APM 346 Lecture 13.

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This week we continue our study of Fourier theory with a look at the Fourier transform. References being used: [IvrXX, §5.1-2] (§5.1, §5.2) and [Str08, Ch.12.3].

1 The Fourier Transform

Those of you who did the "just for fun" reading [Mac78] will have already encountered the Fourier transform, approached from a different perspective than the heuristic via Fourier series we will use below. If you want to know more about this other perspective, and about Fourier transforms more broadly, you should feel free to contact me!

1.1 Heuristic derivation

For the purposes of this class, however, we will take the following heuristic approach to deriving the Fourier transform using Fourier series.

Recall the complex form of the Fourier series for a function f on the interval $[-l, l]$,

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{i\pi n x}{l}} \tag{1}$$

where the Fourier coefficients are given by

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{i\pi n x}{l}} \tag{2}$$

and we have a Plancherel theorem

$$\int_{-l}^{l} |f(x)|^2 dx = 2l \sum_{-\infty}^{+\infty} |c_n|^2. \tag{3}$$

From this we will formally derive the Fourier transform of a function f on $(-\infty, +\infty)$ by taking $l \to \infty$ in the above expressions. Define

$$k_n := \frac{\pi n}{l} \quad \text{and} \quad \Delta k_n := k_n - k_{n-1} = \frac{\pi}{l} \tag{4}$$

Then (1) can be rewritten

$$f(x) = \sum_{n=-\infty}^{+\infty} C(k_n) e^{i k_n x} \Delta k_n \tag{5}$$

where

$$C(k) = \frac{1}{2\pi}\int_{-l}^{l} f(x)e^{-ikx}dx, \tag{6}$$

and the Plancherel theorem becomes

$$\int_{-l}^{l} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{+\infty} |C(k_n)|^2 \Delta k_n. \tag{7}$$

Now, when we take $l \to +\infty$ the following things happen (this is the part that really requires more justification):

- The integrals $\int_{-l}^{l}$ become integrals $\int_{-\infty}^{+\infty}$.
- The Riemann sums $\sum_{-\infty}^{\infty} \phi(k_n)\Delta k_n$ become integrals $\int_{-\infty}^{+\infty} \phi(k)dk$.

So we obtain the following formulae:

$$f(x) = \int_{-\infty}^{\infty} C(k)e^{ikx}dk \tag{8}$$

$$C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{9}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |C(k)|^2 dk \tag{10}$$

1.2 Definition and first properties

Taking a cue from equations (8), (9) and (10), we make the following definitions.

Definition 1.1. The *Fourier transform* of a function $f : \mathbb{R}_x \to \mathbb{C}$ is defined to be

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx. \tag{11}$$

Definition 1.2. The *inverse Fourier transform* of a function $F : \mathbb{R}_k \to \mathbb{C}$ is defined to be

$$\check{F}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk. \tag{12}$$

Proposition 1.1. *The inverse Fourier transform* really is *the inverse to the Fourier transform, i.e.*

$$\check{\hat{f}} = f \quad \textit{and} \quad \hat{\check{F}} = F.$$

Proof. Follows from formulae (9) and (8). □

Theorem 1.2 (Plancherel Theorem)**.** *The Fourier transform preserves the L^2-norm, i.e.*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk. \tag{13}$$

Proof. Follows from equation (10). □

Recall that a function is called L^2 ("square integrable") if it has finite L^2-norm, defined by

$$\|f\|^2 = \int_{\mathbb{R}} |f(x)|^2 dx. \tag{14}$$

Denote by $L^2(\mathbb{R}, \mathbb{C})$ the space of L^2 functions[1] $f : \mathbb{R} \to \mathbb{C}$. This is a vector space over $\mathbb{C}$, and may be equipped with the L^2 inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} dx. \tag{15}$$

Theorem 1.3. *The Fourier transform and inverse Fourier transform are linear, unitary operators $L^2(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C})$.*

Proof. Linearity follows from the linearity of integration. Recalling that unitarity means inner-product preserving, unitarity follows from the Plancherel Theorem (Theorem 1.2) and the polarisation identities (which are an exercise for you to check):

$$\mathrm{Re}\langle f, g \rangle = \frac{\|f+g\|^2 - \|f\|^2 - \|g\|^2}{2} \tag{16}$$

$$\mathrm{Im}\langle f, g \rangle = \frac{\|f+ig\|^2 - \|f\|^2 - \|g\|^2}{2} \tag{17}$$

□

Remark 1.1. There are different normalisation conventions for the Fourier transform and its inverse that show up in the literature (indeed, in the textbook [IvrXX]!) – we are working with the normalisation factor $\frac{1}{\sqrt{2\pi}}$ for the following two reasons:

- Less important reason: it makes the definition of the Fourier transform and its inverse more symmetric (they differ only by the replacement $i \to -i$).
- More important reason: this is the normalisation convention that makes the Fourier transform a *unitary* operator.

A particularly nice feature of the Fourier transform is how it behaves under translations, differentiations, and dilations:

Theorem 1.4. *Let $f(x)$ be a function with Fourier transform $\hat{f}(k)$. Then:*

(a) If $g(x) = f(x-a)$ then $\hat{g}(k) = e^{-ika}\hat{f}(k)$.

(b) If $g(x) = f(x)e^{ibx}$ then $\hat{g}(k) = \hat{f}(k-b)$.

(c) If $g(x) = f'(x)$ then $\hat{g}(x) = ik\hat{f}(k)$.

(d) If $g(x) = xf(x)$ then $\hat{g}(x) = i\frac{d\hat{f}}{dk}(k)$.

(e) If $g(x) = f(\lambda x)$ then $\hat{g}(k) = |\lambda|^{-1}\hat{f}(\lambda^{-1}k)$.

Proof. The proofs of these are left as an exercise, with the following hints:

- (a), (b) and (e) will follow from a change of integration variable.

[1] A careful definition of this space is beyond the scope of this course.

- (c) and (d) will follow from an integration by parts argument.

□

Note from part (e) of the above that:

Corollary 1.5. *f is even/odd if and only if $\hat{f}$ is even/odd.*

1.3 Convolution

Another nice feature of the Fourier transform is the following: it exchanges *pointwise multiplication* with *convolution* of functions.

Definition 1.3. The *convolution* of the functions f and g is the function defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y)dy \tag{18}$$

Theorem 1.6. *Let f and g be functions with Fourier transforms $\hat{f}$ and $\hat{g}$. Then:*

(a) If $h = f * g$ then $\hat{h}(k) = \sqrt{2\pi}\hat{f}(k)\hat{g}(k)$.

(b) If $h(x) = f(x)g(x)$ then $\hat{h} = \sqrt{2\pi}(\hat{f} * \hat{g})$.

Proof. We only prove the first claim, as the second is similar. We have

$$\begin{aligned}\hat{h}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}h(x)dx \\ &= \frac{1}{\sqrt{2\pi}}\iint e^{-ikx}f(x-y)g(y)\,dx\,dy\end{aligned}$$

Letting $z = x - y$, this becomes

$$\begin{aligned}\hat{h}(k) &= \frac{1}{\sqrt{2\pi}}\iint e^{-ik(y+z)}f(z)g(y)\,dz\,dy \\ &= \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int e^{-ikz}f(z)dz\right)\left(\frac{1}{\sqrt{2\pi}}\int e^{-iky}g(y)dy\right) \\ &= \sqrt{2\pi}\hat{f}(k)\hat{g}(k)\end{aligned}$$

□

1.4 Example Fourier transforms

Let's now actually calculate some examples of Fourier transforms:

Example 1. Let $f(x) = e^{-ax}$ for $x > 0$ and $f(x) = 0$ for $x < 0$, for some constant a with $\mathrm{Re}(a) > 0$. Then:

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ax}e^{-ikx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-(a+ik)x}dx \\ &= -\frac{1}{\sqrt{2\pi}}\left(\frac{1}{a+ik}\right)\left[e^{-(a+ik)x}\right]_{x=0}^{x\to\infty} = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a+ik}\right)\end{aligned}$$

Example 2. Let $f(x) = e^{-a|x|}$ where $\text{Re}(a) > 0$. As an exercise, calculate that

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2 + k^2}\right).$$

Hint: Use Example 1.

Example 3. Suppose $f(x) = e^{-\frac{x^2}{2}}$ is a Gaussian. Then

$$f'(x) = -xf(x)$$

and so using the properties of the Fourier transform

$$ik\hat{f}(k) = -i\frac{d\hat{f}}{dk}$$

i.e.

$$-k = \frac{1}{\hat{f}}\frac{d\hat{f}}{dk} = \frac{d}{dk}(\log(\hat{f})).$$

This is solved by

$$\hat{f}(k) = Ce^{-\frac{k^2}{2}}$$

for some constant $C = \hat{f}(0)$. But using the definition of the Fourier transform, we have

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx}_{=\sqrt{2\pi}} = 1.$$

So the Gaussian is its own Fourier transform!

$$f(x) = e^{-\frac{x^2}{2}} \quad \Rightarrow \quad \hat{f}(k) = e^{-\frac{k^2}{2}}.$$

Theorem 1.7 (Poisson summation formula)**.** *Let $f(x)$ be an appropriately continuous function on $\mathbb{R}$ which vanishes for large $|x|$.*[2] *Then for any $a > 0$*

$$\sum_{n=-\infty}^{\infty} f(an) = \frac{\sqrt{2\pi}}{a}\sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{2\pi n}{a}\right). \tag{19}$$

Proof. Define the auxilliary function

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + an),$$

which is periodic with period a. On $\left[-\frac{a}{2}, \frac{a}{2}\right]$ this has complex Fourier series coefficients given by (Exercise!)

$$c_m = \frac{\sqrt{2\pi}}{a}\hat{f}\left(\frac{2\pi m}{a}\right).$$

So, expressing g as its Fourier series gives

$$\sum_{n=-\infty}^{\infty} f(x + an) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{a}\hat{f}\left(\frac{2\pi m}{a}\right)e^{\frac{2\pi inx}{a}},$$

and then evaluating at $x = 0$ yields (19). □

[2] Understanding the exact class of functions for which this theorem holds is beyond the scope of this course – we want to exclude pathological examples such as continuous but nowhere differentiable functions. For those who want to do extra reading, it is sufficient to assume that f is a *Schwarz function*: an infinitely differentiable function whose derivatives all rapidly decay (this has a precise meaning).

Example 4. The calculations in this example are left as an exercise. Consider the function

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & |x| \ge 1 \end{cases}$$

The Fourier transform of $f(x)$ is

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos(k)}{k^2} \right),$$

so the Poisson summation formula for $f(x)$ reads

$$\begin{aligned} \sum_{n=-\infty}^{\infty} f(an) &= \frac{\sqrt{2\pi}}{a} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{2\pi n}{a}\right) \\ &= \frac{1}{a} + \frac{a}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1-\cos\left(\frac{2\pi n}{a}\right)}{n^2} \right) \end{aligned}$$

where we have used that

$$\lim_{k\to 0} \frac{1-\cos(k)}{k^2} = \frac{1}{2}$$

to make sense of the zero frequency mode $\hat{f}(0)$. From this expression, you should be able to derive the following expression:

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Finally, as a bonus exercise: for $a > 1$ derive the expression:

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi n}{a}\right)}{n^2} = \frac{\pi^2}{6a^2}(a^2 - 6a + 6).$$

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Mac78] George W. Mackey. Harmonic analysis as the exploitation of symmetry—a historical survey. *Rice Univ. Stud.*, 64(2-3):73–228, 1978. History of analysis (Proc. Conf., Rice Univ., Houston, Tex., 1977).

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.