APM 346 Lecture 16.

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We will soon encounter problems that will have symmetries we can exploit to find the solution (e.g. spherical symmetry). To prepare for this, let's spend this lecture considering how one would express the Laplace operator in different coordinate systems.

References being used: [IvrXX, §6.3] (§6.3) and [Str08, Ch.6.1].

1 The Laplace operator in different coordinates

Recall that we defined the Laplace operator on $\mathbb{R}^n$ by:

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \tag{1}$$

1.1 Polar coordinates

Suppose that we want to write the 2d Laplace operator in polar coordinates

$$x = r\cos(\theta) \tag{2}$$
$$y = r\sin(\theta) \tag{3}$$

To find the derivatives r_x, r_y, etc., we *could* attempt to invert this coordinate change – up to choosing a branch of arctan, we would get

$$r = \sqrt{x^2 + y^2} \tag{4}$$
$$\theta = \arctan\left(\frac{y}{x}\right) \tag{5}$$

Rather than differentiating (4) and (5), let's differentiate (2) and (3) with respect to x and y. We get the equations

$$\begin{aligned}
1 &= \frac{\partial r}{\partial x}\cos(\theta) - r\sin(\theta)\frac{\partial \theta}{\partial x} \\
1 &= \frac{\partial r}{\partial y}\sin(\theta) + r\cos(\theta)\frac{\partial \theta}{\partial y} \\
0 &= \frac{\partial r}{\partial y}\cos(\theta) - r\sin(\theta)\frac{\partial \theta}{\partial y} \\
0 &= \frac{\partial r}{\partial x}\sin(\theta) + r\cos(\theta)\frac{\partial \theta}{\partial x}
\end{aligned}$$

Multiply the first and third lines by $\cos(\theta)$ and the second and fourth lines by $\sin(\theta)$ to get

$$\begin{aligned}
\cos(\theta) &= \cos^2(\theta)\frac{\partial r}{\partial x} - r\sin(\theta)\cos(\theta)\frac{\partial \theta}{\partial x}\\
\sin(\theta) &= \sin^2(\theta)\frac{\partial r}{\partial y} + r\sin(\theta)\cos(\theta)\frac{\partial \theta}{\partial y}\\
0 &= \cos^2(\theta)\frac{\partial r}{\partial y} - r\sin(\theta)\cos(\theta)\frac{\partial \theta}{\partial y}\\
0 &= \sin^2(\theta)\frac{\partial r}{\partial x} + r\sin(\theta)\cos(\theta)\frac{\partial \theta}{\partial x}
\end{aligned}$$

From here it is an easy exercise to check that

$$\frac{\partial r}{\partial x} = \cos(\theta) \tag{6}$$

$$\frac{\partial r}{\partial y} = \sin(\theta) \tag{7}$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin(\theta)}{r} \tag{8}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos(\theta)}{r} \tag{9}$$

Now, by the chain rule we have

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta},\\
\frac{\partial}{\partial y} &= \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}.
\end{aligned}$$

So (calculations as exercise) the Laplacian in polar coordinates is:

$$\Delta = \left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)^2 + \left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{10}$$

1.1.1 Cylindrical coordinates

In 3d one sometimes encounters problems which have a cylindrical symmetry. For such a problem it is usually prudent to change to cylindrical coordinates:

$$x = r\cos(\theta) \tag{11}$$

$$y = r\sin(\theta) \tag{12}$$

$$z = z \tag{13}$$

Since the z-coordinate remains unchanged, and $(x, y) \to (r, \theta)$ as above in Section 1.1, the Laplacian in cylindrical coordinates is simply

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{14}$$

1.2 Spherical coordinates

Now suppose that our problem exhibits a 3d spherical symmetry. Then a natural choice of coordinates are the spherical coordinates

$$x = \rho\sin(\theta)\cos(\phi) \tag{15}$$

$$y = \rho\sin(\theta)\sin(\phi) \tag{16}$$

$$z = \rho\cos(\theta) \tag{17}$$

Geometrically:

- ρ measures the radial distance from the origin, $0 \leq \rho < \infty$.
- ϕ measures the longitude, $0 \leq \phi \leq 2\pi$.
- θ measures the latitude, $0 \leq \theta \leq \pi$.

Again, up to choosing a branch of arctan, these may be inverted by the formulae:

$$\rho = \sqrt{x^2 + y^2 + z^2} \tag{18}$$

$$\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \tag{19}$$

$$\phi = \arctan\left(\frac{y}{x}\right) \tag{20}$$

These look much less friendly than before, and while we *could* crunch through the calculation it would be messy (and not a great use of class time). Instead, let's be a little creative.

The first observation is that spherical coordinates are *orthogonal*:

- For fixed ϕ, θ we get rays from the origin, which are orthogonal to the spheres we get for each value of r.
- For fixed r, the coordinates ϕ and θ give us meridians (lines of longitude) and θ and parallels (lines of latitude) respectively. These are also orthogonal.

What practical application does this observation have? Suppose we want to calculate the length of an infinitesimal line segment ds. In (orthogonal) cartesian coordinates, this is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

If we instead express this in terms of spherical coordinates then – since they are orthogonal – we will not pick up any cross terms:

$$ds^2 = Ad\rho^2 + Bd\theta^2 + Cd\phi^2$$

It only remains to determine A, B, C – we will not do this in class (though feel free to come talk with me about it during office hours) – to obtain the results:

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2(\theta) d\phi^2. \tag{21}$$

The volume element that we integrate against is therefore[1]

$$\text{dvol} = dxdydz = \rho^2 \sin(\theta) d\rho d\theta d\phi. \tag{22}$$

Next, we want to understand how the *gradient* of a function, ∇u, is expressed in different coordinate systems. Let $d\vec{s} = (dx, dy, dz)$; then ∇u in cartesian coordinates is determined by the property that

$$du = \nabla u \cdot d\vec{s}.$$

The LHS is invariant under changes in coordinates. So, suppose that we change coordinates via some invertible matrix Q,

$$d\vec{s}' = Qd\vec{s}.$$

[1] As an exercise, you should be able to calculate the prefactor here as the absolute value of the Jacobian for the change from cartesian to spherical coordinates.

Write u' for u expressed in the new coordinate system. Then

$$du = \nabla u \cdot d\vec{s} = \nabla u' \cdot d\vec{s}' = \nabla u' \cdot (Q d\vec{s}) = (Q^T \nabla u') \cdot d\vec{s},$$

where Q^T is the transpose of Q, and so

$$\nabla u' = (Q^T)^{-1} \nabla u.$$

So in spherical coordinates we have that

$$\nabla u = \left(\frac{\partial u}{\partial \rho}, \frac{1}{\rho}\frac{\partial u}{\partial \theta}, \frac{1}{\rho \sin(\theta)} \frac{\partial u}{\partial \phi} \right). \tag{23}$$

Now the real fun begins. Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$, and let v be an appropriately differentiable function on Ω that vanishes near $\partial\Omega$. Consider the integral identity

$$\iiint_\Omega \Delta u v d\text{vol} = -\iiint_\Omega \nabla u \cdot \nabla v d\text{vol} \tag{24}$$

(e.g. express in Cartesian coordinates and use integration by parts). Expressing both sides of this equality in terms of spherical coordinates, and integrating by parts using the spherical coordinates, we get

$$\begin{aligned}\iiint_\Omega \Delta u v \rho^2 \sin(\theta) d\rho d\theta d\phi &= -\iiint_\Omega \left(u_\rho v \rho + \frac{1}{\rho^2} u_\theta v_\theta + \frac{1}{\rho^2 \sin^2(\theta)} u_\phi v_\phi \right) \rho^2 \sin(\theta) d\rho d\theta d\phi \\ &= \iiint_\Omega \left(\left(\rho^2 \sin(\theta) u_\rho\right)_\rho + (\sin(\theta) u_\theta)_\theta + \left(\frac{1}{\sin(\theta)} u_\phi \right)_\phi \right) v d\rho d\theta d\phi.\end{aligned}$$

But now we use a trick (that we have seen before): this integral identity holds on *any* domain Ω and for *any* appropriate function v.[2] So the terms in the integrand multiplying v must be equal:

$$\rho^2 \sin(\theta) \Delta u = \left(\rho^2 \sin(\theta) u_\rho\right)_\rho + (\sin(\theta) u_\theta)_\theta + \left(\frac{1}{\sin(\theta)} u_\phi \right)_\phi,$$

so that

$$\Delta u = \frac{1}{\rho^2 \sin(\theta)} \left(\left(\rho^2 \sin(\theta) u_\rho\right)_\rho + (\sin(\theta) u_\theta)_\theta + \left(\frac{1}{\sin(\theta)} u_\phi \right)_\phi \right).$$

As a short exercise, expand this out to obtain the final result:

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot(\theta)\frac{\partial}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}. \tag{25}$$

1.2.1 Laplace operator on the sphere

Now, suppose that we want to study the Laplace operator not on a domain in $\mathbb{R}^2$, but on a sphere. E.g. we might be interested in vibrations of a spherical membrane, in which case we would study the wave equation on the sphere. We can get an equation for this from the work we have already done.

For concreteness let's work with the unit sphere (other radii R will differ by a scaling factor of $\frac{1}{R^2}$). Then in (25) we can set $\rho \equiv 1$, and $\frac{\partial}{\partial \rho} \equiv 0$ (since the radial coordinate is no longer able to change), to obtain the *spherical Laplacian*

$$\Delta_{S^2} \equiv \Lambda := \frac{\partial^2}{\partial \theta^2} + \cot(\theta)\frac{\partial}{\partial \theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2} \tag{26}$$

[2] Eliding questions of in what sense the domain and function should be "nice", which are not important to the result.

1.3 Conformal coordinates on $\mathbb{R}^2$

Finally, let's briefly mention two more coordinate systems: elliptic and parabolic coordinates on $\mathbb{R}^2$. These coordinate systems have a particularly nice property: the new coordinates σ, τ are related to cartesian coordinates by a *conformal transformation*, which means that

$$ds^2 = f(\sigma, \tau)(d\sigma^2 + d\tau^2). \tag{27}$$

From the arguments given in Section 1.2 it is a quick exercise to check that this means that in these coordinates the Laplacian is given by

$$\Delta = \frac{1}{f(\sigma, \tau)} \left(\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial \tau^2} \right). \tag{28}$$

1.3.1 Elliptic coordinates

Elliptic coordinates on $\mathbb{R}^2$ are defined by

$$x = c \cosh(\sigma) \cos(\tau) \tag{29}$$
$$y = c \sinh(\sigma) \sin(\tau) \tag{30}$$

Lines of constant σ yield ellipses, and lines of constant τ yield hyperbolae, both with foci at $(-c, 0)$ and $(c, 0)$. The length of an infinitesimal segment is given by

$$ds^2 = c^2 \left(\sinh^2(\sigma) + \sin^2(\tau) \right) \left(d\sigma^2 + d\tau^2 \right) \tag{31}$$

and so

$$\Delta = \frac{\partial_\sigma^2 + \partial_\tau^2}{c^2(\sinh^2(\sigma) + \sin^2(\tau))}. \tag{32}$$

1.3.2 Parabolic coordinates

Parabolic coordinates on $\mathbb{R}^2$ are defined by

$$x = \sigma\tau \tag{33}$$
$$y = \frac{1}{2}(\sigma^2 - \tau^2) \tag{34}$$

Lines of constant σ and lines of constant τ both yield confocal parabolae. The length of an infinitesimal segment is given by

$$ds^2 = \left(\sigma^2 + \tau^2 \right) \left(d\sigma^2 + d\tau^2 \right) \tag{35}$$

and so

$$\Delta = \frac{\partial_\sigma^2 + \partial_\tau^2}{\sigma^2 + \tau^2}. \tag{36}$$

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.