APM 346 Lecture 9.

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This week we return to the wave equation, only now we wish to look for solutions on a finite interval. We will tackle this problem by introducing the method of separation of variables and Fourier series.

References being used: [IvrXX, §4.1-2] §4.1, §4.2; [Str08, Ch.4].

1 The wave equation on a finite interval

To begin with, let's consider the IBVP for the (homogeneous) 1d-wave equation on $(0, l)$ with homogeneous Dirichlet boundary conditions:

$$\left\{ \begin{array}{rcll|l} u_{tt} - c^2 u_{xx} & = & 0, & 0 < x < l, & \text{PDE} \\ u(0,t) & = & 0, & & \text{BC1} \\ u(l,t) & = & 0, & & \text{BC2} \\ u(x,0) & = & g(x), & & \text{IC1} \\ u_t(x,0) & = & h(x). & & \text{IC2} \end{array} \right. \tag{1}$$

1.1 Separation of variables

To begin with, let us not worry about the initial conditions, and just consider the PDE and BCs from (1). We will attempt to tackle this problem via the method of *separation of variables*. This method proceeds as follows:

(1) We search for *separated solutions* U to our problem which factor into functions of each individual variable, i.e. which are of the form $U(x_1, \ldots, x_N) = f_1(x_1) \cdots f_N(x_N)$.

(2) We use linearity of our differential equation to construct more general solutions as (infinite) sums of the special solutions.

(3) We prove that in fact *every* solution may be written as such an (infinite) sum.

To implement this, let us search for solutions to the BVP component of (1) of the form

$$u(x,t) = X(x)T(t), \tag{2}$$

where $X(x)$ is a function on $(0, l)$ and $T(t)$ is a function on $(-\infty, \infty)$. We want to find non-trivial solutions – solutions where u is not identically zero – which implies that neither of $X(x)$ or $T(t)$ may be identically zero.

Substituting in (2) to the PDE and BC parts of (1) we find

$$X(x)T''(t) = c^2 X''(x)T(t), \tag{3}$$

$$X(0)T(t) = X(l)T(t) = 0, \tag{4}$$

and so dividing through by $X(x)T(t)$ and $T(t)$ respectively, we obtain

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)}, \tag{5}$$

$$X(0) = X(l) = 0. \tag{6}$$

But now we make the crucial observation:

- The LHS of (5) is purely a function of t – it has no dependence on x.
- The RHS of (5) is purely a function of x – it has no dependence on t.
- Therefore, as these expressions are equal, they must depend on neither x nor t – and so they must be constant!

Using this assumption we may rewrite (5) as

$$\frac{X''(x)}{X(x)} = -\lambda, \tag{7}$$

$$\frac{T''(t)}{T(t)} = -c^2\lambda, \tag{8}$$

and so we have turned our PDE problem into a system of ODEs:

$$X'' + \lambda X = 0, \tag{9}$$

$$X(0) = X(l) = 0, \tag{10}$$

$$T'' + c^2\lambda T = 0. \tag{11}$$

1.2 The eigenvalue problem

Let us consider the ODE BVP (9)-(10). We wish to find a nontrivial (not identically zero) solution $X(x)$. This is an example of an *eigenvalue problem.*

Definition 1.1. A solution $X(x)$ to (9)-(10) is called an *eigenfunction* with *eigenvalue* λ.

Where is this terminology coming from? Cast your mind back to linear algebra: suppose that A is an $n \times n$ matrix, and $\vec{v}$ is an n-component vector. Then we call $\vec{v}$ an *eigenvector of A with eigenvalue μ* if $A(\vec{v}) = \mu\vec{v}$.

Compare this with our situation: we have a linear (differential) operator $L := -\frac{d^2}{dx^2}$, which sends a function of x to minus its second derivative. We can add functions, and multiply them by scalars[1] – so functions form a vector space. So an eigenvector for the linear operator L with eigenvalue λ would be a *function* X which satisfies

$$L[X] = \lambda X. \tag{12}$$

But this is exactly (9)! So our eigenvalue problem *really is* just a familiar problem from linear algebra – albeit appearing in a potentially unfamiliar context.

Proposition 1.1. *The eigenvalues and eigenfunctions for* (9)-(10) *are given by*

$$\lambda_n = \frac{\pi^2 n^2}{l^2}, \quad n = 1, 2, \ldots, \tag{13}$$

$$X_n(x) = \sin\left(\frac{\pi n x}{l}\right). \tag{14}$$

[1] There is of course the question of exactly what types of functions we wish to consider. Any "reasonable" choice for us will furnish a vector space.

Proof. Let's consider two different approaches to solving this:

Approach 1: Extreme generality. Recall that to solve a general 2nd order linear ODE with constant coefficients, one makes the educated guess (based on the form of the ODE) that solutions will be of the form e^{kx} for some constants k. Plugging this into our ODE, we find that $(k^2+\lambda)e^{kx}=0$, which can only be satisfied if the the *characteristic equation*

$$k^2+\lambda=0 \tag{15}$$

is satisfied, i.e. $k_\pm=\pm\sqrt{-\lambda}$. So, provided $\lambda\neq 0$, the most general solution to our ODE will be

$$X(x)=Ae^{\sqrt{-\lambda}x}+Be^{-\sqrt{-\lambda}x}. \tag{16}$$

Applying our boundary conditions $X(0)=X(l)=0$ gives the system of linear equations

$$\left\{\begin{array}{rcl} A+B &=& 0,\\ Ae^{\sqrt{-\lambda}l}+Be^{-\sqrt{-\lambda}l} &=& 0,\end{array}\right. \tag{17}$$

and a non-trivial solution $(A,B)\neq(0,0)$ to this exists if and only if its determinant is zero. We calculate

$$\begin{gathered}\det\begin{pmatrix}1 & 1\\ e^{\sqrt{-\lambda}l} & e^{-\sqrt{-\lambda}l}\end{pmatrix}=e^{-\sqrt{-\lambda}l}-e^{\sqrt{-\lambda}l}=0\\ \text{iff}\\ e^{2\sqrt{-\lambda}l}=1\\ \text{iff}\\ 2\sqrt{-\lambda}l=2\pi in,\ \text{for } n=1,2,\ldots\end{gathered}$$

We exclude $n=0$ since that would lead to the excluded $\lambda=0$ case, and we exclude $n<0$ since those values give the same eigenvalues and eigenfunctions as $n>0$. So, we find that

$$\begin{aligned}\lambda_n&=\frac{\pi^2n^2}{l^2},\\ (k_n)_\pm&=\pm\frac{n\pi i}{l}.\end{aligned}$$

Further, from (17) we see that $B=-A$, and so substituting in the derived formula for k_n we obtain

$$X(x)=2Ai\sin\left(\frac{\pi nx}{l}\right)$$

as desired.

It remains to treat the $\lambda=0$ case separately. Here, $X(x)=A+Bx$, so applying our BCs we obtain the linear system

$$\left\{\begin{array}{rcl} A &=& 0,\\ A+Bl &=& 0,\end{array}\right.$$

which leads only to the trivial solution $A=B=0$. So $\lambda=0$ is not an eigenvalue.

Approach 2: Case analysis. Alternately, we can make use of the fact (from a previous calculus or ODEs class) that we already know the solutions to (9) – we are looking for functions which are proportional to their second derivatives, and depending on the sign of λ, they will either be the trigonometric functions sin and cos, of the hyperbolic-trigonometric functions sinh and cosh. Let us proceed case-by-case.

- $\lambda=0$: we have already dealt with.

- $\lambda < 0$: Set $\lambda = -\omega^2$, $\omega > 0$, so we are solving $X'' - \omega^2 X = 0$. This has general solution

$$X(x) = A\sinh(\omega x) + B\cosh(\omega x).$$

Applying $X(0) = 0$ now gives $B = 0$, and applying $X(l) = 0$ gives

$$A\sinh(\omega l) = 0$$

which implies $A = 0$ (since sinh vanishes only at the origin). So there are no nontrivial solutions for $\lambda < 0$.

- $\lambda > 0$: Set $\lambda = \omega^2$, $\omega > 0$, so we are solving $X'' + \omega^2 X = 0$. This has general solution

$$X(x) = A\sin(\omega x) + B\cos(\omega x).$$

Applying $X(0) = 0$ again gives $B = 0$, and applying $X(l) = 0$ gives

$$A\sin(\omega l) = 0$$

which implies that $\omega l = n\pi$ for $n = 1, 2, \ldots$. Then

$$\lambda_n = \omega_n^2 = \frac{n^2\pi^2}{l^2}$$

and

$$X_n(x) = \sin\left(\frac{n\pi x}{l}\right).$$

$\square$

1.3 Standing wave solutions

Having now found the eigenvalues $\lambda_n = \left(\frac{n\pi}{l}\right)^2$, we can substitute them in to (11) to obtain the ODE

$$T'' + \left(\frac{cn\pi}{l}\right)^2 T = 0, \tag{18}$$

which has general solution (same as for X)

$$T_n(t) = A_n\cos\left(\frac{cn\pi}{l}t\right) + B_n\sin\left(\frac{cn\pi}{l}t\right). \tag{19}$$

Putting our solutions together, we find the special solutions

$$u_n(x,t) = \underbrace{\left(A_n\cos\left(\frac{cn\pi}{l}t\right) + B_n\sin\left(\frac{cn\pi}{l}t\right)\right)}_{=T_n(t)} \cdot \underbrace{\sin\left(\frac{n\pi}{l}x\right)}_{X_n(x)}, \quad n = 1, 2, \ldots \tag{20}$$

which represents a *standing wave* (which we have seen before).

1.4 General solutions

Since the wave equation is linear, a linear combination of solutions is also a solution. Hence the function

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{cn\pi}{l}t\right) + B_n\sin\left(\frac{cn\pi}{l}t\right)\right) \cdot \sin\left(\frac{n\pi}{l}x\right) \tag{21}$$

is also a solution to the wave equation.[2]

Question: Can all solutions to (1) be written in the form (21)? In other words, have we found all possible solutions?

The answer to this question is **yes**, although we will not completely justify this claim in this course. We can, however, consider what happens when we substitute the initial conditions of (1) into (21):

$$g(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right), \tag{22}$$

$$h(x) = \sum_{n=1}^{\infty} \frac{cn\pi}{l} B_n \sin\left(\frac{n\pi}{l}x\right). \tag{23}$$

This gives us a clue that, hopefully, one might be able to recover the coefficients A_n and B_n from the initial conditions $g(x)$ and $h(x)$ – specifically, that we might be able to extract them as the coefficients in the *Fourier series* representation of these functions.

2 Boundary conditions for the wave equation

In Section 1 we considered the wave equation on a finite interval with Dirichlet BCs on each end (1). This led to the eigenvalue problem

$$X'' + \lambda X = 0, \tag{24}$$

$$X(0) = X(l) = 0, \tag{25}$$

the solutions of which were given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \dots, \tag{26}$$

$$X_n(x) = \sin\left(\frac{\pi n x}{l}\right). \tag{27}$$

Let's now consider other possible BCs for the wave equation on a finite interval.

Remark 2.1. Note that in all the examples that follow the ODE that we need to solve stays the same – it is always (9). It is only the BCs that will change.

2.1 Neumann-Neumann BCs

Suppose instead that we considered Neumann boundary conditions on both ends of the interval (we allow the strings to move freely up and down at the ends of the interval). This translates to the BCs

$$X'(0) = X'(l) = 0. \tag{28}$$

A similar analysis to that performed in Lecture 9 yields the eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 0, 1, 2, \dots, \tag{29}$$

$$X_n(x) = \cos\left(\frac{\pi n x}{l}\right). \tag{30}$$

Note that in constrast to the Dirichlet-Dirichlet eigenvalue problem, now the $n = 0$ case gives a nontrivial solution.

[2]Sidestepping – for the moment – questions of convergence.

2.2 Dirichlet-Neumann BCs

Suppose now that we impose Dirichlet BCs at $x = 0$, but Neumann BCs at $x = l$ (we fix one end of the string in place, and allow the other end to move freely). This translates to the BCs

$$X(0) = X'(l) = 0. \tag{31}$$

Suppose that $\lambda > 0$ – we leave as an exercise the remaining cases – and set $\lambda = \omega^2$, $\omega > 0$. Then (9) has general solution

$$X(x) = A\sin(\omega x) + B\cos(\omega x),$$

and imposing (31) yields $B = 0$ and

$$A\omega\cos(\omega l) = 0.$$

This has nontrivial solutions given by

$$\omega_n = \frac{(2n+1)\pi}{2l}, \quad n = 0, 1, 2, \ldots, \tag{32}$$

so the corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2, \quad n = 0, 1, 2, \ldots, \tag{33}$$

$$X_n(x) = \sin\left(\frac{(2n+1)\pi}{2l}x\right). \tag{34}$$

Reversing which end has the Dirichlet and with end has the Neumann BC would result in the same eigenvalues, and the replacement $\sin \to \cos$ in (34).

2.3 Periodic BCs

Suppose now we apply periodic BCs (you can think that we are now solving the wave equation on a circle rather than a finite interval – our string is now a loop):

$$X(0) = X(l), \quad X'(0) = X'(l). \tag{35}$$

We can automatically rule out the hyperbolic solutions to (9), as well as $X(x) = x$, since they are not periodic. The constant function *is* periodic, however, so we obtain

$$\lambda_0 = 0, \qquad X_0 = 1. \tag{36}$$

For $\lambda > 0$, set $\lambda = \omega^2$, $\omega > 0$, with solution

$$X(x) = Ae^{i\omega x} + Be^{-i\omega x},$$

and derivative

$$X'(x) = i\omega Ae^{i\omega x} - i\omega Be^{-i\omega x}.$$

Imposing periodic BCs gives

$$A + B = Ae^{i\omega l} + Be^{-i\omega l}$$
$$A - B = Ae^{i\omega l} - Be^{-i\omega l}$$

So,

$$2A = 2Ae^{i\omega l}$$
$$2B = 2Be^{-i\omega l}$$

both conditions which imply that $\omega = \frac{2\pi n}{l}$ for some integer n. There are no further constraints to impose, and so we see that

$$\lambda_n = \left(\frac{2\pi n}{l}\right)^2, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots, \tag{37}$$

$$X_n(x) = \exp\left(\frac{2\pi i n}{l} x\right) \tag{38}$$

Remark 2.2. The eigenfunctions (38) are complex valued – it is reasonable to ask if we wind up with fewer eigenfunctions if we require our solutions to be real.

In fact we do not: for $n > 0$ we may take the linear combinations

$$\frac{X_n + X_{-n}}{2} = \cos\left(\frac{2\pi n}{l} x\right)$$

$$\frac{X_n - X_{-n}}{2i} = \sin\left(\frac{2\pi n}{l} x\right)$$

so that the two independent solutions $X_{\pm n}$ yield the two independent solutions $\cos\left(\frac{2\pi n}{l}x\right)$ and $\sin\left(\frac{2\pi n}{l}x\right)$.

2.4 Quasi-Periodic BCs

Now let's consider the case of *quasi*-periodic boundary conditions – now the solution is not quite periodic, instead differing from the periodic case by some fixed multiplicative factor. For instance, suppose we take as our BCs

$$X(0) = -X(l), \quad X'(0) = -X'(l). \tag{39}$$

In what follows let's take the length of our interval to be 1, i.e. $0 < x < 1$. It is not difficult to show that $\lambda \leq 0$ still leads only to the trivial solution, so consider $\lambda = \omega^2$, $\omega > 0$. Then the generals solution to (9) is

$$X(x) = Ae^{i\omega x} + Be^{-i\omega x}$$

$$X'(x) = i\omega Ae^{i\omega x} - i\omega Be^{-i\omega x}$$

so imposing (39) yields the system of linear equations

$$A + B = -Ae^{i\omega} - Be^{-i\omega}$$

$$A - B = -Ae^{i\omega} + Be^{-i\omega}$$

which can be manipulated to give

$$2A = -2Ae^{i\omega}$$

$$2B = -2Be^{-i\omega}$$

We can therefore achieve nontrivial solutions precisely when $e^{i\omega} = -1$, i.e. for

$$\omega_n = (2n+1)\pi, \quad n \in \mathbb{Z}.$$

You can check that $e^{i\omega_n x} = e^{-i\omega_{-n-1}x}$, so that in fact the ω_n for $n < 0$ are redundant. So, we have that our eigenvalues and eigenfunctions are given by

$$\lambda_n = (2n+1)^2\pi^2, \tag{40}$$

$$X_n^{\pm}(x) = e^{\pm i(2n+1)\pi x} \tag{41}$$

where $n = 0, 1, 2, 3, \ldots$.

Remark 2.3. Just as in the periodic case, by taking sums and differences of the $X_n^{\pm}$ we can see that a collection of real-valued eigenfunctions is given by

$$C_n(x) = \cos((2n+1)\pi x),$$

$$S_n(x) = \sin((2n+1)\pi x).$$

2.5 Robin BCs

In all the above examples we found explicit formulae for our eigenvalues. Let's now consider a situation where such explicit formulae do not necessarily exist, but we will still be able to obtain quite a lot of information: the case of Robin boundary conditions,

$$X'(0) = \alpha X(0), \quad X'(1) = -\beta X(1). \tag{42}$$

Again we are working on the interval $0 < x < 1$, so as not to be distracted by factors of l.

2.5.1 One-sided Robin BC

Let's begin by considering a slightly simpler problem: fix the end of the string at $x = 0$ (impose a homogeneous Dirichlet BC), and impose a Robin BC at $x = 1$:

$$X(0) = 0, \quad X'(1) = -\beta X(1). \tag{43}$$

We will assume that $\beta \neq 0$ (otherwise this is the Dirichlet-Neumann condition we have already studied).

We repeat much of the above: letting $\lambda = \omega^2$, $\omega > 0$, we have a general solution

$$X(x) = A\sin(\omega x) + B\cos(\omega x),$$

and the Dirichlet condition $X(0) = 0$ implies $B = 0$. So we have

$$X'(x) = \omega A\cos(\omega x),$$

and applying the Robin condition at $x = 1$ gives

$$A\omega\cos(\omega) = -A\beta\sin(\omega).$$

This has a non-trivial solution for ω satisfying

$$\omega = -\beta\tan(\omega). \tag{44}$$

We can't solve this equation exactly. However: by plotting both sides of (44) and looking for intersections (Figures 1, 2, 3, 4) we can see that there is one solution ω_n in each interval

$$\frac{2n-1}{2}\pi < \omega < \frac{2n+1}{2}\pi,$$

with the possible exception of $n = 0$. A solution will exist for $n = 0$ if and only if

$$0 < \frac{d}{d\omega}\left(-\beta\tan(\omega)\right)|_{\omega=0} < 1, \tag{45}$$

which occurs precisely when $0 > \beta > -1$. (Exercise: Why is (45) the correct condition to consider?)

Upshot: we have found that there is a string of eigenvalues ω_n, with $n \geq 0$ or $n > 0$ depending on the value of β, with corresponding eigenfunctions

$$X_n(x) = \sin(\omega_n x). \tag{46}$$

Okay – so we're done, right? Not quite – remember that, e.g. for Neumann-Neumann BCs (Section 2.1), the zero eigenvalue could occur.

So, suppose that $\lambda = 0$. Then

$$X(x) = Ax + B$$

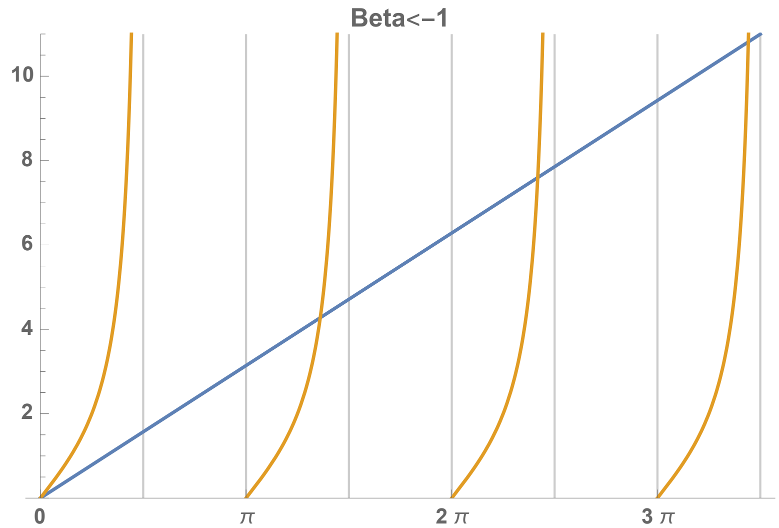

Figure 1: Plotting (44) for $\beta < -1$.

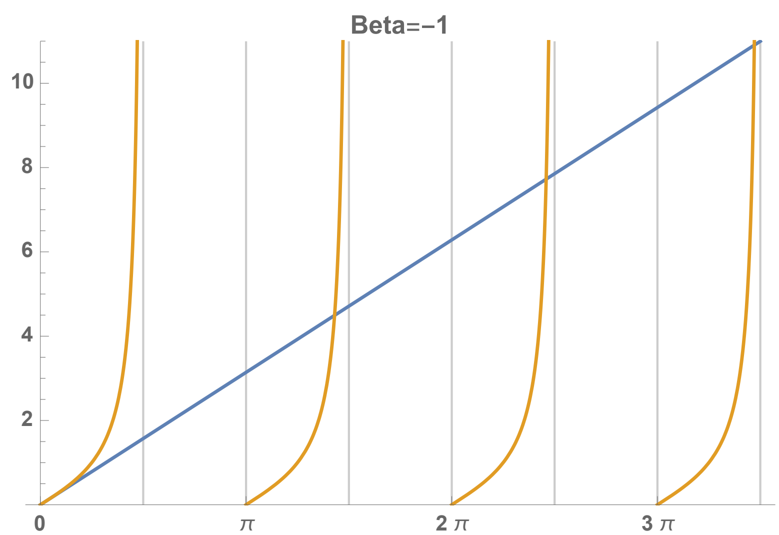

Figure 2: Plotting (44) for $\beta = -1$.

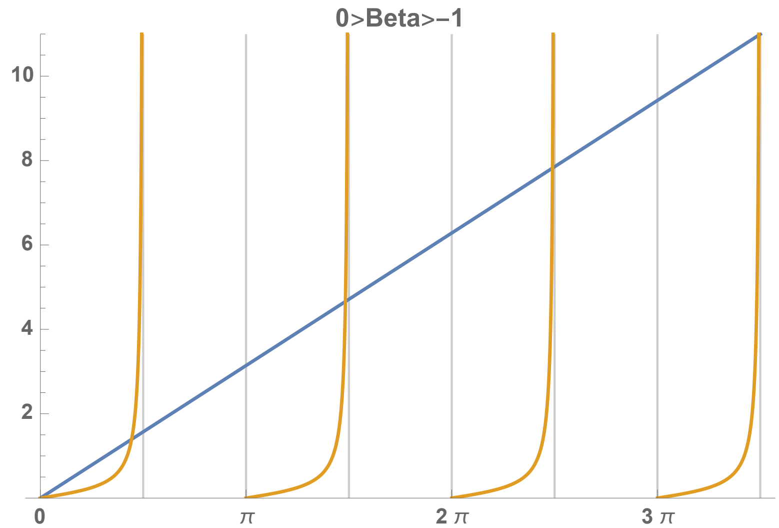

Figure 3: Plotting (44) for $0 > \beta > -1$.

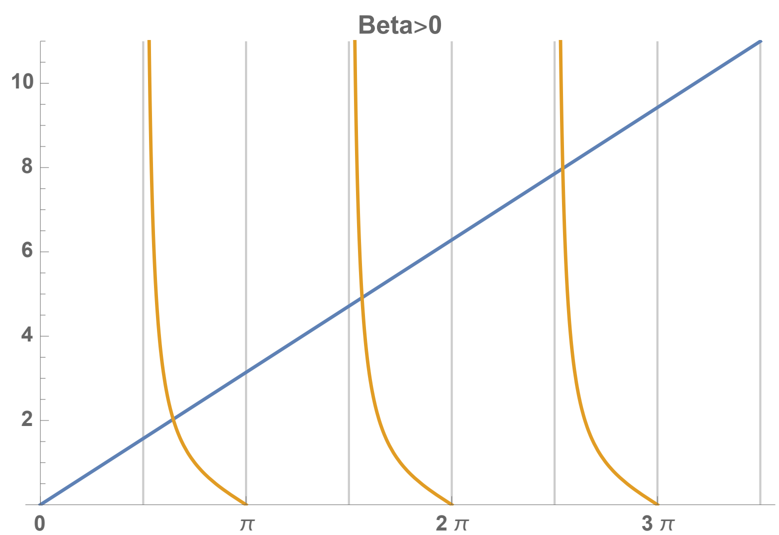

Figure 4: Plotting (44) for $\beta > 0$.

is the general solution to (9). The Dirichlet BC implies that $B = 0$, so that

$$X'(x) = A.$$

Then the Robin condition reads

$$A = -\beta A,$$

and so if $\beta = -1$ we have to supplement the eigenfunctions (46) with the linear eigenfunction

$$X_{\text{lin}}(x) = x, \tag{47}$$

with eigenvalue zero.

Okay – so *now* we're done, right?

Right?

The answer (as you might expect from such a leading question) is still **no**. Unlike in previous examples, here we are going to have to carefully examine the possibility that a negative eigenvalue occurs.

So, let $\lambda = -\omega^2 < 0$, $\omega > 0$. The general solution to (9) is given by

$$X(x) = A\sinh(\omega x) + B\cosh(\omega x),$$

and the Dirichlet BC implies $B = 0$. So,

$$X'(x) = A\omega\cosh(\omega x)$$

and imposing the Robin BC at $x = 1$ yields

$$A\omega\cosh(\omega) = -\beta A\sinh(\omega).$$

This has a non-trivial solution for ω satisfying

$$\omega = -\beta\tanh(\omega). \tag{48}$$

Since we require $\omega > 0$ there can be no solution to this equation for $\beta > 0$. In fact a positive solution to (48) will exist if and only if

$$\frac{d}{d\omega}\left(-\beta\tanh(\omega)\right)|_{\omega=0} > 1; \tag{49}$$

see Figures 5, 6, 7. (Exercise: Why is (49) the correct condition to consider?)

So: if $\beta < -1$, there is a single negative eigenvalue $\lambda_- = -\omega_-^2$, where ω_- is the unique positive solution to (49), with corresponding eigenfunction

$$X_-(x) = \sinh(\omega_- x). \tag{50}$$

Summary: The eigenvalues and eigenfunctions for the BCs (43) are given by

- **If** $\beta > 0$**:**
  - The eigenvalues $\lambda_n = \omega_n^2$, ω_n solving (45), for $n > 0$; trigonometric eigenfunctions $\sin(\omega_n x)$.
- **If** $0 > \beta > -1$**:**
  - The eigenvalues $\lambda_n = \omega_n^2$, ω_n solving (45), for $n \geq 0$; trigonometric eigenfunctions $\sin(\omega_n x)$.
- **If** $\beta = -1$**:**

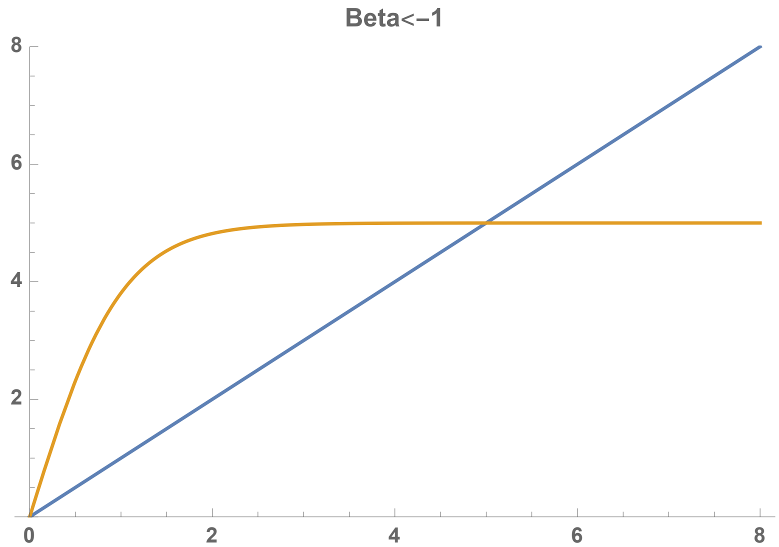

Figure 5: Plotting (48) for $\beta < -1$.

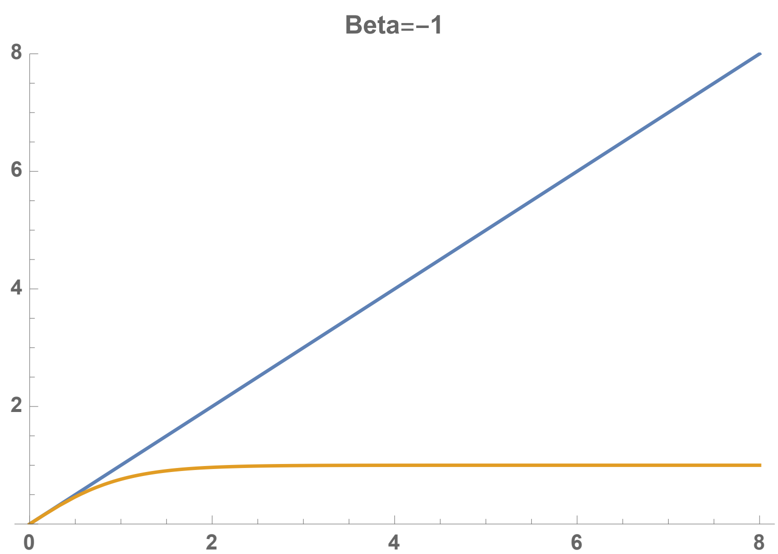

Figure 6: Plotting (48) for $\beta = -1$.

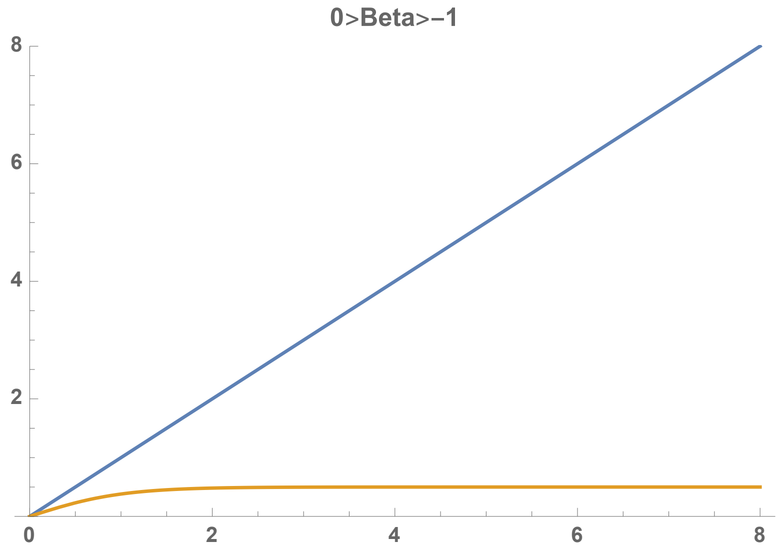

Figure 7: Plotting (48) for $0 > \beta > -1$.

- The eigenvalues $\lambda_n = \omega_n^2$, ω_n solving (45), for $n > 0$; trigonometric eigenfunctions $\sin(\omega_n x)$.
- The zero eigenvalue, with linear eigenfunction x.

- **If $\beta < -1$:**
  - The eigenvalues $\lambda_n = \omega_n^2$, ω_n solving (45), for $n > 0$; trigonometric eigenfunctions $\sin(\omega_n x)$.
  - The unique eigenvalue $\lambda_- = -\omega_-^2$, ω_- solving (49); hyperbolic eigenfunction $\sinh(\omega_- x)$.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.