APM 346 Lecture 17.

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Having seen how to express the Laplace operator in different coordinate systems, let's now study the Laplace equation in some non-rectangular domains.

References being used: [IvrXX, §6.4-5] (§6.4,§6.5) and [Str08, Ch.6].

1 The Laplace equation in polar coordinates

Recall that last lecture we saw that in polar coordinates

$$x = r\cos(\theta) \tag{1}$$
$$y = r\sin(\theta) \tag{2}$$

the Laplace operator takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{3}$$

Remaining agnostic for the moment about the particular domain and boundary conditions we wish to deal with, let's proceed with the method of separation of variables for the Laplace equation in polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{4}$$

Set

$$u(r,\theta) = R(r)\Theta(\theta) \tag{5}$$

and substitute into (4) to obtain

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Divide this expression by $\frac{R\Theta}{r^2}$ to obtain the separated equation:

$$\underbrace{\frac{r^2R'' + rR'}{R}}_{=\lambda} + \underbrace{\frac{\Theta''}{\Theta}}_{=-\lambda} = 0$$

Example 1. Let's begin by determining the solutions to (4) which are θ-independent (so that they only depend on the radial coordinate). Our problem collapses to an ODE:

$$R''(r) + \frac{1}{r}R'(r) = 0$$

We can rearrange this to give

$$\frac{d}{dr}\left(\log(R')\right) = -\frac{1}{r},$$

which integrates to

$$\log(R') = -\log(r) + \log(B),$$

i.e.

$$R'(r) = \frac{B}{r}.$$

We can immediately integrate this equation as well, to arrive at the solution

$$R(r) = A + B\log(r),$$

where A, B are constants.

1.1 Rotationally symmetric domains

Now, let's make our first assumption: suppose θ is a periodic variable ($0 \le \theta \le 2\pi$, so that it literally parametrises one turn around a circle) – so that the function Θ satisfies periodic boundary conditions. Putting this all together, we obtain:

$$r^2R'' + rR' - \lambda R = 0, \tag{6}$$

$$\Theta'' + \lambda\Theta = 0, \tag{7}$$

$$\Theta(0) = \Theta(2\pi), \tag{8}$$

$$\Theta'(0) = \Theta'(2\pi). \tag{9}$$

Now, (7) with periodic BCs (8) and (9) is an eigenvalue problem that we have solved before. The solutions were given by:

$$\text{Eigenvalues}\begin{cases} \lambda_0 &= 0, \\ \lambda_n &= n^2, \end{cases} \tag{10}$$

$$\text{Eigenfunctions}\begin{cases} \Theta_0 &= \frac{1}{2}, \\ \Theta_{c,n} &= \cos(n\theta), \\ \Theta_{s,n} &= \sin(n\theta), \end{cases} \tag{11}$$

where $n = 1, 2, 3, \ldots$.

Now, let's return to the function R. (6) is an *Euler equation*, and so we are looking for solutions of the form $R(r) = r^m$. Differentiating this proposed solution and substituting it into (6) gives the equation

$$m(m-1) + m - \lambda = 0$$

which is solved by $m^2 = \lambda$. Since $\lambda_n = n^2$, this is solved by $m = \pm n$, and so we find that

$$R_n(r) = Ar^n + Br^{-n}, \quad n \neq 0, \tag{12}$$

while for $n = 0$ we have

$$R_0(r) = A + B\log(r). \tag{13}$$

Remark 1.1. Note that R_0 is exactly the θ-independent solution we found in Example 1.

So, the general solution to the Laplace equation on a domain with complete rotational symmetry (a disk, an annulus, the exterior of a disk) is:

$$u(r,\theta) = \frac{1}{2}\left(A_0 + B_0\log(r)\right) + \sum_{n=1}^{\infty}\left(\left(A_nr^n + B_nr^{-n}\right)\cos(n\theta) + \left(C_nr^n + D_nr^{-n}\right)\sin(n\theta)\right). \tag{14}$$

This is as far as we can go without being more specific about the domain and bonudary conditions of our solution. For rotationally symmetric domains, we have the following possibilities:

- **The disc ($r < a$).** If our domain is a disc, we need to specify a boundary condition at $r = a$ and discard terms which are singular as $r \to 0$:

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + C_n \sin(n\theta)\right). \tag{15}$$

- **The disc exterior ($r > a$).** If our domain is the exterior of a disc, we need to specify boundary conditions at $r = a$ and impose the boundedness condition $\max |u| < \infty$ (so discard terms which are singular as $r \to \infty$):

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n} \left(B_n \cos(n\theta) + D_n \sin(n\theta)\right). \tag{16}$$

- **The annulus ($a < r < b$).** If our domain is an annulus, we need to specify boundary conditions at both $r = a$ and $r = b$. There are no boundedness/singularity considerations, and so we do not discard any terms from (14).

1.2 Dirichlet BCs on the disc (Poisson formula)

Now, suppose that we consider Dirichlet boundary conditions for the Laplace equation on a disc:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad r < a, \tag{17}$$

$$u(a,\theta) = g(\theta), \qquad r = a. \tag{18}$$

Subsituting (18) into the general disc solution (15) we find

$$g(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos(n\theta) + C_n \sin(n\theta)\right). \tag{19}$$

Using our old friendly formulae for the Fourier coefficients, we find

$$A_n = \frac{1}{\pi}a^{-n}\int_0^{2\pi} g(\theta')\cos(n\theta')d\theta', \tag{20}$$

$$C_n = \frac{1}{\pi}a^{-n}\int_0^{2\pi} g(\theta')\sin(n\theta')d\theta'. \tag{21}$$

Now, something excellent happens – we are able to explicitly sum up our series solution! Let's see how this works. Substituting the Fourier coefficients back into the general form of the solution gives

$$u(r,\theta) = \int_0^{2\pi} G(r,\theta,\theta')g(\theta')d\theta'$$

where

$$\begin{aligned}
G(r,\theta,\theta') &:= \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} r^n a^{-n}\left(\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\right)\right) \\
&= \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} r^n a^{-n}\cos(n(\theta-\theta'))\right) \\
&= \frac{1}{2\pi}\left(1 + 2\mathrm{Re}\sum_{n=1}^{\infty}\left(ra^{-1}e^{i(\theta-\theta')}\right)^n\right) \\
&= \frac{1}{2\pi}\left(1 + 2\mathrm{Re}\left(\frac{ra^{-1}e^{i(\theta-\theta')}}{1 - ra^{-1}e^{i(\theta-\theta')}}\right)\right).
\end{aligned}$$

In the last line we are allowed to use the familiar expression for the geometric series since

$$\left|ra^{-1}e^{i(\theta-\theta')}\right| = \frac{r}{a} < 1$$

as our domain is the disc $r < a$.

We want to put this into a more useful form. Let's clear inverses and eliminate the imaginary component of the denominator:

$$\begin{aligned}
\mathrm{Re}\left(\frac{a(a-re^{-i(\theta-\theta')})}{a(a-re^{-i(\theta-\theta')})}\frac{ra^{-1}e^{i(\theta-\theta')}}{1-ra^{-1}e^{i(\theta-\theta')}}\right) &= \mathrm{Re}\left(\frac{a-re^{-i(\theta-\theta')}}{a-re^{-i(\theta-\theta')}}\frac{re^{i(\theta-\theta')}}{a-re^{i(\theta-\theta')}}\right)\\
&= \mathrm{Re}\left(\frac{are^{i(\theta-\theta')}-r^2}{a^2-ar(e^{i(\theta-\theta')}+e^{-i(\theta-\theta')})+r^2}\right)\\
&= \mathrm{Re}\left(\frac{are^{i(\theta-\theta')}-r^2}{a^2-2ar\cos(\theta-\theta')+r^2}\right)\\
&= \frac{ar\cos(\theta-\theta')-r^2}{a^2-2ar\cos(\theta-\theta')+r^2}
\end{aligned}$$

Let's substitute this expression back into our formula for G:

$$\begin{aligned}
G(r,\theta,\theta') &= \frac{1}{2\pi}\left(1+2\frac{ar\cos(\theta-\theta')-r^2}{a^2-2ar\cos(\theta-\theta')+r^2}\right)\\
&= \frac{1}{2\pi}\left(\frac{a^2-2ar\cos(\theta-\theta')+r^2+2ar\cos(\theta-\theta')-2r^2}{a^2-2ar\cos(\theta-\theta')+r^2}\right)\\
&= \frac{1}{2\pi}\frac{a^2-r^2}{a^2-2ar\cos(\theta-\theta')+r^2}
\end{aligned}$$

So we have found a fairly nice integral form for the solution to (17)–(18). We can repackage this information, however. The following definition was mentioned way back at the beginning of the course:

Definition 1.1. If u satisfies the Laplace equation $\Delta u = 0$, we call u a *harmonic function.*

Putting this altogether we have:

Theorem 1.1 (Poisson Formula for Harmonic Functions (2d))**.** *If u is a harmonic function on the disc of radius a centred at the origin of $\mathbb{R}^2$, its values in the interior of the disc may be recovered from its values on the boundary via the formula*

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{u(a,\theta')}{a^2-2ar\cos(\theta-\theta')+r^2}d\theta'. \tag{22}$$

Now Theorem 1.1 has a nice, immediate consequence: suppose that we want to know the value of our harmonic function u at the origin. Then plugging $r = 0$ into (22) we obtain

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(a,\theta)d\theta, \tag{23}$$

i.e. the value of u at the origin is the average of its values over the boundary circle $\{r = a\}$.

We can do even better than this, however: writing the Laplace operator in cartesian coordinates

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

we can see that the form of the operator is unchanged under translations

$$\vec{x} \to \vec{x} + \vec{v}$$

for a constant vector $\vec{v}$. The practical effect of this for polar coordinates is as follows: we could have chosen *any* point in $\mathbb{R}^2$ to be the origin of our polar coordinate system (where $r = 0$)!

So, let p be a point in $\mathbb{R}^2$ and write

$$D_R(p) = \{\vec{x} \in \mathbb{R}^2 \,|\, |p - \vec{x}| \leq R\}, \tag{24}$$

$$S_R(p) = \partial D_R(p) = \{\vec{x} \in \mathbb{R}^2 \,|\, |p - \vec{x}| = R\}, \tag{25}$$

for the disc and circle of radius R centred at p, respectively.

Corollary 1.2 (Mean Value Formula)**.** *Suppose that u is a harmonic function on a domain Ω, and that $D_R(p) \subset \Omega$. Then the value of u at p is equal to the average of u over $S_R(p)$:*

$$u(p) = \frac{1}{|S_R(p)|} \int_{S_R(p)} u \, dvol_{S_R(p)}. \tag{26}$$

Remark 1.2. In [IvrXX, §6.4] a similar formula is given for bounded harmonic functions of the *exterior* of a disc, $r > a$. You should prove it as an exercise!

1.3 Neumann BCs on the disc

Next, let's consider Neumann boundary conditions for the Laplace equation on a disc:

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, & r < a, \\ u_r(a,\theta) &= h(\theta), & r = a. \end{aligned} \tag{27}$$

In polar coordinates we saw that we could write a harmonic function on the disc as

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + C_n \sin(n\theta)\right),$$

so differentiating with respect to r and setting $r = a$, we find that

$$h(\theta) = \sum_{n=1}^{\infty} n a^{n-1} \left(A_n \cos(n\theta) + C_n \sin(n\theta)\right). \tag{28}$$

But now we encounter two problems!

(I) There is a vanishing constraint on the constant term in the Fourier expansion of h, i.e.

$$\int_0^{2\pi} h(\theta) d\theta = 0. \tag{29}$$

(II) There is no condition that would allow us to determine A_0, and so our solution (if it exists) will be defined only up to a constant.

We have encountered problems like (I) before, and we can solve it in the same way as we did previously:

Suppose that, rather than the Laplace equation, we were studying the homogeneous *heat* equation on the disc, with inhomogeneous Neumann BCs. Well, we might try and solve this by breaking it up into a stationary problem and a homogeneous problem.

As before, the stationary problem will be the Laplace equation with Neumann BCs.

As before, the function $h(\theta)$ is specifying the rate at which heat is entering or leaving the disc through the boundary.

And so – as before – we see that a stationary solution can only exist if the net heat flow through the boundary of the disc vanishes! For otherwise there would always be an accumulation (or exodus) of heat in (or from) the disc, making existence of a stationary solution impossible.

So to solve problem (I) we will simply assume that our function $h(\theta)$ satisfies (29) (but now we can feel somewhat justified in making that assumption).

To deal with problem (II), we will impose the following normalisation condition on our solution:

$$\iint_{D_a(0)} u(x,y)dxdy = 0. \tag{30}$$

Since

$$\iint u = \pi a^2 A_0$$

this fixes the value of A_0 to be zero. The other Fourier coefficients are now given by the standard formulae

$$\begin{aligned} A_n &= \frac{1}{\pi n} a^{1-n} \int_0^{2\pi} h(\theta)\cos(n\theta)d\theta, \\ C_n &= \frac{1}{\pi n} a^{1-n} \int_0^{2\pi} h(\theta)\sin(n\theta)d\theta. \end{aligned}$$

With these Fourier coefficients, we can now express the solution u as

$$u(r,\theta) = \int_0^{2\pi} G(r,\theta,\theta')h(\theta')d\theta', \tag{31}$$

where

$$\begin{aligned} G(r,\theta,\theta') &:= \frac{1}{\pi}\left(\sum_{n=1}^{\infty} \frac{1}{n} r^n a^{1-n} \left(\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\right)\right) \\ &= \frac{1}{\pi}\left(\sum_{n=1}^{\infty} \frac{1}{n} r^n a^{1-n} \cos(n(\theta-\theta'))\right) \\ &= \frac{a}{\pi}\mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}e^{i(\theta-\theta')}\right)^n\right) \\ &= -\frac{a}{\pi}\mathrm{Re}\log\left(1-\frac{r}{a}e^{i(\theta-\theta')}\right) \end{aligned}$$

Here we have used the series expansion $\sum_{n>0} \frac{1}{n} z^n = -\log(1-z)$ for $|z|<1$. Since

$$\mathrm{Re}\log(z) = \log(|z|) = \frac{1}{2}\log(|z|^2),$$

we can rewrite this last expression as

$$\begin{aligned} G(r,\theta,\theta') &= -\frac{a}{2\pi}\log\left((1-ra^{-1}e^{i(\theta-\theta')})(1-ra^{-1}e^{-i(\theta-\theta')})\right) \\ &= -\frac{a}{2\pi}\log\left(\frac{a^2-2ar\cos(\theta-\theta')+r^2}{a^2}\right). \end{aligned}$$

Putting all this together, our solution is given by:

$$u(r,\theta) = -\frac{a}{2\pi}\int_0^{2\pi} \log\left(\frac{a^2 - 2ar\cos(\theta-\theta') + r^2}{a^2}\right)\frac{\partial u}{\partial r}(a,\theta')d\theta'. \tag{32}$$

Exercise: Derive a similar formula for the exterior of the disc (think about what extra constraints you now need).

1.4 Laplace equation in a sector

Up until now we have assumed that our angular coordinate sweeps out a full circle. Let's now consider what happens when our angular coordinate only sweeps out some partial sector.

Specifically, let's consider the following problem:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad r < a,\ 0 < \theta < \alpha \tag{33}$$

$$u(r,0) = u(r,\alpha) = 0, \qquad 0 < r < a \tag{34}$$

I.e. we are solving the Laplace equation in the sector with homogeneous Dirichlet BCs along the radial components of the boundary. We would also have to supplement this with an appropriate BC at $r = a$.

Separation of variables as before leads us to the problem:

$$\begin{aligned} r^2R'' + rR' - \lambda R &= 0, \\ \Theta'' + \lambda\Theta &= 0, \\ \Theta(0) = \Theta(\alpha) &= 0. \end{aligned} \tag{35}$$

The solutions to the eigenvalue problem are

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{\alpha}\right)^2, \\ \Theta_n(\theta) &= \sin\left(\frac{n\pi\theta}{\alpha}\right), \end{aligned}$$

where $n = 1, 2, \ldots$. $\lambda = 0$ is no longer a possibility, so the solutions for R are given by

$$R_n(r) = A_n r^{\frac{\pi n}{\alpha}} + B_n r^{-\frac{\pi n}{\alpha}}.$$

The general form of the solution is therefore:

$$u(r,\theta) = \sum_{n=1}^{\infty}\left(A_n r^{\frac{n\pi}{\alpha}} + B_n r^{-\frac{n\pi}{\alpha}}\right)\sin\left(\frac{n\pi\theta}{\alpha}\right). \tag{36}$$

Now:

- On the wedge $0 \le r < a$ we should set all $B_n = 0$ (non-singular solution at 0).
- On the exterior $a < r < \infty$ we should set all $A_n = 0$ (bounded; non-singular solution at ∞).
- On the sector $a < r < b$ (where $a > 0$, $b < \infty$) we cannot automatically set any coefficients to zero.

The rest of the problem proceeds as in a usual separation of variables problem (i.e. find the Fourier coefficients) – unfortunately, we will not be able to explicitly sum our Fourier series as we could in Sections 1.2 and 1.3.

Example 2. Let's take a look at a basic example, just to see how this works. Suppose we are looking at the sector $0 \leq r < a$, $0 < \theta < \frac{\pi}{4}$. Then before applying a BC at $r = a$ we have

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{4n} \sin(4n\theta).$$

Now suppose that we have the simple Dirichlet BC at $r = a$ given by

$$u(a,\theta) = \sin(8\theta).$$

Then $A_n = 0$ for $n \neq 2$ and for $n = 2$ we have $A_2 = a^{-8}$, so in the end our solution is just

$$u(r,\theta) = \left(\frac{r}{a}\right)^8 \sin(8\theta).$$

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.