APM 346 Lecture 8.

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We continue our study of the heat equation, following [IvrXX, §3.2].

1 A quick word on the heat equation in higher dimensions

Recall from last lecture that the IVP for the 1d heat equation

$$\begin{cases} u_t = ku_{xx}, & -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) = g(x) \end{cases} \tag{1}$$

had solution given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x,y,t)g(y)dy \tag{2}$$

where the Green's function G is

$$G(x,y,t) = \frac{1}{2\sqrt{k\pi t}} e^{-\frac{|x-y|^2}{4kt}}. \tag{3}$$

Similar formulae held for the heat equation on the half-line with homogeneous Dirichlet or Neumann BCs (we just had to change the Green's function and the domain of integration).

Now consider the 2d and 3d heat equations

$$u_t = k(u_{xx} + u_{yy}), \tag{4}$$

$$u_t = k(u_{xx} + u_{yy} + u_{zz}), \tag{5}$$

or more generally the heat equation in n-dimensions

$$u_t = k\Delta u. \tag{6}$$

We claim that the corresponding IVPs/IBVPs have solutions given by

$$u(x,y,t) = \iint G_2(x,y;x',y';t)g(x',y')dx'dy' \tag{7}$$

$$u(x,y,z,t) = \iiint G_3(x,y,z;x',y',z';t)g(x',y',z')dx'\,dy'\,dz' \tag{8}$$

$$u(\vec{x},t) = \int \cdots \int G_n(\vec{x},\vec{x'};t)g(\vec{x'})d\vec{x'} \tag{9}$$

where the *n-dimensional Green's functions* are given by

$$G_n(\vec{x},\vec{x'};t) = \prod_{i=1}^{n} G(x_i,x_i',t), \tag{10}$$

a product over the appropriate 1d Green's functions.

Example 1. If we consider the heat equation on *all* of $\mathbb{R}^n$, then

$$G_n(\vec{x}, \vec{x'}; t) = (4\pi kt)^{-\frac{n}{2}}\, e^{-\frac{|\vec{x}-\vec{x'}|^2}{4kt}}. \tag{11}$$

Justification of the claim. Notice that the function G_n satisfies the same sorts of properties as G, that we used to derive (2) last lecture:

(1) G_n satisfies the heat equation (6). You should check the following calculation as an exercise:

$$\begin{aligned}
\frac{\partial G_n}{\partial t}(\vec{x}, \vec{x'}; t) &= \sum_{i=i}^{n} \left(\frac{\partial G}{\partial t}(x_i, x_i', t) \prod_{j \neq i} G(x_j, x_j', t) \right) \\
&= \sum_{i=1}^{n} \left(k \frac{\partial^2 G}{\partial x_i^2}(x_i, x_i', t) \prod_{j \neq i} G(x_j, x_j', t) \right) \\
&= k \Delta_{\vec{x}} \left(\prod_{i=1}^{n} G(x_i, x_i', t) \right) \\
&= k \Delta_{\vec{x}} G_n(\vec{x}, \vec{x'}; t).
\end{aligned}$$

Here, $\Delta_{\vec{x}}$ denotes the Laplace operator acting on the $\vec{x}$-coordinates (as opposed to the $\vec{x'}$-coordinates).

(2) $G_n(\vec{x}, \vec{x'}; t)$ quickly decays as $|\vec{x} - \vec{x'}| \to \infty$, and $\lim_{t \to 0^+} G_n(\vec{x}, \vec{x'}; t) = 0$ for $\vec{x} \neq \vec{x'}$.

(3) $\int G_n(\vec{x}, \vec{x'}; t) d\vec{x'} \to 1$ as $t \to 0^+$.

(4) $G_n(\vec{x}, \vec{x'}; t) = G_n(\vec{x'}, \vec{x}; t)$.

These properties all essentially follow from the same claims in 1d, and are left as exercises.

2 The maximum principle

Consider a bounded domain $\Omega \subset \mathbb{R}^2_{x,t}$ which has a "flat-ceiling" $C = \{(x, t_*) \,|\, x_* \leq x \leq x_*'\}$ (with fixed $t = t_*$) as its upper boundary. Decompose the boundary of Ω as the disjoint union (see Figure 1)

$$\partial\Omega = C \sqcup \Gamma. \tag{12}$$

Proposition 2.1 (Maximum Principle)**.** *Suppose that u satisfies the heat equation in Ω. Then*

$$\max_{(x,t) \in \Omega} u(x, t) = \max_{(x,t) \in \Gamma} u(x, t). \tag{13}$$

Remark 2.1. Before diving into the proof, let's use some physical intuition to ask: does this make sense? What is the maximum principle saying, physically? (This is for you to discuss, and not for me to write in these notes!)

Proof. Define a family of functions depending on a small parameter $\epsilon > 0$ by

$$v_\epsilon(x, t) := u(x, t) - \epsilon t. \tag{14}$$

Plugging v_ϵ into the heat equation gives

$$\frac{\partial v_\epsilon}{\partial t} - k \frac{\partial^2 v_\epsilon}{\partial x^2} = u_t - \epsilon - k u_{xx} = -\epsilon < 0. \tag{15}$$

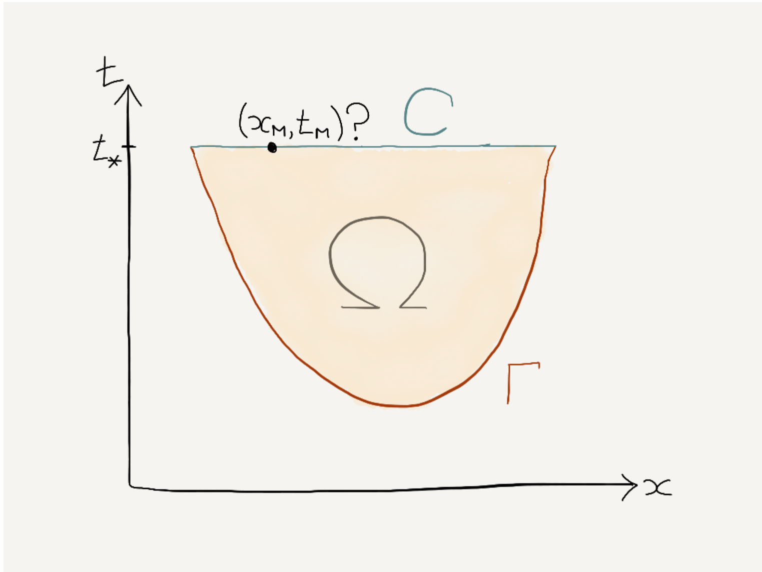

Figure 1: Domain of solution Ω for the heat equation, with boundary decomposed into Γ and C.

Now, suppose that v_ϵ attains its maximum at a point $(x_M, t_M) \in \Omega \setminus \Gamma$. By restricting our domain, we may assume that in fact $(x_M, t_M) \in C$, the "ceiling" of Ω. Then since $v_\epsilon(x, t_M)$ is maximized at (x_M, t_M) we have that

$$\frac{\partial v_\epsilon}{\partial x}(x_M, t_M) = 0, \tag{16}$$

$$\frac{\partial^2 v_\epsilon}{\partial x^2}(x_M, t_M) \leq 0, \tag{17}$$

and similarly in the t-variable we have

$$\frac{\partial v_\epsilon}{\partial t}(x_M, t_M) = \lim_{\delta \to 0^+} \frac{v_\epsilon(x_M, t_M) - v_\epsilon(x_M, t_M - \delta)}{\delta} \geq 0. \tag{18}$$

Putting (16) and (18) together, we find that

$$\frac{\partial v_\epsilon}{\partial t}(x_M, t_M) - k\frac{\partial^2 v_\epsilon}{\partial x^2}(x_M, t_M) \geq 0, \tag{19}$$

contradicting (15). Hence, v_ϵ must attain its maximum on Γ:

$$\max_{(x,t)\in\Omega}(u(x,t) - \epsilon t) = \max_{(x,t)\in\Gamma}(u(x,t) - \epsilon t). \tag{20}$$

(20) holds for arbitrarily small $\epsilon > 0$; taking $\epsilon \to 0^+$ then yields the desired equation (13). □

Remark 2.2. The claim and proof of Proposition 2.1 continue to hold in higher dimensions.

Remark 2.3. A more sophisticated proof would tell us that in fact if u also achieves its maximum in $\Omega \setminus \Gamma$, then u is constant.

Remark 2.4. The maximum principle is not special to the heat equation (as we will rediscover later in the semester when we study harmonic functions). In fact, maximum principles arise in the study of *elliptic* and *parabolic* PDEs (remember those from Lecture 1?), but not in the study of *hyperbolic* PDEs. **Does this make sense given what we have studied so far this semester?**

Corollary 2.2 (Minimum Principle)**.** *Suppose that u satisfies the heat equation in Ω. Then*

$$\min_{(x,t)\in\Omega} u(x,t) = \min_{(x,t)\in\Gamma} u(x,t). \tag{21}$$

Proof. Apply the maximum principle (13) to $-u$ (which also satisfies the heat equation). □

Corollary 2.3. *Suppose that u satisfies the heat equation in Ω, and that $u|_\Gamma = 0$. Then $u = 0$ everywhere on Ω.*

Proof. Combine the maximum and minimum principles: $\max_\Omega u = \min_\Omega u = 0$. □

Corollary 2.4. *Suppose that u and v both satisfy the heat equation on Ω, and that $u|_\Gamma = v|_\Gamma$. Then $u = v$ everywhere on Ω.*

Proof. By linearity, $u - v$ also satisfies the heat equation, and $(u - v)|_\Gamma = 0$. Now apply Corollary 2.3. □

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.