APM 346 Lecture 14.

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We now look at how to apply the Fourier transform to the problem of solving differential equations. References being used: [IvrXX, §5.3] (§5.3) and [Str08, Ch.12.3].

1 Applications of the Fourier Transform to PDEs

The key observation which will allow us to apply the Fourier transform to the study of PDEs is the following: if we take the Fourier transform of a function $f(x)$, call it $\hat{f}(\xi)$, then

$$\widehat{\frac{df}{dx}} = i\xi \hat{f}(\xi).$$

(We saw this last lecture.) So the Fourier transform sends

$$\partial_x \mapsto i\xi, \qquad \partial_x^2 \mapsto -\xi^2, \qquad \text{etc.}$$

In other words: the Fourier transform allows us to turn *differential operators* into *polynomial equations*. Let's see how this can be applied in a number of different situations.

1.1 1d Heat equation

We'll begin by revisiting the 1d heat equation IVP:

$$u_t = ku_{xx}, \qquad t > 0, \quad -\infty < x < \infty, \tag{1}$$

$$u(x,0) = g(x). \tag{2}$$

Taking the Fourier transform of the above equations with respect to the x-coordinate gives:

$$\hat{u}_t(\xi,t) = -k\xi^2 \hat{u}(\xi,t), \tag{3}$$

$$\hat{u}(\xi,0) = \hat{g}(\xi). \tag{4}$$

Now, (3) is a simple ODE, which we may solve to find

$$\hat{u}(\xi,t) = A(\xi)e^{-k\xi^2 t},$$

and evaluating this at $t = 0$ gives $A(\xi) = \hat{g}(\xi)$. So, we have solved for the Fourier transform of our desired solution:

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-k\xi^2 t} \tag{5}$$

Now, recall that if $F(x) = e^{-\frac{x^2}{2}}$ then $\hat{F}(\xi) = e^{-\frac{\xi^2}{2}}$, and that if $G(x) = F(\lambda x)$ then $\hat{G}(\xi) = \frac{1}{|\lambda|}\hat{F}\left(\frac{\xi}{\lambda}\right)$. So, letting

$$F(x) = \frac{1}{\sqrt{2kt}}e^{-\frac{x^2}{4kt}} \quad \text{we have that} \quad \hat{F}(\xi) = e^{-k\xi^2 t},$$

so that (5) becomes

$$\hat{u}(\xi,t) = \hat{g}(\xi)\hat{F}(\xi).$$

Using that the Fourier transform of a convolution $f*g$ is $\sqrt{2\pi}\hat{f}\hat{g}$, we can perform the inverse Fourier transform to obtain

$$u(x,t) = \frac{1}{\sqrt{2\pi}}(F*g)(x) = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)dy, \tag{6}$$

recovering our previous solution to the heat equation IVP!

1.1.1 Multidimensional heat equation

Recall that the heat equation IVP in n dimensions is

$$u_t(\vec{x},t) = k\Delta u, \quad t>0,\, \vec{x}\in\mathbb{R}^n \tag{7}$$

$$u(\vec{x},0) = g(\vec{x}) \tag{8}$$

Write $\vec{x} = (x_1,\dots,x_n)$, and let $|x|^2 = x_1^2+\cdots+x_n^2$. We can take the Fourier transform of u with respect to *all* of the spatial variables, to obtain the equivalent problem

$$\hat{u}_t(\vec{\xi},t) = -k|\vec{\xi}|^2\hat{u}, \tag{9}$$

$$\hat{u}(\vec{\xi},0) = \hat{g}(\vec{\xi}) \tag{10}$$

This can again be solved easily:

$$\hat{u}(\vec{\xi},t) = \hat{g}(\xi)e^{-k|\xi|^2 t}$$

Since $e^{-k|\xi|^2 t}$ factors into a product of single variable functions

$$e^{-k|\xi|^2 t} = \prod_{j=1}^{n} e^{-k\xi_j^2 t}$$

we can take the inverse Fourier transform of each factor in the product separately to obtain the heat kernel/Green's function:

$$\prod_{j=1}^{n}\frac{1}{\sqrt{4\pi kt}}e^{-\frac{x_j^2}{4kt}} = (4\pi kt)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4kt}}$$

1.2 1d Wave equation

Next let's revisit another old friend: the homogeneous 1d wave equation IVP:

$$u_{tt} = c^2u_{xx}, \qquad -\infty<x<\infty, \tag{11}$$

$$u(x,0) = g(x), \tag{12}$$

$$u_t(x,0) = h(x). \tag{13}$$

Again, take the Fourier transform with respect to x:

$$\hat{u}_{tt} = -c^2\xi^2\hat{u}, \tag{14}$$

$$\hat{u}(\xi,0) = \hat{g}(\xi), \tag{15}$$

$$\hat{u}_t(\xi,0) = \hat{h}(\xi). \tag{16}$$

The ODE (14) has general solution

$$\hat{u}(\xi,t) = A(\xi)\cos(c\xi t) + B(\xi)\sin(c\xi t),$$

and the initial conditions (15) and (16) imply that

$$A(\xi) = \hat{g}(\xi)$$
$$B(\xi) = \frac{1}{c\xi}\hat{h}(\xi)$$

so that the Fourier transform of the solution is given by

$$\hat{u}(\xi, t) = \hat{g}(\xi)\cos(c\xi t) + \frac{1}{c\xi}\hat{h}(\xi)\sin(c\xi t). \tag{17}$$

Now: recall from last lecture that if $G(x) = F(x-a)$ then $\hat{G}(\xi) = e^{-i\xi a}\hat{F}(\xi)$. So, writing

$$\cos(c\xi t) = \frac{e^{ic\xi t} + e^{-ic\xi t}}{2}$$

we see that $\hat{g}(\xi)\cos(c\xi t)$ is the Fourier transform of

$$\frac{g(x+ct) + g(x-ct)}{2}.$$

Denote by H a primitive of h. Then

$$\frac{1}{c\xi}\hat{h}(\xi)\sin(c\xi t) = \frac{i}{c}\hat{H}(\xi)\sin(c\xi t) = \frac{1}{2c}\hat{H}(\xi)(e^{ic\xi t} - e^{-ic\xi t})$$

is the Fourier transform of

$$\frac{H(x+ct) - H(x-ct)}{2c} = \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds.$$

Hence we have recovered the D'Alembert formula:

$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds. \tag{18}$$

1.2.1 Multidimensional wave equation

Next, take the homogeneous n-dimensional wave equation IVP:

$$u_{tt} = c^2\Delta u, \tag{19}$$
$$u(\vec{x}, 0) = g(x), \tag{20}$$
$$u_t(\vec{x}, 0) = h(x). \tag{21}$$

As for the heat equation, take the Fourier transform with respect $x_1, \ldots, x_n$:

$$\hat{u}_{tt} = -c^2|\xi|^2\hat{u}, \tag{22}$$
$$\hat{u}(\vec{\xi}, 0) = \hat{g}(\vec{\xi}), \tag{23}$$
$$\hat{u}_t(\vec{\xi}, 0) = \hat{h}(\vec{\xi}). \tag{24}$$

Just as for the 1d wave equation, we can easily solve this problem to obtain

$$\hat{u}(\vec{\xi}, t) = \hat{g}(\vec{\xi})\cos(c|\vec{\xi}|t) + \frac{1}{c|\vec{\xi}|}\hat{h}(|\vec{\xi}|)\sin(c|\vec{\xi}|t). \tag{25}$$

Unfortunately, unlike the heat equation or 1d wave equation, the inverse Fourier transform of (25) is not immediately calculable – we leave this problem alone (at least, for the moment).

1.3 Laplace equation in half-plane (Dirichlet)

Consider the Laplace equation in the half-plane with a Dirichlet boundary condition:

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad y > 0, \quad -\infty < x < \infty, \tag{26}$$
$$u(x,0) = g(x), \tag{27}$$
$$|u| \le M, \tag{28}$$

(The boundedness condition is required for uniqueness of the solution; e.g. $u = \kappa y$ satisfies homogeneous BCs for any constant κ.)

Taking the Fourier transform of the problem gives

$$\hat{u}_{yy} - \xi^2 \hat{u} = 0, \tag{29}$$
$$\hat{u}(\xi, 0) = \hat{g}(\xi). \tag{30}$$

We solve the ODE (29) to find

$$\hat{u}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{|\xi|y}.$$

Since we are looking for a bounded solution, we must have $B(\xi) = 0$, and so $A(\xi) = \hat{g}(\xi)$. So the Fourier transform of the solution is

$$\hat{u}(\xi, y) = \hat{g}(\xi)e^{-|\xi|y}. \tag{31}$$

Now, from last lecture, the inverse Fourier transform of $e^{-y|\xi|}$ is $\frac{1}{\sqrt{2\pi}}\left(\frac{2y}{x^2+y^2}\right)$, so using that

$$\hat{f}\hat{g} = \frac{1}{\sqrt{2\pi}}\widehat{f * g},$$

we find that

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x')\frac{y}{(x-x')^2 + y^2}\,dx'. \tag{32}$$

1.4 Laplace equation in half-plane (Robin)

Consider again the Laplace equation on the half-strip (26). Let us now look for bounded solutions which satisfy a Robin boundary condition:

$$u_y(x,0) - \alpha u(x,0) = h(x). \tag{33}$$

The Fourier transform of this is

$$\hat{u}_y(\xi, 0) - \alpha\hat{u}(\xi, 0) = \hat{h}(\xi). \tag{34}$$

Since we already know that $\hat{u} = A(\xi)e^{-|\xi|y}$, $\hat{u}_y = -|\xi|\hat{u}$ and $\hat{u}(\xi,0) = A(\xi)$, so (34) becomes

$$A(\xi) = -\frac{\hat{h}(\xi)}{|\xi| + \alpha}.$$

So we find the solution

$$\hat{u}(\xi, y) = -\frac{\hat{h}(\xi)}{|\xi| + \alpha}e^{-|\xi|y}. \tag{35}$$

Provided that $\alpha > 0$, this is a perfectly integrable function, and its inverse Fourier transform (while not explicitly calculable) will solve our problem.

1.4.1 Laplace equation in half-plane (Neumann)

Suppose, however, that $\alpha = 0$ – that is, we are considering Neumann BCs. To avoid singular behaviour of

$$-\frac{\hat{h}(\xi)}{|\xi|}e^{-|\xi|y}$$

at $\xi = 0$, we must additionally require that $\hat{h}(0) = 0$. In terms of the original function h, this translates to the requirement

$$\int_{-\infty}^{\infty} h(x)dx = 0. \tag{36}$$

We can justify this physically: suppose, for instance, that we are looking for a stationary solution to the heat equation. Then the function $h(x)$ describes the heat flow through the boundary, and in order for the distribution to be stationary it must be the case that the total flow (given by (36)) is zero.

See the textbook for the explicit inverse Fourier transform in this case [IvrXX, §5.3].

1.5 Laplace equation in a strip

Now let's consider the Laplace equation with Dirichlet BCs on an infinite strip:

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad 0 < y < b, \quad -\infty < x < \infty, \tag{37}$$

$$u(x,0) = g(x), \tag{38}$$

$$u(x,b) = h(x), \tag{39}$$

The general solution for the Fourier transform of the solution is still

$$\hat{u}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{|\xi|y},$$

but we can no longer discard either of these solution via boundedness considerations. Instead, we obtain the linear system of equations:

$$A(\xi) + B(\xi) = \hat{g}(\xi)$$
$$A(\xi)e^{-|\xi|b} + B(\xi)e^{|\xi|b} = \hat{h}(\xi)$$

Subtracting $e^{-|\xi|b}$ times the second row from the first row yields

$$A(\xi)(1 - e^{-2|\xi|b}) = \hat{g}(\xi) - e^{-|\xi|b}\hat{h}(\xi);$$

multiplying through by $e^{|\xi|b}$ gives

$$A(\xi)\underbrace{(e^{|\xi|b} - e^{-|\xi|b})}_{2\sinh(|\xi|b)} = e^{|\xi|b}\hat{g}(\xi) - \hat{h}(\xi).$$

A similar calculation can be performed for $B(\xi)$ – the result is

$$A(\xi) = \frac{e^{|\xi|b}}{2\sinh(|\xi|b)}\hat{g}(\xi) - \frac{1}{2\sinh(|\xi|b)}\hat{h}(\xi) \tag{40}$$

$$B(\xi) = -\frac{e^{-|\xi|b}}{2\sinh(|\xi|b)}\hat{g}(\xi) + \frac{1}{2\sinh(|\xi|b)}\hat{h}(\xi) \tag{41}$$

So the Fourier transform of the solution is

$$\begin{aligned}\hat{u}(\xi,y) &= \left(\frac{e^{|\xi|b}}{2\sinh(|\xi|b)}\hat{g}(\xi) - \frac{1}{2\sinh(|\xi|b)}\hat{h}(\xi)\right)e^{-|\xi|y} + \left(-\frac{e^{-|\xi|b}}{2\sinh(|\xi|b)}\hat{g}(\xi) + \frac{1}{2\sinh(|\xi|b)}\hat{h}(\xi)\right)e^{|\xi|y} \\ &= \frac{e^{|\xi|(b-y)} - e^{-|\xi|(b-y)}}{2\sinh(|\xi|b)}\hat{g}(\xi) + \frac{e^{|\xi|y} - e^{-|\xi|y}}{2\sinh(|\xi|b)}\hat{h}(\xi)\end{aligned}$$

i.e.

$$\hat{u}(\xi,y) = \frac{\sinh(|\xi|(b-y))}{\sinh(|\xi|b)}\hat{g}(\xi) + \frac{\sinh(|\xi|y)}{\sinh(|\xi|b)}\hat{h}(\xi). \tag{42}$$

Once again, while we have an explicit form for $\hat{u}(\xi,y)$, calculating the inverse Fourier transform explicitly may not be possible.

1.6 Summary

To summarise the method we have used in this lecture:

(1) Start with a (well-posed) problem in the variables $-\infty < x_1,\ldots,x_n < \infty$, as well as some other variable – e.g. time $t > 0$.

(2) Take the Fourier transform of the variables $x_i \to \xi_i$, to obtain a problem for an *ordinary* differential equation in t with polynomial coefficients in the ξ_i induced by the x_i-derivatives in the original problem.

(3) Solve the resulting ODE problem.

(4) (Possibly) take the inverse Fourier transform of the ODE solution.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.