APM 346 Lecture 10.

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We continue to study the wave equation on a finite interval via the method of separation of variables and Fourier series.

References being used: [IvrXX, §4.2], §4.2; [Str08, Ch.4].

1 Robin boundary conditions (continued)

Last lecture we finished by solving the ODE

$$X'' + \lambda X = 0, \tag{1}$$

on the interval $0 < x < 1$, subject to the Dirichlet-Robin boundary condition

$$X(0) = 0, \quad X'(1) = -\beta X(1). \tag{2}$$

We found that the eigenvalues and eigenfunctions were given by:

- **If** $\beta > -1$**:**
  - Eigenvalues $\lambda_n = \omega_n^2$, where either $n > 0$ or $n \geq 0$, with trigonometric eigenfunctions $\sin(\omega_n x)$.
- **If** $0 > \beta > -1$**:**
  - Eigenvalues $\lambda_n = \omega_n^2$ where $n \geq 0$, with trigonometric eigenfunctions $\sin(\omega_n x)$.
- **If** $\beta = -1$**:**
  - Eigenvalues $\lambda_n = \omega_n^2$ where $n > 0$, with trigonometric eigenfunctions $\sin(\omega_n x)$.
  - The zero eigenvalue, with linear eigenfunction x.
- **If** $\beta < -1$**:**
  - Eigenvalues $\lambda_n = \omega_n^2$ where $n > 0$, with trigonometric eigenfunctions $\sin(\omega_n x)$.
  - A unique negative eigenvalue $\lambda_- = -\omega_-^2$, with hyperbolic eigenfunction $\sinh(\omega_- x)$.

1.1 Robin-Robin boundary conditions

Now, let's consider solving (1) on $0 < x < 1$, subject to the Robin-Robin BCs

$$X'(0) = \alpha X(0), \quad X'(1) = -\beta X(1). \tag{3}$$

We place the following constraints on α and β:

$$\alpha \neq 0, -1 \tag{4}$$
$$\beta \neq 0, -1 \tag{5}$$
$$\alpha\beta \neq \left(\frac{2n+1}{2}\pi\right)^2, \quad \text{any } n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{6}$$

As an exercise, you are invited to consider what happens when the above constraints are relaxed.

We will need to consider the possibility of positive, negative, and null eigenvalues.

1.1.1 Null eigenvalue

First, let's deal with the case $\lambda = 0$. Then

$$X(x) = Ax + B$$
$$X'(x) = A$$

Applying the BCs (3) gives

$$A = \alpha B$$
$$A = -\beta(A + B)$$

Substituting $A = \alpha B$ into the second equation and rearranging terms, we find:

$$(\alpha + \beta + \alpha\beta)B = 0.$$

If there is a nontrivial eigenfunction, we must therefore have $\alpha + \beta + \alpha\beta = 0$.

Upshot: If $\alpha = \frac{-\beta}{1+\beta}$ there is a linear eigenfunction

$$X_{\text{lin}}(x) = \alpha x + 1 \tag{7}$$

with eigenvalue zero.

1.1.2 Positive eigenvalues

Now let's look for positive eigenvalues. Set $\lambda = \omega^2$, $\omega > 0$. Then (1) is solved by

$$X(x) = A\sin(\omega x) + B\cos(\omega x),$$
$$X'(x) = \omega A\cos(\omega x) - \omega B\sin(\omega x).$$

Imposing (3) gives the system of equations

$$\omega A = \alpha B$$
$$\omega A\cos(\omega) - \omega B\sin(\omega) = -\beta(A\sin(\omega) + B\cos(\omega))$$

Since $\omega \neq 0$, we can substitute $A = \frac{\alpha}{\omega}B$ into the second equation to obtain

$$\alpha B\cos(\omega) - \omega B\sin(\omega) = -\beta\frac{\alpha B}{\omega}\sin(\omega) - \beta B\cos(\omega).$$

The possibility $\cos(\omega) = 0$ is excluded by (6) (Exercise: Why?), so we may rearrange this equation to find that there is a nontrivial solution for those ω solving the equation

$$\tan(\omega) = \frac{(\alpha + \beta)\omega}{\omega^2 - \alpha\beta}. \tag{8}$$

Much as last time, we can obtain information about the eigenvalues by inspecting the intersections of $\tan(\omega)$ and $\frac{(\alpha+\beta)\omega}{\omega^2-\alpha\beta}$ – see Figure 1 for some (not all!) of the possible cases. Visually it is clear that (at least generically) there is a single solution ω_n within each region $\frac{2n-1}{2}\pi < \omega < \frac{2n+1}{2}\pi$, with the possible exception of $n = 0$ (Exercise: When does the $n = 0$ eigenvalue exist? Compare with the Dirichlet-Robin case studied Lecture 9.)

1.1.3 Negative eigenvalues

Finally, let's look for negative eigenvalues. Set $\lambda = -\omega^2$, $\omega > 0$ so that the solution to (1) is given by

$$\begin{aligned} X(x) &= A\sinh(\omega x) + B\cosh(\omega x) \\ X'(x) &= A\omega\cosh(\omega x) + B\omega\sinh(\omega X) \end{aligned}$$

Applying $X'(0) = \alpha X(0)$ gives

$$A = \frac{\alpha}{\omega}B$$

so that the solution becomes

$$\begin{aligned} X(x) &= \frac{B}{\omega}\left(\alpha\sinh(\omega x) + \omega\cosh(\omega x)\right) \\ X'(x) &= B(\alpha\cosh(\omega x) + \omega\sinh(\omega x)) \end{aligned}$$

Imposing $X'(1) = -\beta X(1)$ and performing the same manipulations as in Section 1.1.2 (left as an exercise), we see that a nontrivial solution exists for those ω solving the equation

$$\tanh(\omega) = -\frac{(\alpha+\beta)\omega}{\omega^2+\alpha\beta}. \tag{9}$$

See Figures 2 and 3 for some representative example plots of (9) which show that there can be either 0, 1, or 2 negative eigenvalues, depending on the values of α and β.

We'll stop here having looked at these examples. For a more complete analysis, consult [IvrXX, §4.2] (§4.2) or [Str08, Ch.4.3].

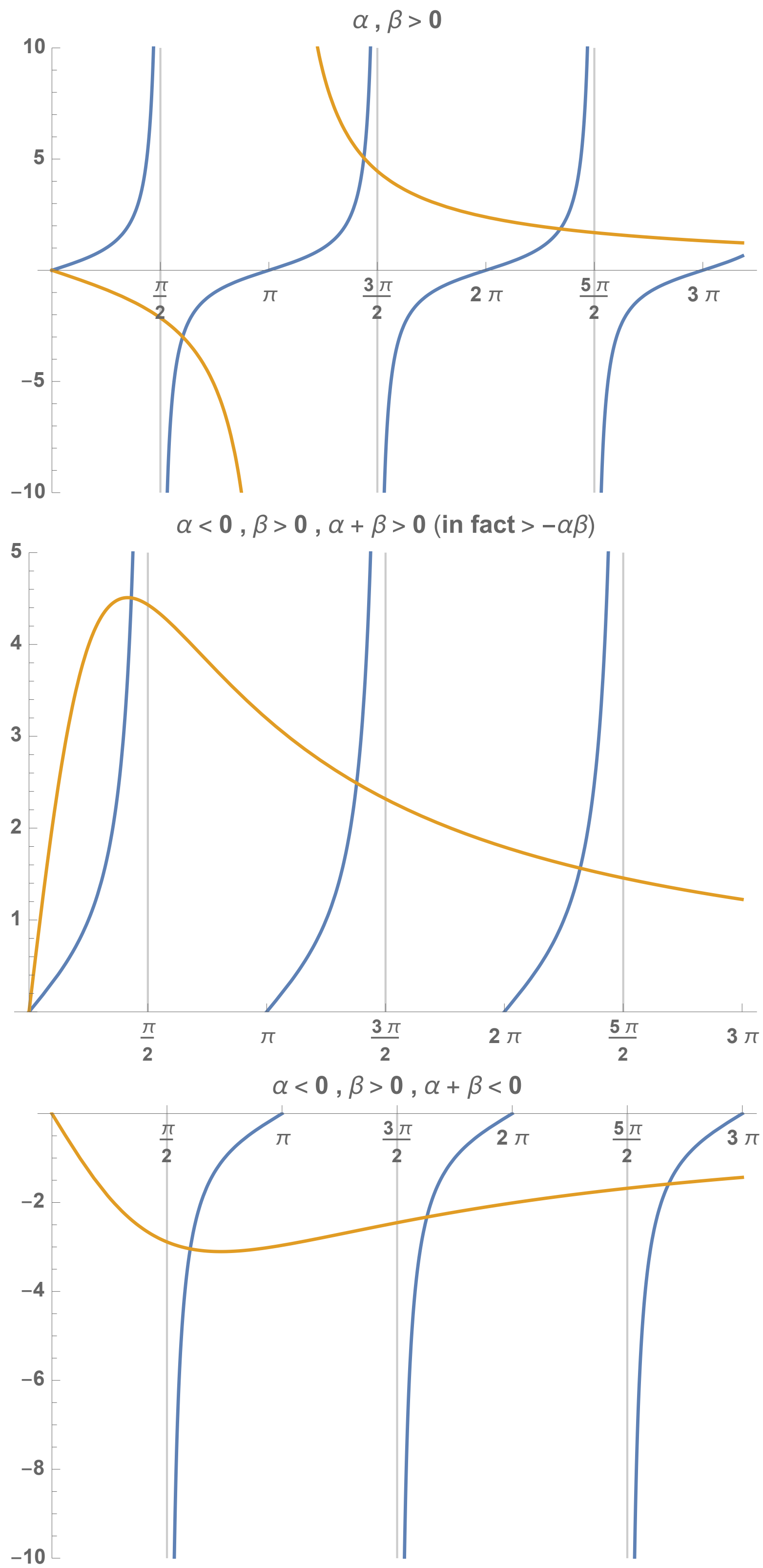

Figure 1: Solving (8) for some different values of α, β.

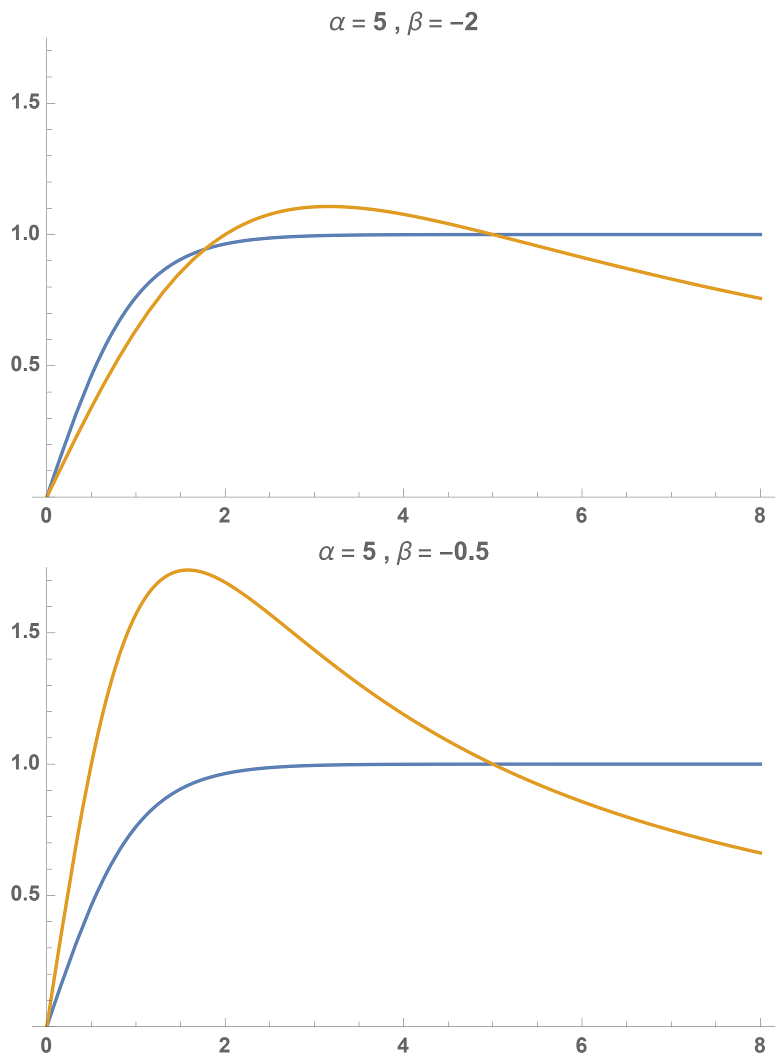

Figure 2: Sample plots of (9) with $\alpha = 5$.

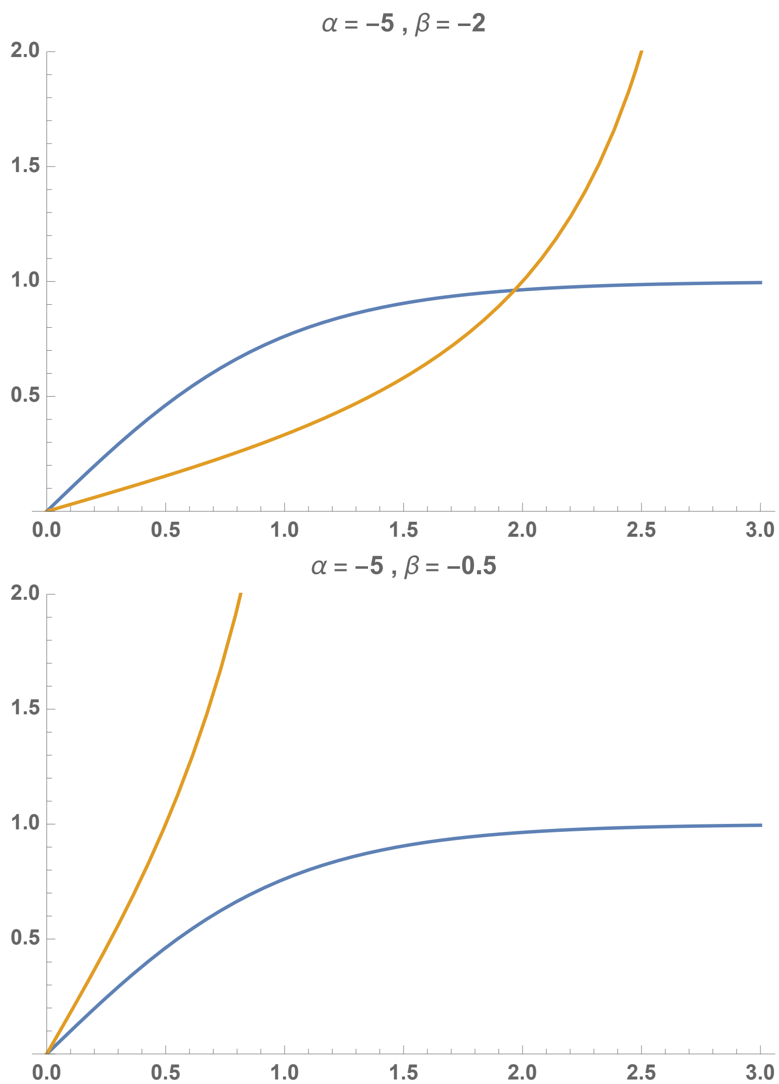

Figure 3: Sample plots of (9) with $\alpha = -5$.

2 An inner product between functions (preview of next lecture)

Recall that last lecture we constructed a *Fourier series* solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < l, & \text{PDE} \\ u(0,t) = 0, & & \text{BC1} \\ u(l,t) = 0, & & \text{BC2} \\ u(x,0) = g(x), & & \text{IC1} \\ u_t(x,0) = h(x), & & \text{IC2} \end{cases} \tag{10}$$

the wave equation on $0 < x < l$ with Dirichlet-Dirichlet boundary conditions, which was of the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{cn\pi}{l} t\right) + B_n \sin\left(\frac{cn\pi}{l} t\right)\right) \cdot \sin\left(\frac{n\pi}{l} x\right). \tag{11}$$

The different boundary conditions we studied last lecture and this lecture will lead to similar Fourier series type solutions, involving the eigenfunctions for the particular BC we wish to study.

I made two claims:

(1) Every solution to (10) is of the form (11). We will not (fully) justify this claim.

(2) The coefficients A_n and B_n in (11) that will solve the IBVP (10) may be explicitly calculated using the initial conditions $g(x)$ and $h(x)$.

These claims will follow from the following facts:

(A) The collection of functions on $0 < x < l$ satisfying Dirichlet-Dirichlet BCs is an (infinite dimensional) vector space. (Exercise.)

(B) The pairing between functions f and g on $0 < x < l$ given by

$$\langle f, g \rangle = \frac{1}{l} \int_0^l f(x) \overline{g(x)} dx \tag{12}$$

defines an inner product on this vector space, where $\overline{a + ib} = a - ib$ is complex conjugation.

(C) The collection of functions $\left\{\sin\left(\frac{n\pi}{l} x\right)\right\}_{n=1,2,\ldots}$ is an (infinite dimensional version of an) orthonormal basis for this vector space with respect to the inner product (12).

To see why these facts should imply the claims, recall what happens for a finite dimensional vector space V:

(1) If $\{v_1, \ldots, v_n\}$ is a basis for V then every vector $\vec{v} \in V$ has a unique expression as a linear combination of the v_i:

$$\vec{v} = A_1 v_1 + \cdots + A_n v_n.$$

(2) If the basis is orthonormal with respect to an inner product $\langle -, - \rangle$, so that $\langle v_i, v_j \rangle = \delta_{ij}$, the coefficients of the v_k in $\vec{v}$ may be recovered as $A_k = \langle \vec{v}, v_k \rangle$.

We'll explore these ideas further next week.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.