APM 346 Lecture Notes 4.

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We will now begin to turn our attention to the 1d wave equation: see [IvrXX, §2.3] (§2.3) and [Str08, Ch.2.1].

1 The Homogeneous 1d Wave Equation

We wish to consider the *homogeneous 1d wave equation*

$$u_{tt} - c^2 u_{xx} = 0 \tag{1}$$

Recall from Lecture 2 that (1) describes the dynamics of a vibrating string where

- $u(x,t)$ is the vertical displacement of the string,
- $c^2 = \frac{T}{\rho}$,
- T is the tension of the string,
- ρ is the mass density of the string,

and c, which has dimensions of speed, may be interpreted as the speed of wave propagation. (For more physical systems modelled by the wave equation, consult [IvrXX, §2.3]).

1.1 The general solution via factorisation of differential operators

We would like to find the general solution to (1). There are multiple ways to go about this – to begin with, let us proceed by factorizing (1) as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0 \tag{2}$$

Now, rather than trying to solve a single second order PDE, we will reduce to solving a *system* of first order PDEs which are subject to some constraints. Namely, set

$$v(x,t) := (\partial_t + c\partial_x)u \tag{3}$$
$$w(x,t) := (\partial_t - c\partial_x)u \tag{4}$$

so that v and w satsify the first order equations

$$v_t - cv_x = 0 \tag{5}$$
$$w_t + cw_x = 0 \tag{6}$$

But there are exactly the sorts of equations we solved in Lecture 3! The general solutions to (5) and (6) are

$$v(x,t) = 2c\Phi(x+ct) \tag{7}$$
$$w(x,t) = -2c\Psi(x-ct) \tag{8}$$

where the factors of $2c$ and $-2c$ are introduced to make our calculations a little easier (as we will soon see). Now, substituting in the definitions of v and w in terms of u we find that

$$u_t + cu_x = 2c\Phi(x+ct) \tag{9}$$
$$u_t - cu_x = -2c\Psi(x-ct) \tag{10}$$

This now looks like a problem in linear algebra! Add/subtract (10) to (9) to obtain

$$\frac{1}{c}u_t = \Phi(x+ct) - \Psi(x-ct) \tag{11}$$
$$u_x = \Phi(x+ct) + \Psi(x-ct) \tag{12}$$

Now, choose primitives ϕ and ψ for the functions Φ and Ψ. Integrating (12), we obtain

$$u(x,t) = \phi(x+ct) + \psi(x-ct) + R(t) \tag{13}$$

where the remainder term $R(t)$ is an arbitrary "constant" of integration (since we are working with partial derivatives, really R is constant in x, but an arbitrary function of t). Taking the partial derivative of (13) with respect to t, we get that

$$u_t(x,t) = c\Phi(x+ct) - c\Psi(x-ct) + R'(t) \tag{14}$$

and comparing this with (11) we see that $R'(t) \equiv 0$, and so R is a constant which may be absorbed into the functions ϕ and ψ. The general solution is therefore given by

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \tag{15}$$

where ϕ and ψ are arbitrary functions.

1.1.1 Interpretation of the general solution

Recall from Lecture 3, when we solved the constant coefficient transport equation, that there a solution of the form

$$u(x,t) = \psi(x-ct)$$

represents a wave propagating to the *right* with constant speed $c > 0$. Similarly, a solution of the form

$$u(x,t) = \phi(x+ct)$$

reprsents a wave propagating to the *left* with constant speed $c > 0$. I.e. we have seen both components of the general solution (15) before – *individually.*

The novelty in (15) is that now we may have both right-moving and left-moving wave fronts propagating together. This potentially small-seeming change to the form of our solution allows for interesting new phenomena to develop, via the interference of left- and right-moving wave fronts.

Example 1. Standing waves: these are waves whose amplitude oscillates in magnitude over time, but whose amplitude profile does not move in space. For example: interference between the left and right moving waves results in nodes of zero amplitude which do not propagate through space. See the Mathematica file or visualisations page for this lecture for an animation.

Example 2. Also in the Mathematica file is an example where the left and right moving waves interfere with eachother, but do not result in a standing wave. The equation for the solution in that Mathematica file is

$$u(x,t) = \underbrace{\frac{3}{5}\left(\frac{x-t}{2}\right)^2 \sin(x-t)}_{\text{right mover}} - \underbrace{\frac{60}{\cosh(x+t)}}_{\text{left mover}}.$$

1.2 Initial value problem for the homogeneous wave equation

Let's now consider a general IVP for the homogeneous 1d wave equation:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= 0 & \text{PDE} \\ u(x,0) &= g(x) & \text{IC1} \\ u_t(x,0) &= h(x) & \text{IC2} \end{cases} \tag{16}$$

From the general solution (15) we have that

$$u(x,t) = \phi(x+ct) + \psi(x-ct), \quad \text{and}$$
$$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct)$$

Substituting the ICs from (16) into these expressions, we find that

$$g(x) = \phi(x) + \psi(x) \tag{17}$$
$$h(x) = c\phi'(x) - c\psi'(x) \tag{18}$$

Integrating (18) from an arbitrary starting point x_0 gives

$$\frac{1}{c}\int_{x_0}^{x} h(s)ds = \int_{x_0}^{x} (\phi'(s) - \psi'(s))\, ds = \phi(x) - \psi(x) - \phi(x_0) + \psi(x_0),$$

and so setting $C(x_0) = \phi(x_0) - \psi(x_0)$, we have that

$$\phi(x) + \psi(x) = g(x) \tag{19}$$
$$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} h(s)ds + C(x_0) \tag{20}$$

A little linear algebra gives that

$$\phi(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_{x_0}^{x} h(s)ds + C(x_0) \tag{21}$$
$$\psi(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_{x_0}^{x} h(s)ds - C(x_0) \tag{22}$$

and so we may conclude that

$$u(x,t) = \underbrace{\frac{1}{2}g(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} h(s)ds + C(x_0)}_{\phi(x+ct)} + \underbrace{\frac{1}{2}g(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} h(s)ds - C(x_0)}_{\psi(x-ct)}$$

i.e.

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds \tag{23}$$

Example 3. Suppose we consider the IVP

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) &= 0 & \text{PDE} \\ u(x,0) &= \frac{20}{\cosh(x)} & \text{IC1} \\ u_t(x,0) &= 1 & \text{IC2} \end{cases} \tag{24}$$

Then we can evaluate (23) with $c = 1$, $g(x) = \frac{20}{\cosh(x)}$ and $h(x) = 1$ to obtain the solution

$$u(x,t) = 10\left(\frac{1}{\cosh(x+t)} + \frac{1}{\cosh(x-t)}\right) + t$$

See the Mathematica file or visualisations page for an animation of this solution.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.