APM 346 Lecture 7.

Richard Derryberry

January 29, 2019

This week we will be considering the heat equation. For this topic you should read all of [IvrXX, Ch.3]. Relevant references in Strauss are [Str08, Ch.2.3-5,Ch.3.1,3,5].

Five minute review exercise for the start of class: Calculate

$$J = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

1 The 1d Heat Equation

Recall that in Lectures 1 and 2 we introduced the *heat equation*, which (in its simplest form and in 1d) is given by

$$u_t = k u_{xx} \tag{1}$$

where, for instance, u might represent a temperature distribution and k is a constant that determines the rate of heat diffusion.

We begin by solving the IVP

$$\begin{cases} u_t &= k u_{xx}, \quad -\infty < x < \infty, \, 0 < t < \infty \\ u(x,0) &= g(x) \end{cases} \tag{2}$$

via the method of looking for self-similar solutions. Later in the semester we will use another approach (the Fourier transform) to solve (2).

1.1 Self-similar solutions of the heat equation

We will make use of the following property of (1), which is left as an **exercise**: *If $S(x,t)$ is a solution of* (1) *then so is*

$$v(x,t) := \int_{-\infty}^{\infty} S(x-y,t) g(y) dy \tag{3}$$

for any function $g(y)$ (modulo issues of convergence for the integral).

As a further exercise, students are invited to consider under which circumstances (3) will have appropriate convergence properties.

Remark 1.1. There is a simple principle at work behind the fact that (3) provides a solution the the heat equation: what is it? (Hint: The heat equation is a linear PDE.)

Using (3), we see we can attempt to solve the heat equation by: (1) finding some sort of ur-solution to (2) and then (2) showing that all other solutions may be obtained by integrating appropriate functions against the ur-solution.

1.1.1 Reduction to self-similar solutions

We begin our search for an appropriate ur-solution by considering the behaviour of solutions to (1) under scalings and dilations. Define

$$u_{\alpha,\beta,\gamma}(x,t) := \gamma u(\alpha x, \beta t). \tag{4}$$

Proposition 1.1. *If $u(x,t)$ is a solution to* (1) *then so is $u_{\alpha,\beta,\gamma}(x,t)$ provided $\beta = \alpha^2$.*

Proof. By straightforward calculation, left as an exercise. □

Definition 1.1. The *total heat energy* of the solution u is

$$I[u](t) \equiv I(t) := \int_{-\infty}^{\infty} u(x,t)dx. \tag{5}$$

In searching for our ur-solution, we now propose the following conservation of energy assumption: *The total heat energy* (5) *is finite and independent of t.*

Now, let us consider what conditions will guarantee that $I[u] = I[u_{\alpha,\beta,\gamma}]$? First we'll assume that $\alpha > 0$. Then

$$\int_{-\infty}^{\infty} u_{\alpha,\beta,\gamma}(x,t)dx = \frac{\gamma}{\alpha}\int_{-\infty}^{\infty} u(x,t)dx, \tag{6}$$

so that to guarantee equality of heat energy we must take $\gamma = \alpha$. Hence, we will now consider the *transformations of similarity* of u,

$$u_\alpha(x,t) = \alpha u(\alpha x, \alpha^2 t). \tag{7}$$

A visualisation of a similarity transform can be seen in the corresponding Mathematica file. (Think: Why do we call this a similarity transform?)

1.1.2 Search for a self-similar solution

We want to find a *self-similar* solution to (1), i.e. a solution u such that

$$u(x,t) = u_\alpha(x,t) = \alpha u(\alpha x, \alpha^2 t), \quad \alpha > 0,\, t > 0. \tag{8}$$

Let us use the constraint of self-similarity to simplify our problem. Setting $\alpha = t^{-\frac{1}{2}}$ gives

$$u(x,t) = t^{-\frac{1}{2}} u(t^{-\frac{1}{2}}x, 1) = t^{-\frac{1}{2}}\phi(t^{-\frac{1}{2}}x) \tag{9}$$

where $\phi(\xi) := u(\xi, 1)$.

We now wish to rewrite (1) in terms of ϕ. Taking derivatives (calculations left as an exercise) gives:

$$u_t = -\frac{1}{2}t^{-\frac{3}{2}}\left(\phi\left(t^{-\frac{1}{2}}x\right) + t^{-\frac{1}{2}}x\phi'\left(t^{-\frac{1}{2}}x\right)\right), \tag{10}$$

$$u_x = t^{-1}\phi'\left(t^{-\frac{1}{2}}x\right), \tag{11}$$

$$u_{xx} = t^{-\frac{3}{2}}\phi''\left(t^{-\frac{1}{2}}x\right). \tag{12}$$

Letting $\xi = t^{-\frac{1}{2}}x$ (and clearing common factors of $t^{-\frac{3}{2}}$) we see that (1) becomes the ODE:

$$-\frac{1}{2}\left(\phi(\xi) + \xi\phi(\xi)\right) = k\phi''(\xi). \tag{13}$$

Observe that the LHS of (13) is in fact a total derivative, since $\phi(\xi) + \xi\phi'(\xi) = \frac{d}{d\xi}(\xi\phi(\xi))$. So we may integrate to obtain the first order ODE

$$-\frac{1}{2}\xi\phi(\xi) = k\phi'(\xi), \tag{14}$$

where we have set the constant of integration to be zero so that ϕ and ϕ' decay quicky at $\pm\infty$ (required for integrability purposes).

Now, (14) may be rewritten as

$$\frac{d}{d\xi}\log(\phi) = -\frac{1}{2k}\xi, \tag{15}$$

which can be immediately integrated to

$$\log(\phi) = -\frac{1}{4k}\xi^2 + \log C \tag{16}$$

which we may exponentiate to obtain

$$\phi(\xi) = Ce^{-\frac{1}{4k}\xi^2}. \tag{17}$$

Putting this all together, we conclude:

Proposition 1.2. *A self-similar solution to the heat equation with total heat energy* $I[u] = 1$ *is given by*

$$G_0(x,t) = \frac{1}{2\sqrt{\pi k t}}e^{-\frac{x^2}{4kt}}. \tag{18}$$

Proof. The form of the equation comes from (9), so the only thing to check is the normalisation factor. This comes from performing the integral

$$I(t) = C\int_{-\infty}^{\infty} t^{-\frac{1}{2}}e^{-\frac{x^2}{4kt}}\,dx = c\sqrt{4k}\int_{-\infty}^{\infty} e^{-z^2}dz = 2c\sqrt{k\pi}.$$

□

Definition 1.2. (18) is called the *fundamental solution* of the heat eqation (on the real line).

1.2 Solving the heat equation IVP

Having found the fundamental solution (18), we would like to be able to show that it is the "ur-solution" from which we may obtain all other solutions to the heat equation.

We immediately encounter two issues that we need to deal with:

(1) If we wish to solve a particular IVP for the heat equation (2), how do we know which function to integrate (18) against (as in (3))?

(2) How can we understand the initial condition for the fundamental solution?

To see why (2) poses a problem, observe that

$$\lim_{t\to 0+} G_0(x,t) = 0, \quad x \neq 0, \tag{19}$$

$$\lim_{t\to 0+} G_0(0,t) = +\infty, \tag{20}$$

$$I[u] = 1, \quad \text{for all } t. \tag{21}$$

I.e. the initial condition for the fundamental solution is a "function" that vanishes away from $x = 0$, is infinite at $x = 0$, and has finite integral equal to 1.

In fact, the initial condition for the fundamental solution is not a function at all, but is something called a *Dirac δ-function*, $\delta(x)$.[1] $\delta(x)$ is an example of a *distribution*, a generalisation of functions which are important in the theory of differential equations, but which are sadly beyond the scope of this course.

We will therefore sidestep the topic of distributions and tackle this problem from another perspective, solving issue (1) at the same time. Consider the integral of the fundamental solution,

$$U(x,t) := \int_{-\infty}^{x} G_0(x',t)dx' \tag{22}$$

Proposition 1.3. *The function $U(x,t)$ satisfies the heat equation* (1)*, with initial condition*

$$U(x,0) = \theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases} \tag{23}$$

Proof. Since $G_0 = U_x$ and G_0 solves the heat equation, we have that

$$(U_t - kU_{xx})_x = 0$$

which we integrate to

$$(U_t - kU_{xx}) = \Phi(t).$$

Since as $x \to -\infty$ U and all its derivatives decay quickly to zero, $\Phi(t) = 0$, so U solves the heat equation.

To see that U has the specific IC, rewrite U using a change of integration variable as

$$U(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz.$$

If $x < 0$ then as $t \to 0+$ the upper limit of integration tends to $-\infty$, and so the integral vanishes. If $x > 0$ then as $t \to 0+$ the upper limit of integration tends to $+\infty$ and so the expression is equal to 1 (we have calculated this previously). □

Remark 1.2. We could have constructed U as a self-similar solution, but with $\gamma = 1$ instead of $\gamma = \alpha$.

Now: suppose that $g(x)$ is a smooth function satisfying $\lim_{x\to-\infty} = 0$. We may rewrite g in an integral form by considering the *convolution* of $g'(x)$ with a step-function:

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} \theta(x-y)g'(y)dy \\ &= \int_{-\infty}^{x} g'(y)dy. \end{aligned} \tag{24}$$

Now: $U(x-y,t)$ solves the heat equation IVP (2) with IC $U(x-y,0) = \theta(x-y)$; hence

$$u(x,t) := \int_{-\infty}^{\infty} U(x-y,t)g'(y)dy \tag{25}$$

solves the heat equation IVP with IC $u(x,0) = g(x)$. Integrating by parts with respect to y gives us

$$u(x,t) = \int_{-\infty}^{\infty} U_x(x-y,t)g(y)dy$$

and so may conclude that:

[1] Using the nomenclature "δ-function" for something that is not actually a function may seem odd at first, but you get used to it. If it particularly upsets you, feel free to use the terminology "Dirac δ-distribution", or even just "Dirac δ".

Proposition 1.4. *The heat equation IVP with initial condition $g(x)$ has solution given by*

$$u(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)dy. \tag{26}$$

Example 1. Suppose that we wish to sold the heat equation IVP with $k = 1$ and

$$u(x,0) = \theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Then

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}} \theta(y)dy \\ &= \frac{1}{2\sqrt{\pi t}} \int_{0}^{\infty} e^{-\frac{|x-y|^2}{4t}} dy \\ &= -\frac{1}{2\sqrt{\pi t}} \int_{x}^{-\infty} e^{-\frac{z^2}{4t}} dz \\ &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-r^2} (2\sqrt{t}dr) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-r^2} dr + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x}{2\sqrt{t}}} e^{-r^2} dr \\ &= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) \end{aligned}$$

where we have made substitutions $z = x - y$ and $r = \frac{z}{2\sqrt{t}}$.

1.3 The heat equation on the half-line

We have just seen that the solution to the heat equation IVP is given by

$$u(x,t) = \int_{-\infty}^{\infty} G_I(x,y,t) g(y)dy \tag{27}$$

where

$$G_I(x,y,t) = G_0(x-y,t) = \frac{1}{2\sqrt{k\pi t}} e^{-\frac{|x-y|^2}{4kt}} \tag{28}$$

(later in the semester we will see that the notation G stands for "Green's function").

Remark 1.3. Note that $G_I(x,y,t) = G_I(y,x,t)$.

I claim that the same approach will work for a heat equation IBVP on the half-line $x > 0$.

1.3.1 Homogeneous boundary conditions on the half-line

Consider the homogeneous Dirichlet and Neumann heat equation IBVPs on the half line:

$$\begin{cases} (u_D)_t - k(u_D)_{xx} = 0, & x > 0,\ t > 0, \\ u_D(x,0) = g(x), & x > 0, \\ u_D(0,t) = 0, & t > 0; \end{cases} \tag{29}$$

$$\begin{cases} (u_N)_t - k(u_N)_{xx} = 0, & x > 0,\ t > 0, \\ u_N(x,0) = g(x), & x > 0, \\ (u_N)_x(0,t) = 0, & t > 0. \end{cases} \tag{30}$$

The claim is that these IBVPs have solutions given by

$$u_D(x,t) = \int_0^\infty G_D(x,y,t)g(y)dy, \tag{31}$$

$$u_N(x,t) = \int_0^\infty G_N(x,y,t)g(y)dy, \tag{32}$$

where the functions G_D and G_N are given by

$$G_D(x,y,t) = G_0(x-y,t) - G_0(x+y,t), \tag{33}$$

$$G_N(x,y,t) = G_0(x-y,t) + G_0(x+y,t). \tag{34}$$

Note that:

- G_D and G_N solve (29) and (30), respectively.
- $G_{N/D} = G_I(x,y,t) \pm G_I(x,-y,t)$.
- G_D and G_N are symmetric in x and y, and have integral 1 with respect to either x or y.

To prove that (31) and (32) solve their respective IBVPs, one constructs G_D and G_N via the *method of continuation*, which you should recall from the wave equation IBVP last week. It is an exercise for you to carry out the following program:

(1) Take the unique odd (for Dirichlet) or even (for Neumann) extension of the IBVPs (29) and (30) to obtain heat equation IVPs for the entire line.

(2) Use the solution (27) to solve the corresponding IVPs.

(3) Rewrite the corresponding solution so that it only involves an integral over the relevant physical domain $0 < x < \infty$.

1.3.2 Inhomogeneous boundary conditions on the half-line

Now consider (29) and (30), but with inhomogeneous BCs

$$u_D(0,t) = p(t), \tag{35}$$

$$(u_N)_x(0,t) = q(t). \tag{36}$$

Consider the integral

$$0 = \int_0^\infty \int_0^{t-\epsilon} G(x,y,t-\tau)\left(-u_\tau(y,\tau) + ku_{yy}(y,\tau)\right) d\tau dy. \tag{37}$$

where G is either G_D or G_N.

Integrate the first term by parts once with respect to τ, and the second term twice with respect to y. We get:

$$0 = \int_0^\infty \int_0^{t-\epsilon} \left(-G_\tau(x,y,t-\tau) + kG_{yy}(x,y,t-\tau)\right) u(y,\tau) d\tau dy \tag{38}$$

$$- \underbrace{\int_0^\infty G(x,y,\epsilon)u(y,t-\epsilon)dy}_{\lim_{\epsilon\to 0} = u(x,t)} + \int_0^\infty G(x,y,t)\underbrace{u(y,0)}_{g(y)} dy \tag{39}$$

$$+ k\int_0^{t-\epsilon} \left(G(x,y,t-\tau)u_y(y,t) + G_y(x,y,t-\tau)u(y,\tau)\right)\big|_{y=0} d\tau \tag{40}$$

and so after rearranging and taking $\epsilon \to 0$:

$$u(x,t) = \int_0^\infty G(x,y,t)g(y)dy + k\int_0^t \left.(G(x,y,t-\tau)u_y(y,t) + G_y(x,y,t-\tau)u(y,\tau))\right|_{y=0} d\tau. \tag{41}$$

To understand the second term, we consider the Dirichlet and Neumann cases separately:

- **Dirichlet:** In this case $G(x,0,t) = 0$, so the second term is

$$k\int_0^t G_y(x,0,t-\tau)\underbrace{u(0,\tau)}_{p(\tau)} d\tau. \tag{42}$$

- **Neumann:** In this case $G_y(x,0,t) = 0$, so the second term becomes

$$-k\int_0^t G(x,0,t-\tau)\underbrace{u_y(0,\tau)}_{q(\tau)} d\tau. \tag{43}$$

So the solutions to the inhomogeneous Dirichlet and Neumann IBVPs are given by:

$$u_D(x,t) = \int_0^\infty G_D(x,y,t)g(y)dy + k\int_0^t (G_D)_y(x,0,t-\tau)p(\tau)d\tau; \tag{44}$$

$$u_N(x,t) = \int_0^\infty G_N(x,y,t)g(y)dy - k\int_0^t G_N(x,0,t-\tau)q(\tau)d\tau. \tag{45}$$

1.4 The inhomogeneous heat equation

Now, instead of considering the homogeneous heat equation (1), consider the *inhomogeneous* heat equation, where we include a source term on the RHS:

$$u_t - ku_{xx} = f(x,t). \tag{46}$$

The solution to (46) is given by (27)/(31)/(32) (depending on the domain $\Omega = \mathbb{R}$ or $\mathbb{R}_{\geq 0}$ we are considering), **plus** an additional term, of the form

$$\int_0^t \int_\Omega G(x,y,t-\tau)f(y,\tau)dy\, d\tau, \tag{47}$$

where G is one of G_I, G_D or G_N.

The proof of this is left as an exercise. Two possible methods of proof are:

(1) Show that the calculations performed in Section 1.3.2 can be used to derive (47).

(2) Imitate the reasoning that we used back in Lecture 5 to solve the wave equation using Duhamel's Principle.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.