APM 346 Lecture 6.

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Today we consider the 1d wave equation with boundary conditions. References being used: [IvrXX, §2.6] (§2.6) and [Str08, Ch.3.2].

1 The wave equation with boundary conditions

1.1 The wave equation on the half-line

Consider the IVP on the half-line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x > 0, t > 0 \\ u(x,0) = g(x) & x > 0 \\ u_t(x,0) = h(x) & x > 0 \end{cases} \tag{1}$$

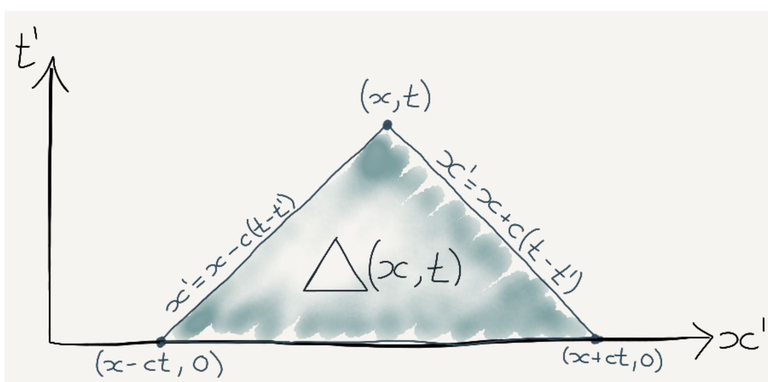

Figure 1: Domain of dependence for a point .

Recall the *domain of dependence* of a point (x,t) (Figure 1). Since we are only specifying initial data on the half-line $x > 0$, the information we have provided is only enough to give us a solution on the domain $\{t > 0, x \geq ct\}$ – for points (x,t) with $x < ct$, the domain intersects the t'-axis before it can reach time $t = 0$, where the initial conditions are specified.

So: for $x \geq ct$ the solution is given by the usual D'Alembert formula

$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')dx', \tag{2}$$

and it remains to solve the problem for $0 < x < ct$.

Claim: We need to specify a single boundary condition at $x = 0$, $t > 0$.

Considering the picture of the domain of dependence, this is (hopefully) an intuitive claim. In these notes, we also supply the following:

"Proof:" The general solution to the wave equation is

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \tag{3}$$

and from Lecture 4 we know that (1) implies

$$\phi(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(s)ds, \quad x > 0, \tag{4}$$

$$\psi(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(s)ds, \quad x > 0. \tag{5}$$

So:

- $\phi(x+ct)$ is defined as $x+ct > 0$ (automatically holds),
- $\psi(x-ct)$ is defined as $x - ct > 0$ (i.e. $x > ct$),
- so it remains to define $\psi(x-ct)$ for $0 < x < ct$.

Therefore we need to define $\psi(x)$ as $x < 0$, and to do so requires a single boundary condition at $x = 0$.

1.1.1 The Dirichlet bounday condition

Consider imposing a Dirichlet BC:

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0, & x > 0, t > 0 \\ u(x,0) &= g(x), & x > 0 \\ u_t(x,0) &= h(x), & x > 0 \\ u(0,t) &= p(t), & t > 0 \end{cases} \tag{6}$$

We want to define $\psi(x)$ for $x < 0$. Impose the BC on (3) to obtain

$$p(t) = \phi(ct) + \psi(-ct), \quad t > 0.$$

Letting $x = -ct < 0$ this is equivalent to

$$\phi(-x) + \psi(x) = p\left(-\frac{x}{c}\right)$$

and so using (4)

$$\psi(x) = p\left(-\frac{x}{c}\right) - \frac{1}{2}g(-x) - \frac{1}{2c}\int_0^{-x} h(s)ds. \tag{7}$$

Hence from (2) and (3) we have that

$$u(x,t) = \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} h(s)ds + p\left(t - \frac{x}{c}\right) - \frac{1}{2}g(ct-x) - \frac{1}{2c}\int_0^{ct-x} h(s)ds, \quad 0 < x < ct. \tag{8}$$

Example 1. If we set $g = h = 0$, so that the wave is generated by the perturbation $p(t)$, we have

$$u(x,t) = \begin{cases} 0, & 0 < ct < x, \\ p\left(t - \frac{x}{c}\right) & 0 < x < ct. \end{cases}$$

Example 2. If we set $h(x) = cg'(x)$ and $p = 0$, the result is an initially left moving wave which is reflected when it hits $x = 0$:

$$u(x,t) = \begin{cases} g(x+ct), & 0 < ct < x, \\ g(x+ct) - g(ct-x), & 0 < x < ct. \end{cases}$$

1.1.2 The Neumann bounday condition

Consider imposing a Neumann BC:

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0, & x > 0, t > 0 \\ u(x,0) &= g(x), & x > 0 \\ u_t(x,0) &= h(x), & x > 0 \\ u_x(0,t) &= q(t), & t > 0 \end{cases} \tag{9}$$

Impose the BC on (3) to obtain

$$q(t) = \phi'(ct) + \psi'(-ct), \quad t > 0.$$

Letting $x = -ct < 0$ and integrating gives[1]

$$\phi(-x) - \psi(x) = c \int_0^{-\frac{x}{c}} q(t')dt', \quad x < 0,$$

and so using (4)

$$\psi(x) = -c \int_0^{-\frac{x}{c}} q(t')dt' + \frac{1}{2} g(-x) + \frac{1}{2c} \int_0^{-x} h(s)ds. \tag{10}$$

Hence from (2) and (3) we have that

$$u(x,t) = \frac{1}{2} g(x+ct) + \frac{1}{2c} \int_0^{x+ct} h(s)ds - c \int_0^{t-\frac{x}{c}} q(t')dt' + \frac{1}{2} g(ct-x) + \frac{1}{2c} \int_0^{ct-x} h(s)ds, \quad 0 < x < ct. \tag{11}$$

Example 3. If we set $g = h = 0$, so that the wave is generated by the perturbation $q(t)$, we have

$$u(x,t) = \begin{cases} 0, & 0 < ct < x, \\ -c \int_0^{t-\frac{x}{c}} q(t')dt' & 0 < x < ct. \end{cases}$$

Example 4. If we set $h(x) = cg'(x)$ and $q = 0$, the result is an initially left moving wave which is reflected when it hits $x = 0$:

$$u(x,t) = \begin{cases} g(x+ct), & 0 < ct < x, \\ g(x+ct) + g(ct-x), & 0 < x < ct. \end{cases}$$

Remark 1.1. Dirichlet and Neumann are of course not the only possible BCs we could impose (e.g. we could impose a Robin BC). Two other examples are considered in [IvrXX, §2.6] – students are invited to read this, and to consider other BCs as an exercise.

1.1.3 The method of continuation

We'll now consider a technique called the *method of continuation*, which exploits our solution to the wave equation on the entire line in the case where we consider homogeneous Dirichlet or Neumann boundary conditions ($u(0,t) = 0$ or $u_x(0,t) = 0$).

So: the problem we are considering is the IBVP

$$\begin{cases} u_{tt} - c^2 u_{xx} &= f(x,t), & x > 0, t > 0, \\ u(x,0) &= g(x), & x > 0, \\ u_t(x,0) &= h(x), & x > 0, \\ u(0,t) \text{ or } u_x(0,t) &= 0, & t > 0. \end{cases} \tag{12}$$

We will exploit the following fact, a proof of which may be found in [IvrXX, §2.6]:

[1] We assume the integration constant vanishes to guarantee continuity of the solution.

Proposition 1.1. *Consider the following IVP on entire real line*

$$\begin{cases} u_{tt} - c^2 u_{xx} &= F(x,t), \quad t > 0, \\ u(x,0) &= G(x), \\ u_t(x,0) &= H(x). \end{cases} \tag{13}$$

If F, G, H are all even (resp. odd)[2] *functions then the solution to* (13) *if also even (resp. odd).*

To solve (12) we now use the following trick. Suppose we consider homogeneous Dirichlet BCs (the Neumann case will be similar, replacing "odd" for "even" in what follows). Let F, G, H be the odd extensions of f, g, h to the entire real line, i.e.

$$G(\pm|x|) = \pm g(|x|), \quad \text{etc.}$$

Then applying D'Alembert's formula (writing the case $f \equiv 0$ for simplicity; consider $f \neq 0$ as an exercise) we find:

$$u(x,t) = \frac{G(x+ct) + G(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} H(s)ds$$

For $x > ct$ this is the usual D'Alembert's formula. For $0 < x < ct$ the integral of H in the above becomes

$$\begin{aligned} \int_{x-ct}^{x+ct} H(s)ds &= \int_0^{x+ct} H(s)ds + \int_{x-ct}^{0} H(s)ds \\ &= \int_0^{x+ct} h(s)ds - \int_{x-ct}^{0} h(-s)ds \\ &= \int_0^{x+ct} h(s)ds + \int_{ct-x}^{0} h(\sigma)d\sigma \\ &= \int_{ct-x}^{ct+x} h(s)ds \end{aligned}$$

and so we can write the solution:

$$u(x,t) = \frac{g(x+ct) - g(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{ct+x} h(s)ds.$$

1.2 The wave equation on a finite interval

The above techniques we used to solve problems for the wave equation on the half-line – using the domain of dependence and the method of continuation – may also be extended to solve the wave equation on a finite interval.

We will not discuss this very much in lecture, as we will study more effective techniques to attack this problem later in the semester. You **should** however read the corresponding sections of [IvrXX, §2.6].

The basic ideas are as follows.

- **Domain of dependence:** To solve the problem on a finite interval, one simply iterates the arguments made for the half-line to produce the appropriate extensions of ϕ and ψ.
- **Method of continuation:** At each end of the interval, extend the data of the problem by another interval in an odd (Dirichlet) or even (Neumann) way. Iterating this, one extends the inital data to the entire real line as a periodic function (with exact periodicity determined by the form of the function and whether the BCs at each end of the interval agree or disagree).

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[2] Recall: A function ϕ is *even* if $\phi(-x) = \phi(x)$ and *odd* if $\phi(-x) = -\phi(x)$

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.