APM 346 Lecture 11.

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Last week we introduced the method of separation of variables and the corresponding eigenvalue problems. This week we will detour to explore the topic of Fourier series, which we will need to finish solving our IBVPs using the method of separation of variables. References being used: [IvrXX, §4.3-§4.5] (§4.3,§4.4,§4.5) and [Str08, Ch.5].

Five minute review exercise for the start of class: Calculate

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(n\theta)\cos(m\theta), \quad n, m \in \mathbb{Z}.$$

1 Orthogonality of eigenfunctions

At the end of last lecture we claimed that

$$\langle f, g\rangle = \int_I f(x)\overline{g(x)}dx \tag{1}$$

defined an inner product on the space of (real or complex valued) functions on I (for real functions the complex conjugation is unnecessary). Recall that an *inner product* on a vector space V is a map

$$\langle -, -\rangle : V \times V \to \mathbb{R} \quad \text{or} \quad \mathbb{C}$$

satisfying:

(a) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$

(b) $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $u, v \in V$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$

(c) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$

(d) $\|u\|^2 := \langle u, u\rangle \geq 0$ for all $u \in V$, with equality if and only if $u = 0$

As an exercise you should check that (1) really does define an inner product.

The claim was that we could use the fact that the eigenfunction solutions for our various ODE eigenvalue problems form (the infinite dimensional version of) an orthogonal basis for the space of solutions to solve our original IBVP for the wave equation.

Let's begin by checking orthogonality of the eigenfunctions in various situations.

1.1 Orthogonality for Robin boundary conditions

Observe the following fact that we discovered (implicitly, and not entirely rigorously) when exploring Robin boundary conditions: every eigenvalue λ_n corresponds to only a *single* eigenfunction X_n. Recall that λ_n and

X_n are related by the expression

$$-X_n'' = \lambda_n X_n \tag{2}$$

On the interval $0 < x < l$, Robin BCs are defined by

$$X'(0) - \alpha X(0) = 0 \tag{3}$$
$$X'(l) + \beta X(l) = 0 \tag{4}$$

where $\alpha, \beta \in \mathbb{R}$. Suppose that X and Y are two functions satisfying (3) and (4). Calculate the following inner products:

$$\begin{aligned}
\langle X'', Y\rangle &= \int_0^l X''(x)\overline{Y(x)}dx \\
&= -\int_0^l X'(x)\overline{Y'(x)}dx + X'(l)\overline{Y(l)} - X'(0)\overline{Y(0)} \\
&= -\langle X', Y'\rangle - \beta X(l)\overline{Y(l)} - \alpha X(0)\overline{Y(0)} && (5) \\
\langle X, Y''\rangle &= \int_0^l X(x)\overline{Y''(x)}dx \\
&= -\int_0^l X'(x)\overline{Y'(x)}dx + X(l)\overline{Y'(l)} - X(0)\overline{Y'(0)} \\
&= -\langle X', Y'\rangle - \overline{\beta} X(l)\overline{Y(l)} - \overline{\alpha} X(0)\overline{Y(0)} && (6)
\end{aligned}$$

Now, since $\alpha, \beta \in \mathbb{R}$, equations (5) and (6) are in fact equal:

$$\langle X'', Y\rangle = \langle X, Y''\rangle \tag{7}$$

Now, suppose that X is an eigenfunction with eigenvalue λ. Then from (7), setting $Y = X$, we have

$$-\lambda\|X\|^2 = -\overline{\lambda}\|X\|^2,$$

and since $\|X\|^2 \neq 0$, $\lambda = \overline{\lambda}$ – i.e. all of the eigenvalues must be real.

Next, suppose that $X = X_n$ and $Y = X_m$ are *distinct* eigenfunctions with eigenvalues λ_n and λ_m. By (2) we have that

$$-\lambda_n\langle X_n, X_m\rangle = -\lambda_m\langle X_n, X_m\rangle,$$

and since $\lambda_n \neq \lambda_m$, we have that

$$\langle X_n, X_m\rangle = 0.$$

Hence we have shown orthogonality for Robin BCs.

Remark 1.1. Take note of what was really crucial in our argument:

- $\langle X'', Y\rangle = \langle X, Y''\rangle$ was used to show that all eigenvalues were real.
- The fact that every eigenvalue corresponded to a unique eigenfunctions was used to show that distinct eigenfunctions must be orthogonal.

1.2 Orthogonality for periodic boundary conditions

Now, consider the ODE

$$X'' + \lambda X = 0 \tag{8}$$

on the domain $-l < x < l$, with periodic BCs

$$X(-l) = X(l) \tag{9}$$
$$X'(-l) = X'(l) \tag{10}$$

In Lecture 9 we saw that the eigenfunctions for this problem were given by

$$\left\{\frac{1}{2}, C_n(x) := \cos\left(\frac{n\pi x}{l}\right), S_n(x) := \sin\left(\frac{n\pi x}{l}\right)\right\}_{n=1,2,\dots}, \tag{11}$$

with eigenvalues

$$\lambda_0 = 0 \qquad \text{corresponding to } \frac{1}{2}, \tag{12}$$
$$\lambda_n = \frac{n^2\pi^2}{l^2} \qquad \text{corresponding to } S_n(x) \text{ and } C_n(x). \tag{13}$$

We would like to see that the eigenfunctions (11) are orthogonal. We still have (7), so our argument that eigenfunctions corresponding to *distinct* eigenvalues are orthogonal still holds. It remains to check the orthogonality of C_n and S_n; while we're at it, we will calculate the norm $\|X\| = \sqrt{\langle X, X\rangle}$ of each of the eigenfunctions.

Since sin is an odd function and cos is an even function, $S_n(x)C_n(x)$ is an odd function. We therefore have

$$\begin{aligned}
\langle S_n(x), C_n(x)\rangle &= \int_{-l}^{l} S_n(x)C_n(x)dx \\
&= \int_0^l S_n(x)C_n(x)dx + \int_{-l}^0 S_n(x)C_n(x)dx \\
&= \int_0^l S_n(x)C_n(x)dx + \int_l^0 S_n(-y)C_n(-y)(-dy) \\
&= \int_0^l S_n(x)C_n(x)dx - \int_0^l S_n(y)C_n(y)dy = 0
\end{aligned}$$

To calculate the norms of the eigenfunctions, we will make use of the trig identities

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$$
$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$$

We have:

$$\begin{aligned}
\langle \frac{1}{2}, \frac{1}{2}\rangle &= \frac{1}{4}\int_{-l}^{l} dx = \frac{l}{2} \\
\langle S_n(x), S_n(x)\rangle &= \int_{-l}^{l} \sin^2\left(\frac{n\pi x}{l}\right) dx \\
&= \frac{1}{2}\int_{-l}^{l} \left(1 - \cos\left(\frac{2n\pi x}{l}\right)\right) \\
&= l - \frac{l}{2n\pi}\left[\sin\left(\frac{2n\pi}{l}\right)\right]_{-l}^{l} = l \\
\langle C_n(x), C_n(x)\rangle &= \int_{-l}^{l} \cos^2\left(\frac{n\pi x}{l}\right) dx \\
&= \frac{1}{2}\int_{-l}^{l} \left(1 + \cos\left(\frac{2n\pi x}{l}\right)\right) = l
\end{aligned}$$

So: assuming that every periodic function may be expanded in a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right), \tag{14}$$

we can read off the coefficients as:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \tag{15}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \tag{16}$$

For instance:

$$\begin{aligned}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx &= \langle f, C_n\rangle \\ &= \langle \frac{1}{2}, C_n\rangle a_0 + \sum_{m=1}^{\infty}\langle C_m, C_n\rangle a_m + \sum_{m=1}^{\infty}\langle S_m, C_n\rangle b_m \\ &= 0 + la_n + 0 = la_n.\end{aligned}$$

Note that we also have the following expression for the norm of a periodic function in terms of its Fourier coefficients:

Proposition 1.1 (Parseval's Theorem)**.** *If $f(x)$ has Fourier decomposition* (14)*, it's norm is given by*

$$\|f\|^2 = \int_{-l}^{l}|f(x)|^2dx = \frac{l}{2}|a_0|^2 + \sum_{n=1}^{\infty} l\left(|a_n|^2 + |b_n|^2\right). \tag{17}$$

Remark 1.2. The results above in fact hold for periodic functions on any interval of length $2l$, not just the special case of the interval $[-l, l]$.

Example 1. Let's compute the Fourier series for the absolute value function $f(x) = |x|$. Since $|x|$ is an even function, we automatically have that there will be no contributing sin terms, so that

$$b_n = 0 \quad \text{for all } n.$$

The other coefficients are given by

$$\begin{aligned}a_0 &= \frac{1}{l}\int_{-l}^{l}|x|dx = \frac{2}{l}\int_0^l xdx = \frac{2}{l}\frac{l^2}{2} = l \\ a_n &= \frac{1}{l}\int_{-l}^{l}|x|\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2l}{n^2\pi^2}\left((-1)^n - 1\right)\end{aligned}$$

So with the exception of a_0 all of the a_{2k} terms vanish, and we conclude that the Fourier series for $|x|$ on the domain $-l < x < l$ is

$$\frac{l}{2} - \frac{4l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\left(\frac{(2k-1)\pi x}{l}\right). \tag{18}$$

2 Fourier series

Although we have stated the results of Section 1.2 only for periodic functions, in fact we can write down a Fourier series for any integrable function on $[-l, l]$:

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right), \tag{19}$$

where a_n and b_n are defined as in (15) and (16):

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

In (19) we have written $f(x)$ is "$\sim$" its Fourier series rather than equal to it. This is quite deliberate: the question of what a Fourier series converges to (or indeed, if it converges at all!) is a rather subtle question, which we will not be able to cover in any detail.

2.1 Convergence of Fourier series

For an example of the sorts of subtleties one might encounter, consider the following:

Theorem 2.1 (Carleson's Theorem)**.** *If $f(x)$ is "square integrable", i.e.*

$$\int_{-l}^{l} |f(x)|^2 dx < \infty,$$

then the Fourier series of f converges to f pointwise almost everywhere:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right) \text{ for almost every } x \in [-l, l].$$

Remark 2.1. Although we will not discuss this in detail, it is worth making a few remarks on Theorem 2.1:

- The terminology "almost everywhere" has a mathematically precise meaning, but for the purpose of this class whatever intuition you have is probably fine. In particular, this theorem does not tell us whether the Fourier series agrees with the original function at any *particular* point of interest $x \in [-l, l]$.
- This theorem turns out to be very hard to prove. To give you some idea of *how* hard: Fourier introduced the notion of a Fourier series in the early 19th century, the statement of the theorem was conjectured (by Luzin) in the early 20th century, and it was proved by Carleson in 1966.

More useful for us is the following theorem (which we will also not prove – see [IvrXX, §4.4, Thm.2] for more information):

Theorem 2.2. *Let $f(x)$ be a periodic function of period $2l$, and suppose that f satisfies the* Dirichlet conditions *on $(-l, l)$:*

(i) f is bounded on $(-l, l)$, i.e. there is some constant M such that $|f(x)| < M$ for all $x \in (-l, l)$.

(ii) f has at most finitely many discontinuities on $(-l, l)$.

(iii) f has a finitely many maxima and minima on $(-l, l)$.

Then at every point x the Fourier series of f converges to

$$\frac{f(x+) + f(x-)}{2}. \tag{20}$$

Here, $f(x\pm) := \lim_{y\to x^\pm} f(y)$ are the right and left limits of f at x.

Remark 2.2. What does Theorem 2.2 mean concretely, on the interval $[-l, l]$?

- If f is continuous at $x \in (-l, l)$, then at x the Fourier series converges to $f(x)$.

- If f is discontinuous at $x \in (-l, l)$, then at x the Fourier series converges to $\frac{f(x+)+f(x-)}{2}$.
- At the end points $x = \pm l$ the Fourier series converges to $\frac{f(-l+)+f(l-)}{2}$.

Example 2. There are plenty of functions that you could cook up that fail to satisfy the Dirichlet conditions (e.g. choose some non-bounded function such as $\frac{1}{x}$). An interesting example is given by $x^2 \sin\left(\frac{1}{x}\right)$, which fails because it has infinitely many minima and maximal near $x = 0$.

Example 3. Let us find the Fourier series for the *sawtooth wave*, defined on the interval $(-l, l)$ by the function $f(x) = x$. Since the function f is odd, all of the a_n coefficients must vanish. The b_n coefficients may be calculated to be

$$\begin{aligned} b_n &= \frac{1}{l}\int_{-l}^{l} x \sin\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2}{l}\int_{0}^{l} x \sin\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2}{n\pi}\left(\left[-x\cos\left(\frac{n\pi x}{l}\right)\right]_0^l + \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx\right) \\ &= \frac{2}{n\pi}\left(-l\cos(n\pi) + \frac{l}{n\pi}\left[\sin\left(\frac{n\pi x}{l}\right)\right]_0^l\right) \\ &= -\frac{2l}{n\pi}\cos(n\pi) = \frac{2l}{n\pi}(-1)^{n+1} \end{aligned}$$

So the Fourier series for the sawtooth wave is

$$x \sim \frac{2l}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right). \tag{21}$$

Now, let's compare this expression for the Fourier series with the results of Theorem 2.2. First, note that at the end points the theorem says that the Fourier series should be equal to

$$\frac{-l+l}{2} = 0,$$

and since $\sin(\pm n\pi) = 0$ for all n, this is indeed true. Next, at the internal point $x = \frac{l}{2}$, Theorem 2.2 tells us that the Fourier series converges to the value of the original function, so that

$$\frac{l}{2} = \frac{2l}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right).$$

For even n $\sin\left(\frac{n\pi}{2}\right) = 0$, so we can simplify this sum into a sum over only odd numbers $n = 2m+1$, $m \geq 0$. We have

$$\sin\left(\frac{(2m+1)\pi}{2}\right) = (-1)^m,$$

and so we can write

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^{2m+2}}{2m+1}(-1)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{22}$$

2.2 Superposition of Fourier series

Note from the definition of the Fourier series that: *the Fourier series for the sum of two functions is the sum of their respective Fourier series.* We can sometimes make use of this to simplify our calculations:

Example 4. Suppose we want to find the Fourier series for the function

$$f(x) = \begin{cases} 0, & -l < x < 0 \\ x, & 0 < x < l \end{cases}$$

We *could* just plug this function into our formulae for the Fourier coefficients (15) and (16) – *or*, we could notice that

$$f(x) = \frac{x + |x|}{2},$$

for which we have already found the Fourier series

$$|x| \sim \frac{l}{2} - \frac{4l}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\left(\frac{(2k-1)\pi x}{l}\right),$$

$$x \sim \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right).$$

Hence the Fourier series for $f(x)$ is

$$f(x) \sim \frac{l}{4} - \frac{2l}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\left(\frac{(2k-1)\pi x}{l}\right) + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right).$$

2.3 Gibbs Phenomenon

We'll end this lecture by returning to the question of convergence of Fourier series. Now, however, we are interested not in the question of *what* the Fourier series converges to, but *how* it converges.

What do we mean by this (and why do we care)? Suppose that we are trying to write down and plot the Fourier series representation of a function in the real world. For practical purposes, we must truncate the Fourier series to some finite number of terms. We would hope, then, that by simply taking more and more terms we could calculate and plot arbitrarily good approximations to our original function.

There is some sense in which this is true: a *finite Fourier series* for $f(x)$,

$$S_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right) \tag{23}$$

minimizes the *squared error*

$$E = \int_{-l}^{l} |f(x) - S_N(x)|^2 dx \tag{24}$$

and so in this sense the finite Fourier series is a good approximation.

In the presence of discontinuities, however, we must be careful.

Example 5. Let's consider what happens when we take finite Fourier series approximations for the sawtooth wave (21). In Figure 1 we can see that as we take more and more terms in the finite Fourier series, the sawtooth wave becomes better approximated almost everywhere.

However suppose now we zoom in on what is happening near the discontinuity, where the sawtooth wave jumps from $+1$ to -1 (Figure 2). As we take more and more terms in our approximation, the amount that our approximation overshoots the value of our original function does not seem to be improving! Instead, the location of the overshoot simply moves closer and closer to the point of discontinuity.

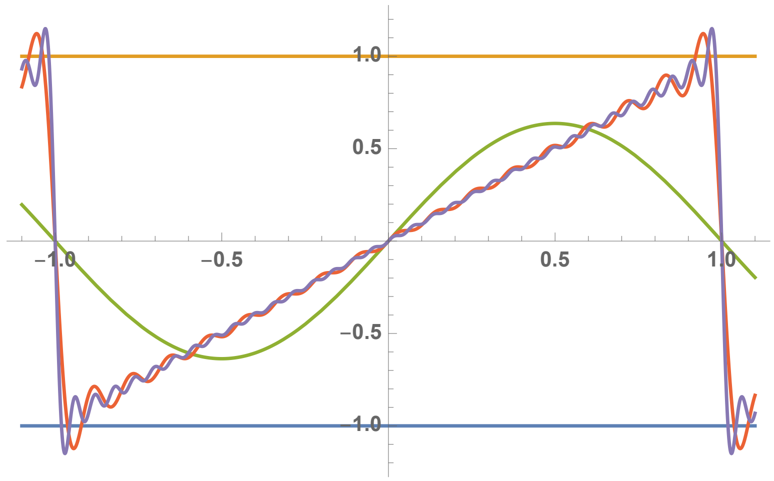

Figure 1: Finite Fourier series approximating the sawtooth wave.

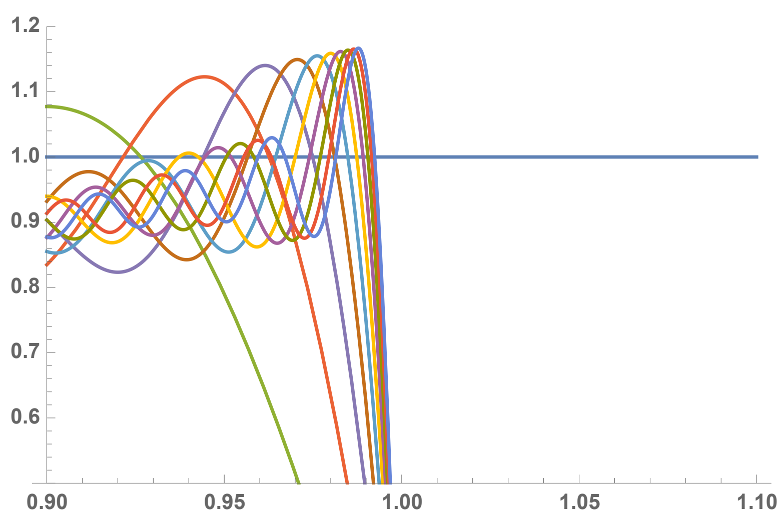

Figure 2: Overshoot of the finite Fourier series at the discontinuity.

The behaviour observed in Example 5 is common to finite Fourier approximations to discontinuous functions, and is known as *Gibbs' phenomenon*: for large N the finite Fourier series $S_N(x)$ overshoots the function $f(x)$ by approximately 9% the size of the discontinuity.

Discuss: Why does Gibbs' phenomenon not contradict what we have already learned about convergence of Fourier series?

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.