APM 346 Lecture 15.

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Right! After that extended detour on Fourier theory, it's time to return to the problem that got us thinking about all this in the first place – the method of separation of variables.

References being used: [IvrXX, §6.1-2] (§6.1, §6.2) and [Str08, Ch.4,Ch.6.2].

1 Return to separation of variables

1.1 1d Heat equation

Let's revisit (again!) the 1d heat equation – but now on a bounded domain. We could choose any of the boundary conditions we considered previously (Dirichlet, Neumann, Robin, periodic, etc.); for concreteness let's study what happens for Dirichlet BCs:

$$\begin{aligned} u_t &= k u_{xx}, \qquad t > 0, \quad 0 < x < l, \\ u(0,t) &= u(l,t) = 0, \\ u(x,0) &= g(x). \end{aligned} \tag{1}$$

We look for a separated solution, i.e. one of the form

$$u(x,t) = X(x)T(t).$$

Plugging this into the PDE from (1) gives

$$X(x)T'(t) = kX''(x)T(t),$$

and dividing through by XT gives

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)}.$$

Since the LHS is purely a function of t and the RHS is purely a function of x, both sides must in fact be constant. Setting $\frac{X''}{X} = -\lambda$ we obtain:

$$X'' + \lambda X = 0, \tag{2}$$

$$T' + k\lambda T = 0, \tag{3}$$

Substitute the separated solution form into the Dirichlet BCs of (1) to get

$$X(0)T(t) = X(l)T(t) = 0,$$

which since T is not identically zero (we want to find a nontrivial solution) implies that

$$X(0) = X(l) = 0 \tag{4}$$

Equations (2) and (4) form an eigenvalue problem that we have solved previously – the eigenvalues and eigenfunctions were given by:

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2, \tag{5}$$

$$X_n(x) = \sin\left(\frac{\pi n x}{l}\right). \tag{6}$$

where $n = 1, 2, \ldots$. Substituting these eigenvalues into the ODE for T (3) gives a collection of ODEs

$$T_n' + k\lambda_n T_n = 0$$

which we can immediately integrate to obtain:

$$T_n(t) = A_n e^{-k\lambda_n t} \tag{7}$$

The separated solutions are therefore given by

$$u_n(x,t) = A_n e^{-k\lambda_n t} \sin\left(\frac{\pi n x}{l}\right), \tag{8}$$

and so we look for a general solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t} \sin\left(\frac{\pi n x}{l}\right). \tag{9}$$

Substitute (9) into the IC from (1) to obtain

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{l}\right),$$

which is just a sine Fourier series on $[0, l]$! The Fourier coefficients may then be calculated (and so the problem solved) by

$$A_n = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{\pi n x}{l}\right) dx. \tag{10}$$

Remark 1.1. Suppose that we wanted to solve the *backwards* heat equation – i.e. we wanted to determined where our initial distribution of heat came *from*. So, set $\tau = -t > 0$, and consider the putative series solution

$$u(x,\tau) = \sum_{n=1}^{\infty} A_n e^{k\lambda_n \tau} \sin\left(\frac{\pi n x}{l}\right). \tag{11}$$

Consider the following sequence of initial conditions:

$$g_N(x) := e^{-k\lambda_N} \sin\left(\frac{N\pi x}{l}\right) \tag{12}$$

Using orthogonality of the eigenfunctions (6) we have that A_n vanishes for $n \neq N$, and $A_N = e^{-k\lambda_N}$. So we find the sequence of solutions:

$$u_N(x,\tau) = e^{k\lambda_N(\tau-1)} \sin\left(\frac{\pi N x}{l}\right) \tag{13}$$

But now notice: for $\tau > 1$, u_N (roughly) blows up exponentially as $N \to \infty$. Meanwhile, $g_N \to 0$ as $N \to \infty$. So the backwards heat equation does not depend continuously on the initial data!

(To make this rigorous, we would have to define what it means for our initial data to vary "continuously" – it turns out that there are many inequivalent possibilities for this.)

1.1.1 1d Heat equation: Inhomogeneous Dirichlet boundary conditions

Suppose now we take the inhomogeneous 1d heat equation on a bounded domain with inhomogeneous Dirichlet boundary conditions, and assume that all inhomogeneities are t-independent:

$$\begin{aligned} u_t &= ku_{xx} + f(x), \qquad t > 0, \quad 0 < x < l, \\ u(0,t) &= a \\ u(l,t) &= b, \\ u(x,0) &= g(x). \end{aligned} \tag{14}$$

How can we solve this? Suppose that we look for a *stationary solution* $u(x,t) \equiv v(x)$, which solves the problem

$$\begin{aligned} \frac{d^2v}{dx} &= -\frac{1}{k}f(x), \quad 0 < x < l, \\ v(0) &= a, \\ v(l) &= b. \end{aligned} \tag{15}$$

See [IvrXX, §6.1] for the explicit solution to (15) in terms of an integral against a Green's function. (You may of course solve (15) using whatever techniques you wish.)

Now, consider the function $\tilde{u} = u - v$. It solves the problem

$$\begin{aligned} \tilde{u}_t &= k\tilde{u}_{xx}, \qquad t > 0, \quad 0 < x < l, \\ \tilde{u}(0,t) = \tilde{u}(l,t) &= 0, \\ \tilde{u}(x,0) &= g(x) - v(x), \end{aligned}$$

which is exactly the homogeneous problem we solved above. In particular, since the solution (9) decays to zero,[1] as $t \to \infty$ we see that the solution $u(x,t)$ to (14) must limit to the stationary solution $v(x)$ as $t \to \infty$.

Example 1. Consider the following problem:

$$\begin{aligned} u_t &= ku_{xx} + \sin(x), \qquad t > 0, \quad 0 < x < \pi, \\ u(0,t) &= 0, \\ u(l,t) &= -\frac{\pi}{k}, \\ u(x,0) &= 0. \end{aligned}$$

The corresponding stationary problem is:

$$\begin{aligned} v''(x) &= -\frac{1}{k}\sin(x), \\ v(0) &= 0, \\ v(\pi) &= -\frac{\pi}{k} \end{aligned}$$

The general solution to the ODE is

$$v(x) = \frac{1}{k}\sin(x) + Ax + B,$$

and using the initial conditions gives

$$v(x) = \frac{\sin(x) - x}{k}.$$

[1]See [IvrXX, §6.1] for a more precise statement.

Setting $\tilde{u} = u - v$, the corresponding homogeneous problem is:

$$\begin{aligned} \tilde{u}_t &= k\tilde{u}_{xx}, \\ \tilde{u}(0,t) = \tilde{u}(\pi,t) &= 0 \\ \tilde{u}(x,0) &= \frac{x - \sin(x)}{k} \end{aligned}$$

The sine Fourier coefficients for $\frac{x}{k}$ on $[0,\pi]$ are given by

$$\frac{2}{nk\pi}(-1)^{n+1},$$

so we have

$$\tilde{u}(x,t) = \sum_{n>0} \frac{2}{nk\pi}(-1)^{n+1} e^{-kn^2 t}\sin(nx) - \frac{1}{k}e^{-kt}\sin(x).$$

Hence solving for $u = \tilde{u} + v$ we obtain

$$u(x,t) = \frac{1}{k}\left[(1 - e^{-kt})\sin(x) + \left(\frac{2}{\pi}\sum_{n>0}\frac{(-1)^{n+1}}{n}e^{-kn^2 t}\sin(nx) - x\right)\right].$$

1.1.2 1d Heat equation: Other inhomogeneous boundary conditions

As mentioned at the start, we can also study the heat equation on an interval with other types of boundary condition – further, we could consider the *inhomogeneous* heat equation with *inhomogeneous* boundary conditions (we will still assume that all inhomogeneities are t-independent).

To solve this we could again try the trick of decomposing our problem into a homogeneous problem and a stationary solution. In doing so, however, we may run into difficulties: there may now be extra constraints that prevent a stationary solution from existing!

Let's see this by considering the case of inhomogeneous Neumann BCs:

$$\begin{aligned} u_t &= ku_{xx} + f(x), \qquad t > 0, \quad 0 < x < l, \\ u_x(0,t) &= a \\ u_x(l,t) &= b, \\ u(x,0) &= g(x). \end{aligned} \tag{16}$$

The corresponding stationary problem is:

$$\begin{aligned} \frac{d^2 v}{dx} &= -\frac{1}{k}f(x), \quad 0 < x < l, \\ v'(0) &= a, \\ v'(l) &= b. \end{aligned} \tag{17}$$

But now we see that there is an extra constraint that must be satisfied if (17) is to have a solution:

$$-\frac{1}{k}\int_0^l f(x)dx = \int_0^l v''(x)dx = v'(l) - v'(0) = b - a. \tag{18}$$

Suppose then that (18) is **not** satisfied. Then we can use the following trick: set

$$p = \frac{1}{l}\int_0^l f(x)dx + \frac{k}{l}(b - a) \tag{19}$$

and consider the function $w(x,t) = u(x,t) - pt$. If u solves our (16) then w solves

$$\begin{aligned} w_t &= kw_{xx} + f(x) - p, \\ w_x(0,t) &= a, \\ w_x(l,t) &= b, \\ w(x,0) &= g(x). \end{aligned}$$

This problem has corresponding stationary problem:

$$\begin{aligned} v''(x) &= \frac{p - f(x)}{k}, \\ v'(0) &= a, \\ v'(l) &= b. \end{aligned}$$

Our judicious choice of p now ensures that a solution to the stationary problem exists. (Exercise: Check this!) So we can solve for the function $w(x,t)$ by separately finding a stationary solution and a solution to the homoegeneous problem, and the solution to (16) is given by

$$u(x,t) = pt + w(x,t).$$

Remark 1.2. We know that $w(x,t)$ approaches a stationary solution $v(x)$ as $t \to \infty$, so for $t >> 0$ let's approximate $w(x,t) \approx v(x)$. Then for large t we have that

$$u(x,t) \approx pt + v(x),$$

so that eventually u given by a stationary heat profile $v(x)$ whose value is either increasing or decreasing linearly with time.

We can give a sensible interpretation of this as follows:

- b gives the rate at which heat is leaving the interval at $x = l$.
- a gives the rate at which heat is entering the interval at $x = 0$.
- $f(x)$ is a source or sink term for the heat energy (e.g. an exothermic chemical reaction).
- p is the average of the difference between the amount of heat created/destroyed by the source/sink term and the flow of heat into and out of the interval.

So, for instance: if $p > 0$, then we see that the source term $f(x)$ is creating heat energy faster than it can flow out the ends of the interval. Hence over time the total heat energy in the interval must be increasing propotionally to the discrepancy between the rate at which heat is being created and the rate at which it is flowing out of the interval.

1.2 1d Wave equation

When we first started discussing separation of variables, it was in reference to the 1d wave equation. Let's now finish that discussion.

Recall that the problem under consideration is:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < l, \\ (\alpha_0 u_x - \alpha u)(0,t) &= (\beta_0 u_x - \beta u)(l,t) = 0, \qquad (20) \\ u(x,0) &= g(x), \\ u_t(x,0) &= h(x). \qquad (21) \end{aligned}$$

We spent a week solving the eigenvalue problem for $X(x)$, so let's take as given that we have:

- eigenvalues λ_n, corresponding to
- eigenfunctions $X_n(x)$.

The equation for $T(t)$ is

$$T_n''(t) + \lambda_n c^2 T_n(t) = 0,$$

so setting

$$\omega_n = c\sqrt{\lambda_n}, \quad \lambda_n > 0, \tag{22}$$

$$\eta_n = c\sqrt{-\lambda_n}, \quad \lambda_n < 0, \tag{23}$$

we obtain

$$T_n(t) = \begin{cases} A_n \cos(\omega_n t) + B_n \sin(\omega_n t), & \lambda_n > 0, \\ A_n + B_n t, & \lambda_n = 0, \\ A_n \cosh(\eta_n t) + B_n \sinh(\eta_n t), & \lambda_n < 0. \end{cases} \tag{24}$$

So the solution is given by

$$u(x,t) = \sum_n T_n(t) X_n(x) \tag{25}$$

with coefficients A_n and B_n given by

$$A_n = \frac{1}{\|X_n\|^2} \int_0^l g(x) X_n(x) dx, \tag{26}$$

$$B_n = \frac{1}{\|X_n\|^2} \int_0^l \kappa_n h(x) X_n(x) dx, \tag{27}$$

where

$$\kappa_n = \begin{cases} \frac{1}{\omega_n}, & \lambda_n > 0, \\ 1, & \lambda_n = 0, \\ \frac{1}{\eta_n}, & \lambda_n < 0. \end{cases} \tag{28}$$

1.3 2d Laplace Equation

Let's now consider the 2d Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0. \tag{29}$$

A fairly general analysis can be found in [IvrXX, §6.2]. Here we focus on some specific examples.

1.3.1 Laplace equation in the half-strip

Consider solving (29) on the half-strip

$$y > 0, \qquad 0 < x < l, \tag{30}$$

and subject to the boundary conditions

$$u_x(0,y) = u_x(l,y) = 0, \tag{31}$$

$$u(x,0) = g(x), \tag{32}$$

and the boundedness constraint $|u| \leq M < \infty$.

Looking for a separated solution $u(x, y) = X(x)Y(y)$ yields

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ Y''(y) - \lambda Y(y) &= 0, \\ X'(0) = X'(l) &= 0. \end{aligned}$$

We have solved this eigenvalue problem before: the solutions are given by

$$\begin{aligned} \lambda_n &= \omega_n^2, \\ \omega_n &= \frac{n\pi}{l} \\ X_n(x) &= \cos(\omega_n x), \end{aligned}$$

where $n = 0, 1, 2, \ldots$

For $\lambda = 0$ we solve the equation for Y to find

$$Y_0(y) = A + By,$$

and by the boundedness constraint $B = 0$. So this leads to the constant solution.

For $\lambda > 0$ we solve the equation for Y to find

$$Y_n(y) = A_n e^{-\omega_n y} + B_n e^{\omega_n y},$$

and again by boundedness $B_n = 0$. So the solution is

$$u(x, y) = \frac{A_0}{2} + \sum_{n>0} A_n e^{-\omega_n y} \cos(\omega_n x).$$

where by imposing (32) we find the Fourier coefficients are given by

$$A_n = \frac{2}{l} \int_0^l g(x) \cos(\omega_n x) dx.$$

1.3.2 Laplace equation in the rectangle

Consider solving (29) on the rectangle

$$0 < y < b, \qquad 0 < x < a, \tag{33}$$

and subject to the boundary conditions

$$\begin{aligned} u_x(0, y) = u_x(a, y) &= 0, & (34) \\ u(x, 0) &= 0, & (35) \\ u(x, b) &= x, & (36) \end{aligned}$$

Again we have

$$\begin{aligned} \lambda_n &= \omega_n^2, \\ \omega_n &= \frac{n\pi}{a} \\ X_n(x) &= \cos(\omega_n x), \end{aligned}$$

where $n = 0, 1, 2, \ldots$, and

$$Y_0(y) = \frac{A_0}{2} + B_0 y,$$
$$Y_n(y) = A_n e^{-\omega_n y} + B_n e^{\omega_n y}.$$

The solution therefore looks like

$$u(x,y) = \frac{A_0}{2} + B_0 y + \sum_{n>0} \left(A_n e^{-\omega_n y} + B_n e^{\omega_n y}\right)\cos(\omega_n x).$$

Plugging in the y BCs gives[2]

$$\frac{A_0}{2} + \sum_{n>0} (A_n + B_n)\cos(\omega_n x) = 0, \tag{37}$$

$$\frac{A_0}{2} + B_0 b + \sum_{n>0} \left(A_n e^{-\omega_n b} + B_n e^{\omega_n b}\right)\cos(\omega_n x) = x = a - \frac{4a}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}\cos(\omega_{2m+1} x). \tag{38}$$

From (37) we find that $A_0 = 0$ and $A_n = -B_n$ for $n > 0$. We can use this to rewrite (38) as

$$B_0 b + 2\sum_{n>0} B_n \sinh(\omega_n b)\cos(\omega_n x) = a - \frac{4a}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(\omega_{2m+1}x).$$

Matching Fourier coefficients gives:

$$B_0 = \frac{a}{b},$$
$$B_{2m} = 0,$$
$$B_{2m+1} = \frac{-2a}{\pi^2(2m+1)^2\sinh(\omega_{2m+1}b)}.$$

Putting this altogether we arrive at the solution

$$u(x,y) = \frac{a}{b}y - \frac{4a}{\pi^2}\sum_{m=0}^{\infty}\frac{\sinh(\omega_{2m+1}y)}{\sinh(\omega_{2m+1}b)}\frac{\cos(\omega_{2m+1}x)}{(2m+1)^2}.$$

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.

[2] Using here that the cosine Fourier coefficients for x on $[0, a]$ are $A_0 = a$, $A_n = \frac{2a}{n^2\pi^2}((-1)^n - 1)$.