APM 346 Lecture 12.

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We continue our study of Fourier series. References being used: [IvrXX, §4.5] (§4.5) and [Str08, Ch.5].

1 Even/odd functions and periodic extensions

Last lecture we saw that every function $f(x)$ on an interval I of length $2l$ could be decomposed into a Fourier series:

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{\pi n x}{l}\right) + b_n \sin\left(\frac{\pi n x}{l}\right)\right) \tag{1}$$

$$a_n = \frac{1}{l}\int_I f(x)\cos\left(\frac{n\pi x}{l}\right) dx \tag{2}$$

$$b_n = \frac{1}{l}\int_I f(x)\sin\left(\frac{n\pi x}{l}\right) dx \tag{3}$$

$$\int_I |f(x)|^2 dx = \frac{l}{2}|a_0|^2 + \sum_{n=1}^{\infty} l\left(|a_n|^2 + |b_n|^2\right) \tag{4}$$

We did not explicitly use this terminology last lecture, but one says that the system of functions

$$\left\{ \frac{1}{2}, \quad \cos\left(\frac{n\pi x}{l}\right), \quad \sin\left(\frac{n\pi x}{l}\right)\right\}_{n=1,2,\ldots} \tag{5}$$

forms a *complete orthogonal system* on the interval I.

Today we will consider some other naturally arising orthogonal systems.

1.1 Fourier series for even and odd functions

First consider the following Lemma, which we implicitly used last lecture.

Lemma 1.1. *Let $I = [-l, l]$. Then*

(a) $f(x)$ is even iff $b_n = 0$ for all n

(b) $f(x)$ is odd iff $a_n = 0$ for all n

Proof. Follows from (1) the fact that cos/sin is an even/odd function, and (2) that we are integrating over a symmetric interval. □

1.2 Cosine Fourier series

Consider a function $f(x)$ on the interval $[0, l]$. Take the even extension of f to a function on $[-l, l]$,

$$F(x) := f(|x|), \quad -l \le x \le l. \tag{6}$$

By (1) and Lemma 1.1, F has a Fourier series

$$F(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \tag{7}$$

where the coefficients are calculated as in (2). Restricting to the interval $[0, l]$, this provides a decomposition of $f(x)$ as a cosine Fourier series; using the fact that we took an even extension, the coefficients a_n may be calculated as

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx. \tag{8}$$

Parseval's equality becomes

$$\int_0^l |f(x)|^2 dx = \frac{l}{4}|a_0|^2 + \sum_{n=1}^{\infty}\frac{l}{2}|a_n|^2. \tag{9}$$

Upshot:

- The resulting Fourier series is even and $2l$-periodic.
- For "nice" f (see last lecture)[1] it provides an (even, $2l$-periodic) extension of our original function.
- Taking an even and periodic extension does not introduce new discontinuities into our function.
- This is a decomposition of f with respect to the the orthogonal system

$$\left\{\frac{1}{2}, \quad \cos\left(\frac{n\pi x}{l}\right)\right\}_{n=1,2,\ldots}. \tag{10}$$

Example 1. Let's determine the cosine Fourier series for $f(x) = x$, on the interval $[0, 1]$. For $n > 0$ the coefficients are

$$a_n = 2\int_0^1 x\cos(n\pi x)dx = \frac{2}{n^2\pi^2}((-1)^n - 1)$$

which vanish for even n and equal $-\frac{4}{n^2\pi^2}$ for odd n. For $n = 0$ we have

$$a_0 = 2\int_0^1 x dx = 1.$$

So we have

$$x \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos((2n+1)\pi x)}{(2n+1)^2}.$$

See Figure 1 for the periodic extension to the entire line.

[1] E.g. if the function satisfies the Dirichlet conditions.

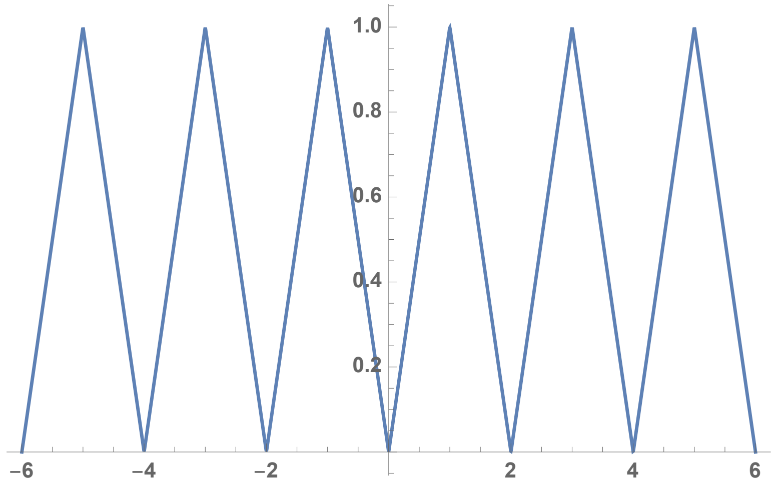

Figure 1: Cosine Fourier series extension of $f(x) = x$.

1.3 Sine Fourier series

Consider a function $f(x)$ on the interval $[0, l]$. Take the odd extension of f to a function on $[-l, l]$,

$$F(\pm|x|) := \pm f(|x|), \quad -l \leq x \leq l. \tag{11}$$

By (1) and Lemma 1.1, F has a Fourier series

$$F(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \tag{12}$$

where the coefficients are calculated as in (3). Restricting to the interval $[0, l]$, this provides a decomposition of $f(x)$ as a sine Fourier series; using the fact that we took an odd extension, the coefficients b_n may be calculated as

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx. \tag{13}$$

Parseval's equality becomes

$$\int_0^l |f(x)|^2 dx = \sum_{n=1}^{\infty} \frac{l}{2} |b_n|^2. \tag{14}$$

Upshot:

- The resulting Fourier series is odd and $2l$-periodic.
- For "nice" f it provides an (odd, $2l$-periodic) extension of our original function.
- Taking an even and periodic extension does not introduce new discontinuities into our function if and only if $f(0) = f(l) = 0$.
- This is a decomposition of f with respect to the the orthogonal system

$$\left\{\sin\left(\frac{n\pi x}{l}\right)\right\}_{n=1,2,\ldots}. \tag{15}$$

Example 2. Let's determine the sine Fourier series for $f(x) = x$, on the interval $[0, 1]$. The coefficients are

$$b_n = 2\int_0^1 x\sin(n\pi x)dx = -\frac{2\cos(n\pi)}{n\pi} = (-1)^{n+1}\frac{2}{n\pi}.$$

So we have

$$x \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

See Figure 2 for the periodic extension to the entire line.

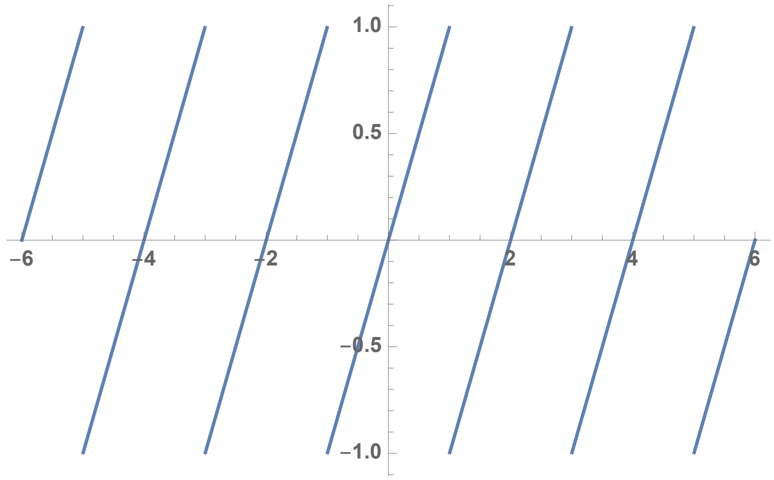

Figure 2: Sine Fourier series extension of $f(x) = x$.

1.4 Half-integer Sine Fourier series

Consider a function $f(x)$ on the interval $[0, l]$. Now: extend it as an even function around the endpoint $x = l$ to obtain a function on $[0, 2l]$ that satisfies

$$f(x) = f(2l - x) \quad \text{for } x \in [l, 2l].$$

Next, make an odd continuation about $x = 0$, to obtain a function on $[-2l, 2l]$. This function has a Fourier series given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n' \sin\left(\frac{n\pi x}{2l}\right), \tag{16}$$

where

$$b_n' = \frac{1}{2l}\int_{-2l}^{2l} f(x)\sin\left(\frac{n\pi x}{2l}\right)dx \tag{17}$$

Now, using the periodicity of sin, we have that

$$f(2l - x) = \sum_{n=1}^{\infty} b_n' \sin\left(\frac{n\pi x}{2l}\right)(-1)^{n+1},$$

and so since our original continuation around $x = l$ was even we have that all of the even index coefficients b_{2m} must vanish, leaving us with

$$f(x) \sim \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi x}{2l}\right) \tag{18}$$

where $b_n = b'_{2n+1}$. These coefficients can be calculated as

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{(2n+1)\pi x}{2l}\right)dx, \quad n \geq 0.$$

Parseval's equality becomes

$$\int_0^l |f(x)|^2 dx = \sum_{n=1}^{\infty} \frac{l}{2}|b_n|^2. \tag{19}$$

Upshot:

- The resulting Fourier series is odd and $4l$-periodic.
- For "nice" f it provides an (odd, $4l$-periodic) extension of our original function.
- Taking this even/odd extension does not introduce new discontinuities into our function if and only if $f(0) = 0$.
- This is a decomposition of f with respect to the the orthogonal system

$$\left\{\sin\left(\frac{(2n+1)\pi x}{2l}\right)\right\}_{n=0,1,2,\dots}. \tag{20}$$

Example 3. Let's determine the half-integer sine Fourier series for $f(x) = x$, on the interval $[0, 1]$. The coefficients are

$$b_n = 2\int_0^1 x\sin\left(\frac{2n+1}{2}\pi x\right)dx = \frac{8\cos(n\pi)}{(2n+1)^2\pi^2} = 8\frac{(-1)^n}{(2n+1)^2\pi^2}$$

So we have

$$x \sim \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left(\frac{(2n+1)\pi x}{2}\right).$$

See Figure 3 for the periodic extension to the entire line.

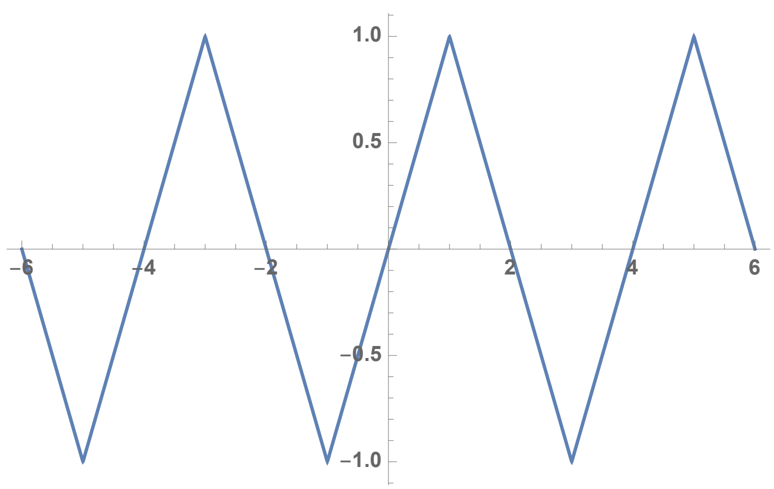

Figure 3: Half-integral sine Fourier series extension of $f(x) = x$.

2 Complex Fourier series

Recall that we can write sine and cosine in terms of complex exponentials as

$$\cos\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{n\pi i}{l}x} + e^{-\frac{n\pi i}{l}x}}{2}, \tag{21}$$

$$\sin\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{n\pi i}{l}x} - e^{-\frac{n\pi i}{l}x}}{2i}. \tag{22}$$

Using this decomposition we can perform a change of basis to rewrite the Fourier series for $f(x)$ (1) as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{n\pi i}{l}x} \tag{23}$$

where

$$c_0 = \frac{1}{2}a_0, \tag{24}$$

$$c_n = \frac{1}{2}(a_n - ib_n), \quad n = 1, 2, \dots \tag{25}$$

$$c_n = \frac{1}{2}(a_{-n} + ib_{-n}), \quad n = -1, -2, \dots \tag{26}$$

or generally:

$$c_n = \frac{1}{2l}\int_I f(x)e^{-\frac{n\pi i}{l}x}dx. \tag{27}$$

Parseval's equality becomes

$$\int_I |f(x)|^2 dx = 2l \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{28}$$

This is a decomposition of f with respect to the the orthogonal system

$$\left\{X_n(x) := e^{\frac{n\pi i}{l}x}\right\}_{n=\dots,-2,-1,0,1,2,\dots}. \tag{29}$$

Exercise: Show that

$$\int_I X_n(x)\overline{X_m(x)}dx = 2l\delta_{mn}.$$

Remark 2.1. Note that for complex Fourier series, we require that both

$$\sum_{n=0}^{\infty} c_n X_n(x) \quad \text{and} \quad \sum_{n=0}^{-\infty} c_n X_n(x)$$

converge, which is a stronger requirement than convergence of a trigonometric Fourier series. E.g. if f is a piecewise differentiable function, then the complex Fourier series for f converges only at points where f is continuous, *not* at discontinuities.

3 Integration and differentition of Fourier series

Finally, let's consider the following problem: suppose that we have a function $f(x)$ defined on the interval $[-l, l]$, with corresponding Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{\pi n x}{l}\right) + b_n \sin\left(\frac{\pi n x}{l}\right)\right).$$

When can we obtain the Fourier series for the functions

$$\int_{x_0}^{x} f(y)dy \quad \text{and} \quad f'(x)$$

via term-by-term integration or differentiation of the Fourier series for $f(x)$?

It turns out that term-by-term *integration* of a Fourier series is permissible. You should think that this is (roughly) because the series obtained via term-by-term integration has *better* convergence properties than the original series – in the n^{th} term integration brings down a factor of $\frac{1}{n}$, improving the convergence.

Example 4. We've seen previously that the Fourier series for the function $f(x) = x$ on the interval $[-l, l]$ is given by

$$x \sim \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right). \tag{30}$$

So to find the Fourier series for x^2 on the interval $[-l, l]$, we may integrate

$$\begin{aligned} x^2 &= 2\int_0^x t dt \\ &\sim 2\int_0^x \left(\frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi t}{l}\right)\right) dt \\ &= \frac{4l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^x \sin\left(\frac{n\pi t}{l}\right) dt \\ &= \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(\cos\left(\frac{n\pi x}{l}\right) - 1\right) \\ &= \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{l}\right) \end{aligned} \tag{31}$$

To evaluate the first term, we can calculate the a_0 Fourier term directly:

$$a_0 = \frac{1}{l} \int_{-l}^{l} x^2 dx = \frac{1}{l} \left[\frac{x^3}{3}\right]_{-l}^{l} = \frac{1}{l} \left(\frac{l^3}{3} + \frac{l^3}{3}\right) = \frac{2}{3} l^2$$

So using $x^2 \sim \frac{1}{2} a_0 + \cdots$ we find:

$$x^2 \sim \frac{1}{3} l^2 + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{l}\right) \tag{32}$$

Now, let's consider the more subtle question of term-by-term differentiation. First, let's assume that $f(x)$ is once differentiable, so that $f'(x)$ exists and is continuous. Let the Fourier coefficients of $f(x)$ be denoted a_n, b_n. Then write the Fourier series for $f'(x)$ as

$$f'(x) \sim \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right) \tag{33}$$

We can calculate these Fourier coefficients:

$$A_0 = \frac{1}{l}\int_{-l}^{l} f'(x)dx = \frac{f(l) - f(-l)}{l}$$

$$A_n = \frac{1}{l}\int_{-l}^{l} f'(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right]_{-l}^{l} + \frac{n\pi}{l^2}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{f(l) - f(-l)}{l}\cos(n\pi) + \frac{n\pi}{l}b_n$$

$$B_n = \frac{1}{l}\int_{-l}^{l} f'(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right]_{-l}^{l} - \frac{n\pi}{l^2}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = -\frac{n\pi}{l}a_n$$

From this we see: *If $f(x)$ is once (continuously) differentiable on $[-l, l]$, the Fourier series for $f'(x)$ may be obtained from the Fourier series of $f(x)$ via term-by-term differentiation if and only if $f(l) = f(-l)$.*

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.