APM 346 Lecture 18.

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We continue our study of the Laplace equation.

References being used: [IvrXX, §6.3] (§6.3) and [Str08, Ch.6].

1 Uniqueness of solutions to the 2d Laplace equation

Last lecture we proved the existence of a variety of solutions to the 2d Laplace equation. We will not be able to prove a general existence theorem in this class. We can, however, tackle the problem of uniqueness.

1.1 First go around: The Maximum Principle

As a first pass, we will prove a uniqueness result by making use of a maximum principle (recall that we did this previously for the heat equation). For harmonic functions, the statement is as follows:

Proposition 1.1 (Maximum Principle (Harmonic Functions))**.** *Let Ω be a connected, bounded, open*[1] *set in $\mathbb{R}^2$. Suppose that u is a harmonic function on Ω which is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$. Then,*

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x), \tag{1}$$

$$\min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x), \tag{2}$$

and if the maximum or minimum of u is attained at an interior point $x_* \in \Omega$ then u is constant.

Proof. Recall the statement of the mean value formula for a harmonic function, that we proved last time:

Suppose that u is a harmonic function on a domain Ω, and that $B_R(p) \subset \Omega$. Then the value of u at p is equal to the average of u over $S_R(p)$:

$$u(p) = \frac{1}{|S_R(p)|}\int_{S_R(p)} u\, dvol_{S_R(p)}. \tag{3}$$

Here we are using the notation

$$B_R(p) = \{\vec{x} \in \mathbb{R}^2 \,|\, |p - \vec{x}| \leq R\}, \tag{4}$$

$$S_R(p) = \partial D_R(p) = \{\vec{x} \in \mathbb{R}^2 \,|\, |p - \vec{x}| = R\}, \tag{5}$$

for the ball and circle of radius R centred at $p \in \mathbb{R}^2$, respectively.

[1] As topology is not a prerequisite to this class, students should consider “open” to mean “around every point in the set is an open ball fully contained in the set”.

It is sufficient to prove the claim involving the maximum. First, since u is assumed to be continuous is must attain its maximum, M, at *some* point $x_M \in \overline{\Omega}$.

Suppose that $x_M \in \Omega$. For some radius R we have the containment $B_R(x_M) \subset \Omega$, and so applying the mean value formula (3) we find

$$M = u(x_M) = \frac{1}{|S_R(x_M)|} \int_{x \in S_R(x_M)} u(x) d\text{vol}.$$

Since $S_R(x_M) \subset \Omega$ we have that for every $x \in S_R(x_M)$,

$$u(x) \leq M;$$

but in order for the average of the function to be equal to its maximum, the function must be constant! Hence, $u|_{S_R(x_M)}$ is constant. As the same argument could be applied to every radius $r < R$, this implies that $u|_{B_R(x_M)} \equiv M$ is constant.

But now: choose some other point $x'_M \in B_R(x_M)$ – which by the above is also a maximum for u – and repeat the argument above for a new disc centred at x'_M. We find that u must be constant on this new disc as well. In fact, since by assumption Ω is connected, given *any* point $x \in \Omega$ we can find a chain of discs connecting x to x_M, such that u is constant on every disc in the chain. Therefore u must be constant on the entire domain Ω. □

We can now use the maximum principle to prove unicity for the Dirichlet problem:

Theorem 1.2 (Unicity of Dirichlet problem)**.** *Let Ω be as above, and consider the Dirichlet problem*

$$\Delta u(x) = f(x), \qquad x \in \Omega, \tag{6}$$
$$u(x) = g(x). \qquad x \in \partial\Omega. \tag{7}$$

Suppose that u_1 and u_2 solve (6)–(7)*. Then $u_1 \equiv u_2$.*

Proof. Set $v := u_1 - u_2$. Then $\Delta v = 0$ on Ω, and $v|_{\partial\Omega} \equiv 0$. Since v must achieve its maximum and minimum on $\partial\Omega$ it must be identically zero. Hence, $u_1 \equiv u_2$. □

1.2 More general boundary conditions

Now, consider the problem

$$\Delta u - cu = 0, \qquad \text{in } \Omega, \tag{8}$$
$$u = 0, \qquad \text{on } \Gamma_-, \tag{9}$$
$$\frac{\partial u}{\partial \nu} - \alpha u = 0, \qquad \text{on } \Gamma_+, \tag{10}$$

where

- Ω is a connected bounded domain with smooth boundary $\partial\Omega$,
- $\partial\Omega = \Gamma_+ \cup \Gamma_-$, and $\Gamma_+ \cap \Gamma_- = \emptyset$,
- ν is an inward pointing unit normal vector field along $\partial\Omega$, and $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$,
- c and α are real-valued functions.

Then

$$\begin{aligned}0 &= -\int_\Omega u(\Delta u - cu)d\text{vol}_\Omega \\ &= \int_\Omega \left(|\nabla u|^2 + cu^2\right) + \int_{\partial\Omega} u\frac{\partial u}{\partial \nu} d\text{vol}_{\partial\Omega} \\ &= \int_\Omega \left(|\nabla u|^2 + cu^2\right) + \int_{\Gamma_+} \alpha u^2 d\text{vol}_{\partial\Omega}.\end{aligned}$$

Therefore, under the additional assumption that

$$c \geq 0 \qquad \text{and} \qquad \alpha \geq 0, \tag{11}$$

we may conclude that $\nabla u = 0$, and so u is constant. Moreover, provided

$$c \not\equiv 0 \qquad \text{or} \qquad \alpha \not\equiv 0 \qquad \text{or} \qquad \Gamma_- \neq \emptyset, \tag{12}$$

we may conclude that $u \equiv 0$.

Theorem 1.3 (Unicity of non-Neumann Laplace problem)**.** *With all notation as above, consider the problem*

$$\Delta u - cu = f, \qquad \textit{in } \Omega, \tag{13}$$

$$u = g, \qquad \textit{on } \Gamma_-, \tag{14}$$

$$\frac{\partial u}{\partial \nu} - \alpha u = h, \qquad \textit{on } \Gamma_+, \tag{15}$$

Suppose that (11) *and at least one of the conditions from* (12) *holds. Then if* u_1 *and* u_2 *are solutions to* (13)–(15)*,* $u_1 \equiv u_2$*.*

Proof. As in the previous proof, follows from linearity of the PDE and BCs – setting $v = u_1 - u_2$, v solves (8)–(10) subject to (11) and (12), and so $v \equiv 0$. □

Now, suppose that none of the conditions from (12) hold. Then the homogeneous problem we are considering is

$$\Delta u = 0, \qquad \text{in } \Omega, \tag{16}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad \text{on } \partial\Omega. \tag{17}$$

The arguments above tell us that the solutions to (16)–(17) are given by the constant functions $u \equiv C$ – however the problem does not specify a particular value of C.

So, let's consider the non-homogeneous version of this problem:

$$\Delta u = f, \qquad \text{in } \Omega, \tag{18}$$

$$\frac{\partial u}{\partial \nu} = h, \qquad \text{on } \partial\Omega. \tag{19}$$

Just as we have encountered before, we can see that there is an additional constraint on existence of solutions to this problem, since

$$\int_\Omega f\, d\text{vol}_\Omega = \int_\Omega \Delta u\, d\text{vol}_\Omega = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\text{vol}_{\partial\Omega}.$$

Theorem 1.4 (Unicity of Neumann problem up to constants)**.** *Assume that*

$$\int_\Omega f\, dvol_\Omega + \int_{\partial\Omega} h\, dvol_{\partial\Omega} = 0. \tag{20}$$

Then if u_1 *and* u_2 *are solutions to* (18)–(19)*,* $u_1 \equiv u_2 + C$ *for some constant* C*.*

Proof. Same arguments as before, with the boundary condition giving the form of the constraint (20). □

2 The Laplace equation in higher dimensions

Let's now turn our attention to the Laplace equation in higher dimensions, and see what we can learn (and in particular, what carries over from the $n = 2$ case). To start, recall the *divergence theorem*:

$$\int_\Omega \nabla \cdot \vec{V}\, d\mathrm{vol}_\Omega = -\int_{\partial\Omega} \vec{V} \cdot \nu\, d\mathrm{vol}_{\partial\Omega} \tag{21}$$

Here:

- Ω is a bounded domain with boundary $\partial\Omega$.
- ν is an inward-pointing unit normal vector field.
- $\vec{V}$ is a vector field with divergence $\nabla \cdot \vec{V}$.

Taking $\vec{V} = \nabla u$ gives:

$$\int_\Omega \Delta u\, d\mathrm{vol}_\Omega = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\mathrm{vol}_{\partial\Omega} \tag{22}$$

Let w be another scalar function, and set $\vec{V} = w\nabla u$ to get

$$I[w,u] := \int_\Omega (w\Delta u + \nabla u \cdot \nabla w)\, d\mathrm{vol}_\Omega = -\int_{\partial\Omega} w\frac{\partial u}{\partial \nu}\, d\mathrm{vol}_{\partial\Omega}. \tag{23}$$

Antisymmetrise (23) to get

$$I[w,u] - I[u,w] = \int_\Omega (w\Delta u - u\Delta w)\, d\mathrm{vol}_\Omega = \int_{\partial\Omega} \left(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\right) d\mathrm{vol}_{\partial\Omega} \tag{24}$$

Now, choose a point $y \in \Omega \setminus \partial\Omega$. If we are working in n-dimensions, define a function w by

$$w(x) := \begin{cases} |x-y|^{2-n}, & n \neq 2, \\ -\log|x-y|, & n = 2. \end{cases} \tag{25}$$

For the remainder of this section we will assume that $n \neq 2$ – the $n = 2$ statements are left as an exercise.

We cannot immediately apply (24) to the function w, since w is singular at $y \in \Omega$. So, excise a small ball of radius ϵ around y, $B_\epsilon(y) \subset \Omega$, and call the resulting domain $\Omega_\epsilon := \Omega \setminus B_\epsilon(y)$. Then $\partial\Omega_\epsilon = \partial\Omega \cup S_\epsilon(y)$, and we can apply (24) to get

$$\int_{\Omega_\epsilon} (u\Delta u - u\Delta w) d\mathrm{vol}_{\Omega_\epsilon} = \int_{\partial\Omega} \left(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\right) d\mathrm{vol}_{\partial\Omega} + \int_{S_\epsilon(y)} \left(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\right) d\mathrm{vol}_{S_\epsilon(y)} \tag{26}$$

Let's think about what happens to the integral over $S_\epsilon(y)$ as $\epsilon \to 0$. $\partial_\nu u$ is bounded by some finite constant M on $B_\epsilon(y)$, and $w|_{S_\epsilon(y)} = \epsilon^{2-n}$, so

$$\left|\int_{S_\epsilon(y)} w\frac{\partial u}{\partial \nu},\, d\mathrm{vol}\right| \leq \int_{S_\epsilon(y)} \left|w\frac{\partial u}{\partial \nu}\right| d\mathrm{vol} \leq C\epsilon^{2-n} \underbrace{\mathrm{vol}\,(S_\epsilon(y))}_{\sigma_n \epsilon^{n-1}} = C\sigma_n\epsilon,$$

where σ_n is the volume of the unit n-sphere.[2] So, as $\epsilon \to 0$, this term limits to zero.

[2]In particular, $\sigma_2 = 2\pi$ and $\sigma_3 = 4\pi$.

Since $\frac{\partial w}{\partial \nu}\big|_{S_\epsilon(y)} = (n-2)\epsilon^{1-n}$, the same argument does not work for the first term in this integral. Instead, write

$$\int_{S_\epsilon(y)} u\frac{\partial w}{\partial \nu},\ \mathrm{dvol} = \int_{S_\epsilon(y)} (u-u(y))\frac{\partial w}{\partial \nu},\ \mathrm{dvol} + u(y)\int_{S_\epsilon(y)} \frac{\partial w}{\partial \nu},\ \mathrm{dvol}.$$

We have

$$\left|\int_{S_\epsilon(y)} (u-u(y))\frac{\partial w}{\partial \nu},\ \mathrm{dvol}\right| \le (n-2)\epsilon^{1-n}\sigma_n\epsilon^{n-1}\max_{x\in S_\epsilon(y)}|u(x)-u(y)| = (n-2)\sigma_n\max_{x\in S_\epsilon(y)}|u(x)-u(y)|$$

which again tends to zero as $\epsilon \to 0$. So we are left with

$$\lim_{\epsilon\to 0}\int_{S_\epsilon(y)} u\frac{\partial w}{\partial \nu},\ \mathrm{dvol} = \lim_{\epsilon\to 0}\left(u(y)\int_{S_\epsilon(y)}\frac{\partial w}{\partial \nu}\mathrm{dvol}\right) = \lim_{\epsilon\to 0}\left((n-2)\epsilon^{1-n}\sigma_n\epsilon^{n-1}u(y)\right) = (n-2)\sigma_n u(y).$$

Taking the limit of (26) as $\epsilon \to 0$ then gives

$$\int_\Omega w\Delta u \mathrm{dvol}_\Omega = \int_{\partial\Omega}\left(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\right)\mathrm{dvol}_{\partial\Omega} + (n-2)\sigma_n u(y).$$

Theorem 2.1. *If Ω is a bounded domain with boundary $\partial\Omega$ and $y \in \Omega$ (in particular not on the boundary), then*

$$u(y) = \int_{x\in\Omega} G(x,y)\Delta u(x)\, dvol_\Omega + \int_{x\in\partial\Omega}\left(-u(x)\frac{\partial G}{\partial \nu_x}(x,y) + G(x,y)\frac{\partial u}{\partial \nu}(x)\right) dvol_{\partial\Omega} \tag{27}$$

where $G(x,y)$ is given by

$$G(x,y) = \begin{cases} -\frac{|x-y|^{2-n}}{(n-2)\sigma_n}, & n \neq 2, \\ \frac{1}{2\pi}\log|x-y|, & n = 2. \end{cases} \tag{28}$$

In particular we have

$$\frac{1}{2}|x-y| \qquad \text{and} \qquad -\frac{1}{4\pi|x-y|}$$

for $n = 1$ and $n = 3$ respectively.

Remark 2.1. How would one cook up the function $G(x,y)$ if it were not handed down to you from on high? As an exercise, you should prove that $G(x,y)$ satisfies the equation

$$\Delta_x G(x,y) = \delta(x-y),$$

where Δ_x is the Laplacian in the x-coordinate, and $\delta(x-y)$ is defined by the property that

$$\int_\Omega f(y)\delta(x-y) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

In particular, $\Delta_x G(x,y) = 0$ for $x \neq y$.

So, given the (overdetermined!) problem

$$\begin{aligned} \Delta u &= f, && \text{in } \Omega, \\ u &= g, && \text{on } \partial\Omega, \\ \partial_\nu u &= h, && \text{on } \partial\Omega, \end{aligned}$$

we can write

$$u(y) = \int_{x\in\Omega} G(x,y)f(x)\,\mathrm{dvol}_\Omega + \int_{x\in\partial\Omega} G(x,y)h(x)\mathrm{dvol}_{\partial\Omega} - \int_{x\in\partial\Omega} g(x)\frac{\partial G}{\partial \nu_x}(x,y)\mathrm{dvol}_{\partial\Omega}. \tag{29}$$

Remark 2.2. If we let $\Omega \to \mathbb{R}^n$ and assume that f decays fast enough at infinity that the boundary terms in (29) vanish, we obtain a solution to the Poisson equation $\Delta u = f$ as

$$u(y) = \int_{\mathbb{R}^n} G(x,y)\Delta u(x)dx_1\cdots dx_n.$$

For a more precise statement see [IvrXX, §7.2].

2.1 Mean-value and maximum principles

Suppose that u is a harmonic function on $B_r(y)$.

Theorem 2.2 (Mean-value Theorem). *$u(y)$ is given by the average of u over $S_r(y)$ and $B_r(y)$:*

$$u(y) = \frac{1}{vol(S_r(y))}\int_{S_r(y)} u(x)dx = \frac{1}{vol(B_r(y))}\int_{B_r(y)} u(x)dx. \tag{30}$$

Proof. The claim for $B_r(y)$ follows from the claim for $S_r(y)$, since

$$\begin{aligned}\int_{B_r(y)} u(x)dx &= \int_0^r \left(\int_{S_\rho(y)} u d\mathrm{vol}_{S_\rho(y)}\right) d\rho \\ &= u(y)\mathrm{vol}(S_1)\int_0^r \rho^{n-1} d\rho = u(y)\mathrm{vol}(B_r(y)).\end{aligned}$$

To prove the claim for $S_r(y)$, consider (27):

$$u(y) = \int_{B_r(y)} G(x,y)\underbrace{\Delta u(x)}_{=0}\, dx + \int_{S_r(y)} \underbrace{G(x,y)}_{\text{constant on } S_r(y)} \frac{\partial u}{\partial \nu}(x)dx - \int_{S_r(y)} u(x) \underbrace{\frac{\partial G}{\partial \nu_x}(x,y)}_{\text{constant on } S_r(y)}\, dx$$

Then the claim follows since by construction $\frac{\partial G}{\partial \nu} \equiv \frac{1}{\mathrm{vol}(S_r(y))}$ on $S_r(y)$, and

$$\int_{S_r(y)} \frac{\partial u}{\partial \nu}(x)dx = -\int_{B_r(y)} \Delta u(x)dx = 0.$$

□

The proof we gave for the maximum principle in 2d using the mean-value theorem carries over to n-dimensions, so we have:

Theorem 2.3 (Maximum and minimum principles). *If u is a harmonic function on the bounded domain Ω then*

$$\max_{\Omega} u = \max_{\partial\Omega} u \quad \text{and} \quad \min_{\Omega} u = \min_{\partial\Omega} u.$$

Moreover, if Ω is connected and u attains its maximum or minimum at some interior point of Ω, then u is constant.

Finally, just as before, we can use the max/min principle to prove a uniqueness result for the Dirichlet problem.

Theorem 2.4 (Unicity for Dirichlet problem). *If Ω is a bounded domain, the solution to the Dirichlet problem for the Laplace equation in Ω is unique. If Ω is unbounded, we obtain uniqueness after specifying the extra condition $|u| \to 0$ as $|x| \to \infty$.*

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.