, the problem with the function G_0 is that it is insensitive to the boundary conditions for our problems (it was, after, constructed in a BC insensitive manner). Let's first try to remedy this for the Dirichlet problem.

Let $g(x,y)$ be a solution to the problem

$$\begin{aligned} \Delta_x g(x,y) &= 0, && x \in \Omega, \\ g(x,y) &= -G_0(x,y), && x \in \partial\Omega, \end{aligned} \tag{4}$$

with the further condition that $g \to 0$ as $|x| \to \infty$ if the domain Ω is unbounded.

[1] Given appropriate conditions at ∞, these also made sense on an unbounded domain.

[2] I.e. the inhomogeneous Laplace equation.

Recall the identity

$$\int_\Omega (w\Delta u - u\Delta w)\,\mathrm{dvol}_\Omega = \int_{\partial\Omega}\left(u\frac{\partial w}{\partial\nu} - w\frac{\partial u}{\partial\nu}\right)\,\mathrm{dvol}_{\partial\Omega}. \tag{5}$$

Setting $w_y(x) = g(x,y)$ gives

$$\int_\Omega (g(x,y)\Delta u(x) - u(x)\underbrace{\Delta_x g}_{=0})\,\mathrm{dvol}_\Omega = \int_{\partial\Omega}\left(u\frac{\partial g}{\partial\nu_x} - g(x,y)\frac{\partial u}{\partial\nu}\right)\,\mathrm{dvol}_{\partial\Omega},$$

and so

$$0 = \int_\Omega g(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega - \int_{\partial\Omega} u(x)\frac{\partial g}{\partial\nu_x}(x,y)\mathrm{dvol}_{\partial\Omega} + \int_{\partial\Omega} g(x,y)\frac{\partial u}{\partial\nu}\,\mathrm{dvol}_{\partial\Omega}. \tag{6}$$

Adding (6) to (1) gives

$$\begin{aligned}
u(y) &= \int_{x\in\Omega} G_0(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega + \int_{x\in\partial\Omega}\left(-u(x)\frac{\partial G_0}{\partial\nu_x}(x,y) + \underbrace{G_0(x,y)}_{=-g}\frac{\partial u}{\partial\nu}(x)\right)\mathrm{dvol}_{\partial\Omega}\\
&\qquad + \int_\Omega g(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega - \int_{\partial\Omega} u(x)\frac{\partial g}{\partial\nu_x}(x,y)\mathrm{dvol}_{\partial\Omega} + \int_{\partial\Omega} g(x,y)\frac{\partial u}{\partial\nu}\,\mathrm{dvol}_{\partial\Omega}\\
&= \int_{x\in\Omega}(G_0+g)(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega - \int_{x\in\partial\Omega} u(x)\frac{\partial(G_0+g)}{\partial\nu_x}(x,y)\mathrm{dvol}_{\partial\Omega}
\end{aligned}$$

Setting $G(x,y) := G_0(x,y) + g(x,y)$, we have shown:

Proposition 1.1. *The function*

$$u(y) := \int_\Omega G(x,y)f(x)\,dvol_\Omega - \int_{\partial\Omega}\frac{\partial G}{\partial\nu_x}(x,y)\phi(x)\,dvol_{\partial\Omega}$$

is a solution to the Dirichlet problem

$$\begin{aligned}
\Delta u &= f, && \text{in}\quad \Omega,\\
u &= \phi, && \text{on}\quad \partial\Omega.
\end{aligned}$$

Remark 1.1. Observe that by construction the function G satisfies the homogeneous Dirichlet boundary condition $G|_{x\in\partial\Omega} \equiv 0$.

1.2 Green's function for the Robin problem

Suppose instead of (4) we considered a Robin boundary condition for g of the form

$$\left(\frac{\partial g}{\partial\nu_x} - \alpha g\right)(x,y) = -\left(\frac{\partial G_0}{\partial\nu_x} - \alpha G_0\right)(x,y), \qquad x\in\partial\Omega. \tag{7}$$

where $\alpha \neq 0$ (i.e. not the pure Neumann BC case).

Rewrite (1) as

$$\begin{aligned}
u(y) &= \int_{x\in\Omega} G_0(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega + \int_{x\in\partial\Omega}\left(-u(x)\frac{\partial G_0}{\partial\nu_x}(x,y) + G_0(x,y)\frac{\partial u}{\partial\nu}(x)\right)\mathrm{dvol}_{\partial\Omega}\\
&= \int_{x\in\Omega} G_0(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega + \int_{x\in\partial\Omega}\left(-u(x)\left(\frac{\partial G_0}{\partial\nu_x} - \alpha G_0\right)(x,y) + G_0(x,y)\left(\frac{\partial u}{\partial\nu}(x) - \alpha u(x)\right)\right)\mathrm{dvol}_{\partial\Omega}
\end{aligned}$$

and (6) as

$$\int_\Omega g(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega = \int_{\partial\Omega}\left(u(x)\frac{\partial g}{\partial \nu_x}(x,y) - g(x,y)\frac{\partial u}{\partial \nu}(x)\right)\mathrm{dvol}_{\partial\Omega}$$
$$= \int_{\partial\Omega}\left(u(x)\left(\frac{\partial g}{\partial \nu_x} - \alpha g\right)(x,y) - g(x,y)\left(\frac{\partial u}{\partial \nu} - \alpha u\right)(x)\right)\mathrm{dvol}_{\partial\Omega}.$$

Adding these together as in the Dirichlet case gives

$$u(y) = \int_{x\in\Omega} G_0(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega + \int_{x\in\partial\Omega}\left(-u(x)\left(\underbrace{\frac{\partial G_0}{\partial \nu_x} - \alpha G_0}_{=-(\partial_\nu g - \alpha g)}\right)(x,y) + G_0(x,y)\left(\frac{\partial u}{\partial \nu} - \alpha u\right)(x)\right)\mathrm{dvol}_{\partial\Omega}$$
$$+ \int_\Omega g(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega - \int_{\partial\Omega} u(x)\left(\frac{\partial g}{\partial \nu_x} - \alpha g\right)(x,y)\,\mathrm{dvol}_{\partial\Omega} + \int_{\partial\Omega} g(x,y)\left(\frac{\partial u}{\partial \nu} - \alpha u\right)(x)\,\mathrm{dvol}_{\partial\Omega}$$
$$= \int_\Omega (G_0+g)(x,y)\Delta u(x)\,\mathrm{dvol}_\Omega + \int_{\partial\Omega}(G_0+g)(x,y)\left(\frac{\partial u}{\partial \nu} - \alpha u\right)(x)\,\mathrm{dvol}_{\partial\Omega}.$$

So again, setting $G(x,y) = G_0(x,y) + g(x,y)$ – which will now satisfy the desired homogeneous Robin BCs – we have shown

Proposition 1.2. *The function*

$$u(y) := \int_\Omega G(x,y)f(x)\,dvol_\Omega + \int_{\partial\Omega} G(x,y)\psi(x)\,dvol_{\partial\Omega}$$

is a solution to the Robin problem

$$\begin{aligned} \Delta u &= f, && \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} - \alpha u &= \psi, && \text{on} \quad \partial\Omega. \end{aligned}$$

1.2.1 Green's function for mixed problems

The above Propositions 1.1 and 1.2 can be combined to give a solution to a problem with mixed boundary conditions

$$\begin{aligned} \Delta u &= f, && \text{in} \quad \Omega, \\ u &= \phi, && \text{on} \quad \Gamma_1, \\ \frac{\partial u}{\partial \nu} - \alpha u &= \psi, && \text{on} \quad \Gamma_2, \end{aligned} \tag{8}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. If we define g to be the harmonic function on Ω satisfying the mixed boundary conditions

$$\begin{aligned} g(x,y) &= -G_0(x,y), && \text{on} \quad \Gamma_1, \\ \left(\frac{\partial g}{\partial \nu_x} - \alpha g\right)(x,y) &= -\left(\frac{\partial G_0}{\partial \nu_x} - \alpha G_0\right)(x,y), && \text{on} \quad \Gamma_2, \end{aligned}$$

and we set $G = G_0 + g$, then we find:

Corollary 1.3. *The function*

$$u(y) := \int_\Omega G(x,y)f(x)\,dvol_\Omega - \int_{\Gamma_1} \frac{\partial G}{\partial \nu_x}(x,y)\phi(x)\,dvol_{\partial\Omega} + \int_{\Gamma_2} G(x,y)\psi(x)\,dvol_{\partial\Omega}$$

is a solution to the mixed problem (8).

1.3 Final comments on Green's functions

There are two final comments to make before we finish up our discussion of Green's functions.

1.3.1 Green's function for Neumann problems

Above we have covered the cases of Dirichlet, Robin and mixed boundary conditions. We will not cover the case of pure Neumann BCs, except to mention that there are extra complications that arise.

First, as we should expect by now, there is an extra constraint on the data in the Neumann problem

$$\begin{aligned} \Delta u &= f, && \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} &= \psi, && \text{on} \quad \partial\Omega, \end{aligned}$$

which (as we have seen in the past) is the statement that the contributions from the source term f and the boundary rates ψ must "balance out":

$$\int_\Omega f\, d\mathrm{vol}_\Omega + \int_{\partial\Omega} \psi\, d\mathrm{vol}_{\partial\Omega} = 0 \tag{9}$$

The second thing to be aware of is that the Neumann Green's function satisfies a different differential equation to the Dirichlet and Robin Green's functions. Those satisfy the differential equation

$$\Delta_x G(x,y) = \delta(x-y),$$

while the Neumann Green's function $G_N(x,y)$ satisfies

$$\Delta_x G_N(x,y) = \delta(x-y) - C$$

for some constant C depending on the geometry of our problem (e.g. for the interval of length L, $C = \frac{1}{L}$ [Fra12]). See [IvrXX, §7.3] and [Fra12] for more information.

1.3.2 Symmetry of Green's functions

Suppose that $G(x,y)$ is a Green's function for Dirichlet or Robin boundary conditions. Then one can show (but we won't) that G is symmetric in x and y,

$$G(x,y) = G(y,x).$$

(Note that this does not have to hold for a Neumann Green's function.)

2 Separation of variables in spherical coordinates

We'll now return to the study of the Laplace equation via separation of variables, this time for problems that have some degree of spherical symmetry. Recall that the Laplace operator in spherical coordinates

$$\begin{aligned} x &= \rho \sin\theta \cos\phi \\ y &= \rho \sin\theta \sin\phi \qquad (10) \\ z &= \rho \cos\theta \qquad (11) \end{aligned}$$

is given by

$$\Delta = \partial_\rho^2 + \frac{2}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_{S^2} \tag{12}$$

where

$$\Delta_{S^2} \equiv \Lambda := \partial_\theta^2 + \cot(\theta)\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2 \tag{13}$$

is the Laplace operator on the unit sphere.

Let's begin by separating out the radial variable from the angular variables. Write

$$u(\rho,\theta,\phi) = R(\rho)Y(\theta,\phi)$$

so that the equation $\Lambda u = 0$ becomes

$$R''(\rho)Y(\theta,\phi) + \frac{2}{\rho}R'(\rho)Y(\theta,\phi) + \frac{R(\rho)}{\rho^2}\Delta_{S^2}Y(\theta,\phi) = 0.$$

Divide through by $\frac{R(\rho)Y(\theta,\phi)}{\rho^2}$ to obtain the separated equation

$$\frac{\rho^2 R''(\rho) + 2\rho R'(\rho)}{R(\rho)} + \frac{\Delta_{S^2}Y(\theta,\phi)}{Y(\theta,\phi)} = 0.$$

As always, we observe that for this equation to hold the two summands must both be constant. Setting the Y-summand equal to $-\lambda$, we obtain the system of equations

$$\rho^2 R''(\rho) + 2\rho R'(\rho) - \lambda R(\rho) = 0, \tag{14}$$

$$\Delta_{S^2}Y(\theta,\phi) + \lambda Y(\theta,\phi) = 0. \tag{15}$$

2.1 The radial ODE

Let's first consider the radial ODE (14). This is an Euler type equation (we encountered this previously when studying the polar Laplace equation in 2d), so we expect solutions to be of the form

$$R(\rho) = \rho^l.$$

Substituting this in to (14) gives

$$(l(l-1) + 2l - \lambda)r^l = 0,$$

i.e.

$$\lambda = l^2 + l = l(l+1). \tag{16}$$

Now, suppose we are working in a solid ball – so that in particular $\rho = 0$ is in our domain. Then u must be infinitely differentiable at the origin,[3] and this will only be true if $l = 0, 1, 2, \ldots$.

2.2 Spherical harmonics: the Laplace operator on the sphere

Let's now move on to our problem on the sphere (15),

$$\Delta_{S^2}Y(\theta,\phi) + l(l+1)Y(\theta,\phi) = 0$$

with the implicit boundary conditions

[3] This is a fact about harmonic functions that we unfortunately did not have time to cover; consult e.g. [Str08, Ch.6].

- $Y(\theta, \phi)$ is periodic of period 2π in ϕ,
- $Y(\theta, \phi)$ is finite at the poles $\theta = 0, \pi$.

Separating the angular variables $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$ and using the definition of Δ_{S^2} gives

$$\Theta''(\theta)\Phi(\phi) + \cot(\theta)\Theta'(\theta)\Phi(\phi) + \frac{1}{\sin^2\theta}\Theta(\theta)\Phi''(\phi) + l(l+1)\Theta(\theta)\Phi(\phi) = 0.$$

Divide through by $\frac{\Theta\Phi}{\sin^2\theta}$ to obtain the separated equation

$$\frac{\Phi''}{\Phi} + \frac{\sin^2(\theta)\Theta'' + \sin(\theta)\cos(\theta)\Theta'}{\Theta} + l(l+1)\sin^2(\theta) = \frac{\Phi''}{\Phi} + \frac{\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right)}{\Theta} + l(l+1)\sin^2(\theta) = 0.$$

Setting the Φ-summand to be equal to $-\mu$, we have our old friend

$$\Phi'' + \mu\Phi = 0$$

with periodic BCs; so the eigenvalues are $\mu = m^2$ for $m = 0, 1, 2, \ldots$, and the eigenfunctions are

$$\Phi_m(\phi) = A_m\cos(m\phi) + B_m\sin(m\phi).$$

So it remains to study the equation for Θ, which is of the form

$$\frac{\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right)}{\Theta} + l(l+1)\sin^2(\theta) - m^2 = 0.$$

After some minor algebraic manipulations this becomes

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2(\theta)}\right)\Theta = 0,$$

with the condition that Θ is finite at 0 and π.

To deal with this ODE we'll use a clever change of variable. For fixed values of ρ and ϕ, the parameter θ traces out a closed interval – a line of longitude starting at the north pole and ending at the south pole. So, let's take this line of longitude and identify it with a closed interval via the change of coordinate

$$s = \cos(\theta), \quad -1 \le s \le +1. \tag{17}$$

In these coordinates we have

$$\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - s^2,$$

and

$$\frac{d}{d\theta} = \frac{ds}{d\theta}\frac{d}{ds} = -\sin(\theta)\frac{d}{ds},$$

i.e.

$$\frac{d}{ds} = -\frac{1}{\sin(\theta)}\frac{d}{d\theta}.$$

So we have

$$\begin{aligned}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2(\theta)}\right)\Theta &= -\frac{d}{ds}\left(-\sin^2(\theta)\frac{d\Theta}{ds}\right) + \left(l(l+1) - \frac{m^2}{1-s^2}\right)\Theta \\ &= \frac{d}{ds}\left((1-s^2)\frac{d\Theta}{ds}\right) + \left(l(l+1) - \frac{m^2}{1-s^2}\right)\Theta.\end{aligned}$$

The ODE

$$\frac{d}{ds}\left((1-s^2)\frac{d\Theta}{ds}\right) + \left(l(l+1) - \frac{m^2}{1-s^2}\right)\Theta = 0 \tag{18}$$

is called the *associated Legendre equation*; we supplement it with the additional constraint the $\Theta(s)$ is finite at $s = \pm 1$.

In order for our solution to be nonsingular at ± 1, it turns out that the values of m are constrained to be $0 \le m \le l$. So in the end we will get a collection of solutions

$$P_l^m(s), \quad l \in \mathbb{Z}_{\ge 0}, \quad m \text{ integral, satisfying } |m| \le l. \tag{19}$$

Definition 2.1. For $m = 0$ the functions $P_l(s) := P_l^0(s)$ are called *Legendre polynomials*. For $m \neq 0$ the functions $P_l^m(s)$ are called *associated Legendre functions*.

We will not derive the solutions $P_l^m(s)$ – for a derivation see e.g. [Str08, Ch.10.6] (or use the Google machine). Instead we claim that the Legendre polynomials are given by *Rodrigues' Formula*

$$P_l(s) = \frac{1}{2^l l!}\frac{d^l}{ds^l}(s^2-1)^l, \tag{20}$$

where the normalisation factor ensures that $P_l(1) = 1$; the associated Legendre functions are then given by

$$P_l^m(s) = (-1)^m (1-s^2)^{\frac{m}{2}} \frac{d^m}{ds^m} P_l(s). \tag{21}$$

Remark 2.1. The $(-1)^m$ in this definition is called the *Condon-Shortley phase*. It is of course not required (since eigenfunctions are defined up to scale, and orthonormality is insensitive to phase), but depending on what application you need spherical harmonics for it can be convenient. Some references will omit it.

Let's now put all of this together. First, let's replace the eigenfunctions

$$\cos(m\phi), \sin(m\phi) \quad \text{by} \quad e^{im\phi};$$

the restriction on m now becomes $|m| \le l$, and we must write $P_l^{|m|}$ instead of P_l^m in our expression for spherical harmonics.

Second, let's reintroduce explicit θ-dependence to our functions be remembering that $s = \cos(\theta)$. The following result shows that the resulting functions are the analogue for the sphere of trigonometric functions on the circle:

Theorem 2.1. *The functions*

$$Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)e^{im\theta}, \qquad 0 \le l < \infty,\ |m| \le l, \tag{22}$$

are called spherical harmonics. *Spherical harmonics are orthogonal to each other with respect to the L^2-inner product on the sphere,*

$$\langle Y_l^m, Y_{l'}^{m'}\rangle = \int_{S^2} Y_l^m \overline{Y_{l'}^{m'}}\, dvol_{S^2} = \int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\phi)\overline{Y_{l'}^{m'}(\theta,\phi)}\sin(\theta)\, d\theta\, d\phi \propto \delta^{mm'}\delta_{ll'}. \tag{23}$$

Any square-integrable function on the sphere can be expanded as a series in the spherical harmonics,

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} f_m^l Y_l^m(\theta,\phi), \tag{24}$$

for a unique collection of coefficients $\{f_m^l\}$.

Example 1. To give an idea of what spherical harmonics look like, consider the first few examples (expressed in terms of trigonometic functions instead of exponentials, and up to unimportant constants factors):

l	**m**	$P_l^{\lvert m \rvert}$
0	0	1
1	0	$\cos(\theta) = \frac{z}{r}$
	± 1	$\sin(\theta)\cos(\phi) = \frac{x}{r}$
		$\sin(\theta)\sin(\phi) = \frac{y}{r}$
2	0	$3\cos^2(\theta) - 1 = \frac{3z^2 - r^2}{r^2}$
	± 1	$\sin(\theta)\cos(\theta)\cos(\phi) = \frac{xz}{r^2}$
		$\frac{yz}{r^2}$
	± 2	$\sin^2(\theta)\cos(2\phi) = \frac{x^2 - y^2}{r^2}$
		$\frac{xy}{r^2}$

(For a visualisation of these spherical harmonics, take a look at the Wikipedia page for spherical harmonics.)

2.2.1 Spherical harmonics and spherical symmetry

Let's finish up this lecture by making an observation that will be familiar to those of you who read the Mackey article I recommended at the start of the semester [Mac78]: we have discovered spherical harmonics via an exploitation of symmetry!

What does this actually mean? Consider the process by which we arrived at the formulae for spherical harmonics: we searched for functions Y_l^m which satisfied:

$$\Delta_{S^2} Y_l^m = -l(l+1) Y_l^m,$$
$$\frac{\partial}{\partial \phi} Y_l^m = -m^2 Y_l^m.$$

In other words, we were attempting to simultaneously solve two eigenvalue problems.

That's all well and good – but now we want to relate this to spherical symmetry. To do this, we reinterpret the differential operators of our eigenvalue problems as follows:[4]

- Recall the idea of an "integral curve" from our study of method of characteristics – this associated to a first order differential operator D a collection of curves along each of which D restricted to ordinary differentiation.
- So: what are the curves along which $\frac{\partial}{\partial \phi}$ restricts to ordinary differentiation? They are exactly the latitudes on the sphere; moving around one of these curves is exactly rotation around the z-axis.
- So we say: $\frac{\partial}{\partial \phi}$ *generates rotations around the z-axis.* Physicists might write $\frac{\partial}{\partial \phi} =: L_z$.
- Now, just looking at eigenfunctions for rotation around the z-axis is not sufficient to produce a basis – the eigenspaces for L_z are not all one-dimensional. So we need another operator that will produce a basis for each L_z-eigenspace.
- Recall a result from elementary linear algebra: in order to for two linear transformations A, B to have simultaneous eigenvectors, the transformations must *commute* with each other: $AB = BA$.
- We can use rotational symmetry to find another commuting operator!
  - Rotations about the x- and y-axes do **not** commute with rotation about the z-axis. So we can't take the "generators" of these rotations, L_x and L_y to be our second operator.

[4]If you have taken a quantum mechanics course, you may be familiar with this particular slight of hand.

- But we *can* achieve any rotation of the sphere via some combination of rotations around the x-, y-, and z-axes. So let's look for an expression which is a polynomial in these "symmetry generators"!
- It turns out that the only possibilites are a combination of (1) polynomials in L_z (boring – these won't give us anything new), and (2) polynomials in $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$. (Physicists might recognise this as the operator that measures the *total angular momentum* of a system).
- And it turns out (exercise at least for the physicists), $\vec{L}^2 = \Delta_{S^2}$.

This is the beginning of a very interesting story in *representation theory* that tells us how we can decompose the space of L^2-functions on a highly symmetric space (such as the sphere) using a collection of commuting operators that includes the Laplace operator.

References

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