APM 346 Lecture 5.

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This week we continue our study of the 1d wave equation. References are: [IvrXX, §2.4-6] (§2.4, §2.5, §2.6) and [Str08, Ch.2.1-2,Ch.3.2,Ch.3.4].

1 Solving the wave equation via characteristic coordinates

Recall that to solve the transport equation $au_t + bu_x = 0$ we made the observation that any solution must be constant along certain *characteristic curves* (which are straight lines in the constant coefficient situation).

This approach may be extended to our analysis of the wave equation as follows. Consider the *characteristic lines*

$$x + ct = \text{ const.} \quad \text{and} \quad x - ct = \text{ const.}$$

which are parametrised by the *characteristic coordinates*

$$\begin{cases} \xi &= x + ct \\ \eta &= x - ct \end{cases} \tag{1}$$

Proposition 1.1. *The LHS of the wave equation may be rewritten as*

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} \tag{2}$$

Proof. (1) may be rewritten as

$$x = \frac{1}{2}(\xi + \eta) \quad \text{and} \quad t = \frac{1}{2c}(\xi - \eta)$$

and hence by the chain rule

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t}$$

and there is a similar calculation for $\frac{\partial}{\partial \eta}$ (exercise). So,

$$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right) = \frac{1}{4c^2}\left(c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)$$

and an algebraic manipulation completes the proof. □

So, in characteristic coordinates the 1d homogeneous wave equation $u_{tt} - c^2 u_{xx}$ becomes

$$u_{\xi\eta} = 0. \tag{3}$$

But we already solved this equation back in Lecture 1! The general solution to (3) is given by

$$u = \phi(\xi) + \psi(\eta) = \phi(x + ct) + \psi(x - ct) \tag{4}$$

which agrees with the general solution we found in Lecture 4.

Example 1. Consider the *Goursat problem*

$$\begin{cases} u_{\xi\eta}(\xi,\eta) &= 0 & \xi>0,\eta>0 & \text{PDE} \\ u(\xi,0) &= g(\xi) & \xi>0 & \text{C1} \\ u(0,\eta) &= h(\eta) & \eta>0 & \text{C2} \\ g(0) &= h(0) & & \text{compatibiility condition} \end{cases} \tag{5}$$

It is an easy exercise to show that the solution to (5) is given by $u(\xi,\eta) = g(\xi) + h(\eta) - g(0)$.

1.1 The inhomogeneous wave equation and the d'Alembert formula

Let us now consider an application of characteristic coordinates to the *inhomogeneous* wave equation. Consider the IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} &= f(x,t) & \text{PDE} \\ u(x,0) &= g(x) & \text{IC1} \\ u_t(x,0) &= h(x) & \text{IC2} \end{cases} \tag{6}$$

Let us first solve (6) under the assumption the $g \equiv h \equiv 0$. Changing coordinates from (x,t) to (ξ,η) and using Proposition 1.1, we may rewrite the inhomogeneous wave equation as

$$u_{\xi\eta}(\xi,\eta) = -\frac{1}{4c^2} f(\xi,\eta). \tag{7}$$

Now, we may integrate both sides to obtain

$$u_\xi(\xi,\eta) = -\frac{1}{4c^2}\int_{\eta_0}^{\eta} f(\xi,\eta')d\eta' + u_\xi(\xi,\eta_0). \tag{8}$$

Since we may choose the lower limit of integration of η_0, let us set it equal to ξ. Then

$$u_\xi(\xi,\eta) = -\frac{1}{4c^2}\int_{\xi}^{\eta} f(\xi,\eta')d\eta' + u_\xi(\xi,\xi). \tag{9}$$

Now, the line $t=0$ (where we apply our initial conditions) becomes in characteristic coordinates the line $\xi=\eta$. So applying the initial conditions of (6) with $g=h=0$ we have that $u_\xi(\xi,\xi)=0$ and so

$$u_\xi(\xi,\eta) = -\frac{1}{4c^2}\int_{\xi}^{\eta} f(\xi,\eta')d\eta' = \frac{1}{4c^2}\int_{\eta}^{\xi} f(\xi,\eta')d\eta'. \tag{10}$$

Integrating (10) with respect to ξ, and using the initial condition $u(\eta,\eta)=0$ to choose the lower limit of integration to be $\xi_0=\eta$, we obtain

$$u(\xi,\eta) = \frac{1}{4c^2}\int_{\eta}^{\xi}\int_{\eta}^{\xi'} f(\xi',\eta')d\eta' d\xi' \tag{11}$$

We wish to transform this into a integral over some domain in the (x,t)-plane. Assume that $\xi>\eta$ (this holds for $t>0$). In the (ξ',η')-plane we are integrating over a right angle triangle with (Figure 1):

- Base: the horizontal line $\eta \le \xi' \le \xi$, at height $\eta' = \eta$.
- Side: the vertical line $\eta \le \eta' \le \xi$, at horizontal position $\xi' = \xi$.
- Hypotenuse: the diagonal line $\eta' = \xi'$ with $\eta \le \eta' \le \xi$.

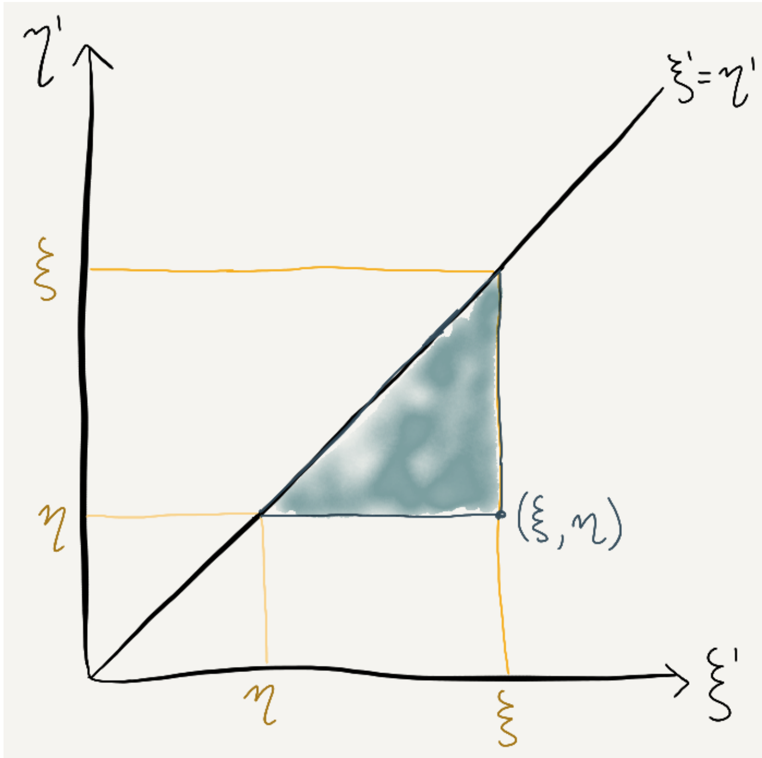

Figure 1: Domain of integration in characteristic coordinates.

Under the linear change of coordinates

$$\xi' = x' + ct' \quad \text{and} \quad \eta' = x' - ct'$$

this triangle will be transformed into a trangle in the (x', t')-plane. In the new triangle, which we will denote by $\Delta(x,t)$ (Figure 2):

- The hypotenuse becomes: the horizontal segment with $t' = 0$ and $x - ct \le x' \le x + ct$.
- The base becomes: the diagonal line $x' = x - c(t - t')$ with $0 \le t' \le t$.
- The side becomes: the diagonal line $x' = x + c(t - t')$ with $0 \le t' \le t$.

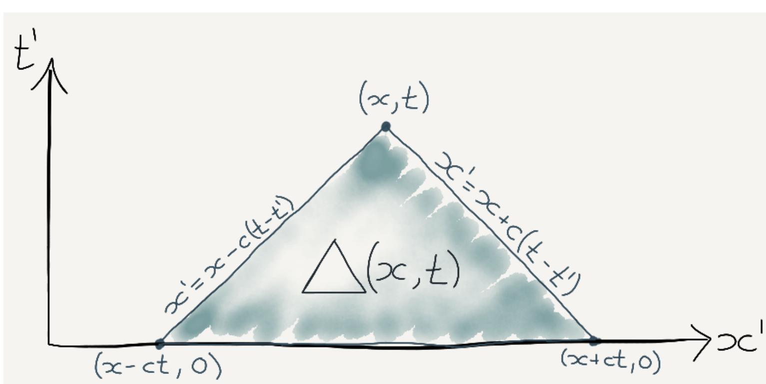

Figure 2: Domain of dependence.

So, changing coordinates back from (ξ', η') to (x', t'), and making sure to include the factor coming from the Jacobian, the solution to (6) with $g = h = 0$ becomes

$$u(x,t) = \frac{1}{2c}\iint_{\Delta(x,t)} f(x',t')dx'dt'. \tag{12}$$

Example 2. Back in Lecture 2 we derived that the vibration of a string acted on externally by Earth's gravity is described by the equation

$$u_{tt} - c^2 u_{xx} = -g$$

where $g \simeq 9.8\text{m/sec}^2$ (and c depends on the tension and mass density of the string). Suppose that initially the string is flat and stationary, i.e. $u(x,0) = u_t(x,0) = 0$. Then according to (12), the vertical displacement of the string is given by

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} (-g)dx'd\tau \\ &= -\frac{g}{2c}\int_0^t \left((x + c(t-\tau)) - (x - c(t-\tau))\right)d\tau \\ &= -\frac{g}{2c}\int_0^t 2c(t-\tau)d\tau \\ &= g\int_t^0 \tau' d\tau' \\ &= -\frac{g}{2}t^2 \end{aligned}$$

Is this solution realistic? Why or why not?

1.1.1 Incorporating nonzero initial conditions

Recall the IVP (6)

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{PDE} \\ u(x,0) = g(x) & \text{IC1} \\ u_t(x,0) = h(x) & \text{IC2} \end{cases}$$

only now consider the situation where none of $f(x,t)$, $g(x)$ or $h(x)$ are assumed to be zero.

Since our equation is linear, if we can find solutions u_H and u_P solving the problems[1]

$$\begin{cases} (u_P)_{tt} - c^2 (u_P)_{xx} = f(x,t) & \text{PDE} \\ u_P(x,0) = 0 & \text{IC1} \\ (u_P)_t(x,0) = 0 & \text{IC2} \end{cases} \tag{13}$$

and

$$\begin{cases} (u_H)_{tt} - c^2 (u_H)_{xx} = 0 & \text{PDE} \\ (u_H)(x,0) = g(x) & \text{IC1} \\ (u_H)_t(x,0) = h(x) & \text{IC2} \end{cases} \tag{14}$$

then their sum $u(x,t) = u_H(x,t) + u_P(x,t)$ will solve the original problem (6).

But now: We determined the solution u_H in Lecture 4, and we have determined the solution u_P in (12)! So we can write down the solution to our IVP as

$$u(x,t) = \underbrace{\frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds}_{u_H(x,t)} + \underbrace{\frac{1}{2c}\iint_{\Delta(x,t)} f(x',t')dx'dt'}_{u_P(x,t)}. \tag{15}$$

[1]Here H stands for "homogeneous solution" and P stands for "particular solution".

Definition 1.1. Equation (15) is called *D'Alembert's Formula.*

Example 3. Consider the IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} &= (\theta(t-1) - \theta(t-2))\cos(x) \\ u(x,0) &= 0 \\ u_t(x,0) &= 0 \end{cases} \tag{16}$$

where θ is the Heaviside step function

$$\theta(t) := \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t > 0 \end{cases} \tag{17}$$

I.e. we are turning on an external force $\cos(x)$ at time $t = 1$ and turning it off again at time $t = 2$. It is an exercise to show that (16) is solved by the function:

$$u(x,t) = \begin{cases} 0, & \text{if } t < 1, \\ \frac{\cos(x)}{c^2} - \frac{\cos(x+c(t-1)) + \cos(x-c(t-1))}{2c^2} & \text{if } 1 < t < 2, \\ \frac{\cos(x+c(t-2)) + \cos(x-c(t-2))}{2c^2} - \frac{\cos(x+c(t-1)) + \cos(x-c(t-1))}{2c^2} & \text{if } t > 2. \end{cases} \tag{18}$$

See the Mathematica file/visualisations page for an animation of this solution.

2 The Duhamel Formula

Above we derived the solution (12) to (13) by making a judicious change-of-coordinates using the method of characteristics. Let us now consider another derivation of this solution, but from a different perspective.

Recall that the problem we are considering is

$$\left\{ \begin{array}{rcl|l} u_{tt} - c^2 u_{xx} &=& f(x,t) & \text{PDE} \\ u(x,0) &=& 0 & \text{IC1} \\ u_t(x,0) &=& 0 & \text{IC2} \end{array} \right. \tag{19}$$

Define an auxilliary function

$$U(x,t,\tau), \quad 0 < \tau < t$$

as the solution to the auxilliary problem

$$\begin{aligned} U_{tt} - c^2 U_{xx} &= 0, \\ U|_{t=\tau} &= 0, \\ U_t|_{t=\tau} &= f(x,\tau) \end{aligned} \tag{20}$$

Proposition 2.1. (The Duhamel Formula) *The function*

$$u(x,t) = \int_0^t U(x,t,\tau)d\tau \tag{21}$$

is a solution to (19).

Proof. The variable x appears only in the integrand, so we immediately have

$$u_{xx}(x,t) = \int_0^t U_{xx}(x,t,\tau)d\tau. \tag{22}$$

Applying the formula

$$\frac{d}{dt}\left(\int_{\alpha(t)}^{\beta(t)} F(t,\tau)d\tau\right) = F(t,\beta(t))\frac{d\beta}{dt} - F(t,\alpha(t))\frac{d\alpha}{dt} + \int_{\alpha(t)}^{\beta(t)} \frac{\partial F}{\partial t}(t,\tau)d\tau \tag{23}$$

to (21) gives

$$u_t(x,t) = U(x,t,t) + \int_0^t U_t(x,t,\tau)d\tau \tag{24}$$

and by (20) $U(x,t,t) = 0$, so that

$$u_t(x,t) = \int_0^t U_t(x,t,\tau)d\tau. \tag{25}$$

Differentiating again with respect to t gives

$$u_{tt}(x,t) = U_t(x,t,t) + \int_0^t U_{tt}(x,t,\tau)d\tau, \tag{26}$$

and since by (20) $U_t(x,t,t) = f(x,t)$,

$$u_{tt}(x,t) = f(x,t) + \int_0^t U_{tt}(x,t,\tau)d\tau. \tag{27}$$

But now

$$u_{tt} - c^2 u_{xx} = f(x,t) + \int_0^t \underbrace{\left(U_{tt} - c^2 U_{xx}\right)}_{=0} d\tau = f(x,t)$$

so that u satisfies the desired PDE. It satisfies the desired initial conditions since

$$u(x,0) = \int_0^0 U d\tau = 0 \quad \text{and} \quad u_t(x,0) = \int_0^0 U_t d\tau = 0.$$

□

Remark 2.1. Let's pause for a moment to consider what the Duhamel Formula (21) actually *means*:

- We want to solve an *inhomogeneous* problem for the wave equation (19).
- I.e. we are trying to solve for the dynamics of a wave (u) that is subject to some external force (f). (Recall in Lecture 2 we saw that this is exactly the equation describing a string vibrating in the prescence of an external force determined by f).
- Change focus to the *effect* of the external force f. Recalling that the final result in our derivation of the wave equation was

  $$u_{tt} - c^2 u_{xx} = \frac{\text{external force}}{\text{mass density}}$$

  we see from Newton's Second Law that the function $f(x,t)$ represents an externally imposed *acceleration* of the string.
- Consider imposing this acceleration $f(x,\tau)$ over the time period $[\tau - \Delta\tau, \tau]$. The string will acquire a velocity of $f(x,\tau)\Delta\tau$ and will be displaced by $f(x,\tau)\frac{\Delta\tau^2}{2}$. Assume that $\Delta\tau$ is very small, so that we may approximate $\Delta\tau^2 \simeq 0$.
- Hence we have that to evolve a solution of (19) from time $\tau - \Delta\tau$ to time τ, one must add the $u(x, \tau - \Delta\tau)$ ($\Delta\tau$ times) a solution to the problem, defined for $t \geq \tau$,

  $$\begin{cases} U_{tt} - c^2 U_{xx} &= 0 \\ U|_{t=\tau} &= 0 \\ U_t|_{t=\tau} &= f(x,\tau) \end{cases} \tag{28}$$

  i.e. we have successfully moved the force from the PDE to the initial conditions, by explicitly considering *how* the force is acting on the string at each moment in time.

- Therefore, to evolve a solution of (19) from time 0 to time t, one simply adds up the contributions coming from each time interval,

$$u(x,t) \sim \underbrace{u(x,0)}_{=0} + U(x,t,\Delta\tau)\Delta\tau + U(x,t,2\Delta\tau)\Delta\tau + \cdots$$

Taking $\Delta\tau \to d\tau$ to be infinitesimal, this is exactly the integral (21)

$$u(x,t) = \int_0^t U(x,t,\tau)d\tau.$$

Now: we have solved the auxilliary problem (20) before (it is a homogeneous wave equation, studied in Lecture 4), so we know that the solution is

$$U(x,t,\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(x',\tau)dx' \tag{29}$$

and so

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(x',\tau)dx'd\tau, \tag{30}$$

in agreement with (12).

3 Domains of dependence and influence

Let's now think a little harder about what D'Alembert's Formula (15) is telling us about the solution to (6). Recall that this formula is

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds + \frac{1}{2c}\iint_{\Delta(x,t)} f(x',t')dx'dt'$$

Proposition 3.1. *The solution $u(x,t)$ to the IVP for the inhomogeneous wave equation* (6) *depends only on:*

- *the values of the function $f(x,t)$ on the domain $\Delta(x,t)$, and*
- *the values of the initial data g and h on the base of $\Delta(x,t)$.*

Definition 3.1. $\Delta(x,t)$ is called the *domain of dependence* for the point (x,t).

Remark 3.1. The terminology "domain of dependence" makes sense in a very literal fashion: the value of the solution u at the point (x,t) can only depend on data in the *domain* $\Delta(x,t)$.

Remark 3.2. [IvrXX, §2.5] calls $\Delta(x,t)$ the *triangle* of dependence. We use the terminology *domain* of dependence since, as noted in [IvrXX, Remark 3], the concept is applicable to more general situations where $\Delta(x,t)$ is no longer a triangle.

Conversely, we could consider not the domain of points that will influence the solution at (x,t), but the domain of points that data at (x,t) will itself influence (Figure 3)

$$\Delta^+(x,t) := \{(x',t') \mid (x,t) \in \Delta(x',t')\} \tag{31}$$

Definition 3.2. $\Delta^+(x,t)$ is called the *domain of influence* for the point (x,t).

Considering the domains of influence and dependence for any given point (x,t), we see that:

Proposition 3.2. *The solution to the wave equation propagates at finite speed not exceeding c.*

For instance: it will take at least t_* units of time for data at position x to have an effect on the solution at position $x + ct_*$.

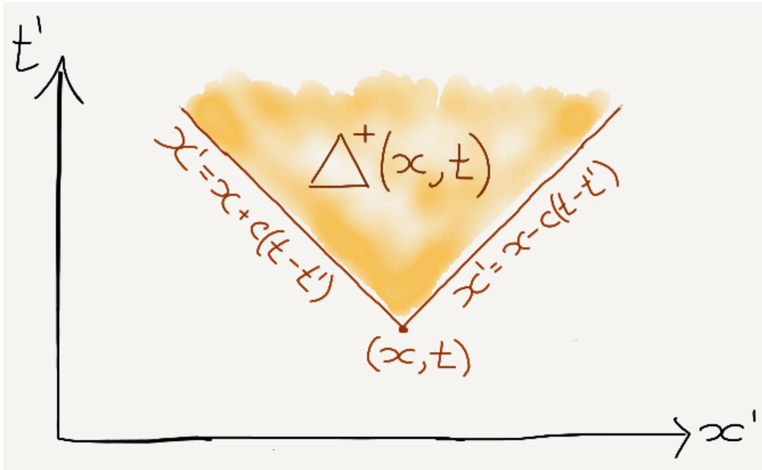

Figure 3: Domain of influence.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.