APM 346 Lecture 3.

Richard Derryberry

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This week we will focus on first order partial differential equations, and the method of characteristics technique. Relevant textbook sections are [IvrXX, §2.1-2.2]: §2.1; §2.2. You may also wish to consult [Str08, §1.2 and §14.1].

1 Solving the Transport Equation

Consider the PDE

$$au_t + bu_x = 0 \tag{1}$$

where for the moment we leave the precise properties of a and b unspecified. If we let $\vec{l}(t, x) := (a, b)$, then the LHS of (1) may be rewritten as

$$\vec{l} \cdot \nabla u. \tag{2}$$

(1) and (2) taken together tell us that the function u is *constant in the $\vec{l}$ direction*, and therefore is constant along the *integral curves* of $\vec{l}$ – curves in the (t, x)-plane which satisfy the equation

$$\frac{dt}{a} = \frac{dx}{b} \tag{3}$$

i.e. curves such that the tangent vector to the curve at every point is given by $\vec{l}$.

Remark 1.1. If $\|\vec{l}\| = 1$ we say that (2) is the *directional derivative of u in the $\vec{l}$-direction.*

Remark 1.2. Compare this to the *transport equation* we derived in Lecture 2,

$$u_t(x, t) + V(x, t)u_x(x, t) = S(x, t, u). \tag{4}$$

where

- u is the density of the transported substance,
- V is the velocity of the transported substance, and
- S is an external source/sink term.

Then comparing equations we have that $S = 0$,

$$V(x, t) = \frac{b}{a} = \frac{dx}{dt},$$

and we may interpret (1) and (2) as saying that *in the absence of external sources and sinks, the density u remains unchanged from the point of view of an observer travelling along with the substance at velocity $V(x, t)$.*

1.1 Constant coefficient case

As a warmup, let's consider the case where a and b are both constant; further let's assume that $a \neq 0$ (we solved the $a = 0$ version of this equation in Lecture 1). Writing $c = \frac{b}{a}$, the characteristic curve equation becomes

$$\frac{dx}{dt} = c \tag{5}$$

which we may solve to obtain the integral curves $x - ct = C$, where C is a constant along integral curves and (provided we consider the entire (x, t)-plane) labels them (Figure 1).

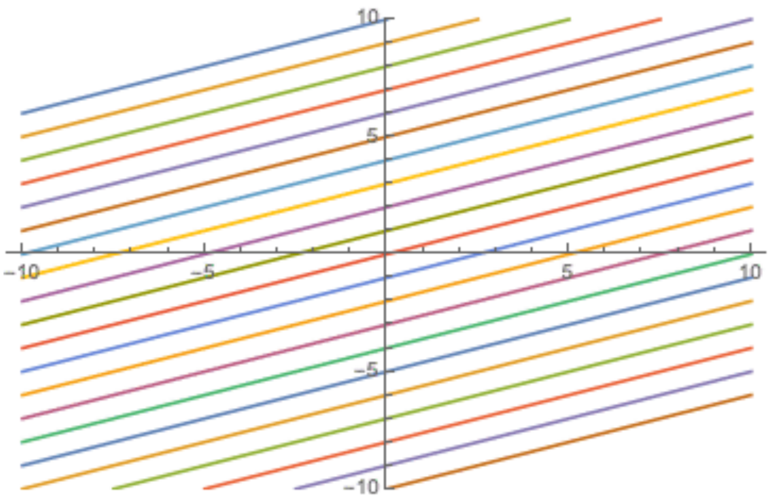

Figure 1: Example of straight line characteristics.

Therefore, u depends only on C, and the general solution to our equation is

$$u(x, t) = \phi(x - ct) \tag{6}$$

where ϕ is an arbitrary function.

Definition 1.1. The solutions $\phi(x - ct)$ are called *running waves*, and c is called the *propagation speed*.

1.1.1 Constant coefficient IVP

Consider the IVP

$$\begin{cases} u_t + cu_x &= 0, \\ u(x,0) &= f(x) \end{cases} \tag{7}$$

We know that the general solution is given by $u(x, t) = \phi(x - ct)$ for an arbitrary function ϕ; using the initial condition of (7) gives

$$u(x, 0) = \phi(x) = f(x)$$

and so the solution to the IVP (7) is

$$u(x, t) = f(x - ct). \tag{8}$$

1.2 Variable coefficient case

Suppose now that in (1) the coefficients a and b are functions of the independent variables: $a = a(x,t)$ and $b = b(x,t)$. It still makes sense to look for integral curves along which the solution u is constant – the difference is that now the integral curves will not necessarily[1] be straight lines.

Example 1. Consider the IVP

$$\begin{cases} u_t + tu_x &= 0, \\ u(x,0) &= f(x) \end{cases} \tag{9}$$

The integral curve equation is

$$\frac{dx}{dt} = t,$$

hence the integral curves are $x - \frac{1}{2}t^2 = C$ for C a constant (Firgure 2), and the solution to the IVP is

$$u(x,t) = f\left(x - \frac{1}{2}t^2\right).$$

Example 2. Reversing the roles of x and t, consider the IVP

$$\begin{cases} u_t + tu_x &= 0, \\ u(0,t) &= g(t) \end{cases} \tag{10}$$

This is an ill-posed problem. First: the characteristic curves are still given by $x - \frac{1}{2}t^2 = C$, and so at $x = 0$

$$t^2 = -2C$$

which has:

- Two real solutions for $C < 0$.
- One solution for $C = 0$.
- No real solutions for $C > 0$.

Consider those curves that do intersect the t-axis: with the exception of $x = \frac{1}{2}t^2$, these all intersect at *two* points $\pm t_C(0)$. Since any solution is constant along these curves, we require that $g(-t) = g(t)$ i.e. that g is an even function. If g is not an even function, no solution can exist.

Now: suppose that g is even. Then we can solve for u along any characteristic curve that intersects the t-axis – but we do not have any constraints on those curves that do not intersect the t-axis! Consequently, in the region $x > \frac{1}{2}t^2$ we are unable to single out a particular solution, and so uniqueness fails.

1.3 Including a source/sink term

Suppose that we include a source/sink term S on the RHS of (1), so that we are considering the PDE

$$au_t + bu_x = S(x,t,u). \tag{11}$$

We can rewrite this as

$$(a,b) \cdot \nabla u = S(x,t,u), \tag{12}$$

[1] It is possible that some of the integral curves will still be straight lines, while others will be more complicated curves.

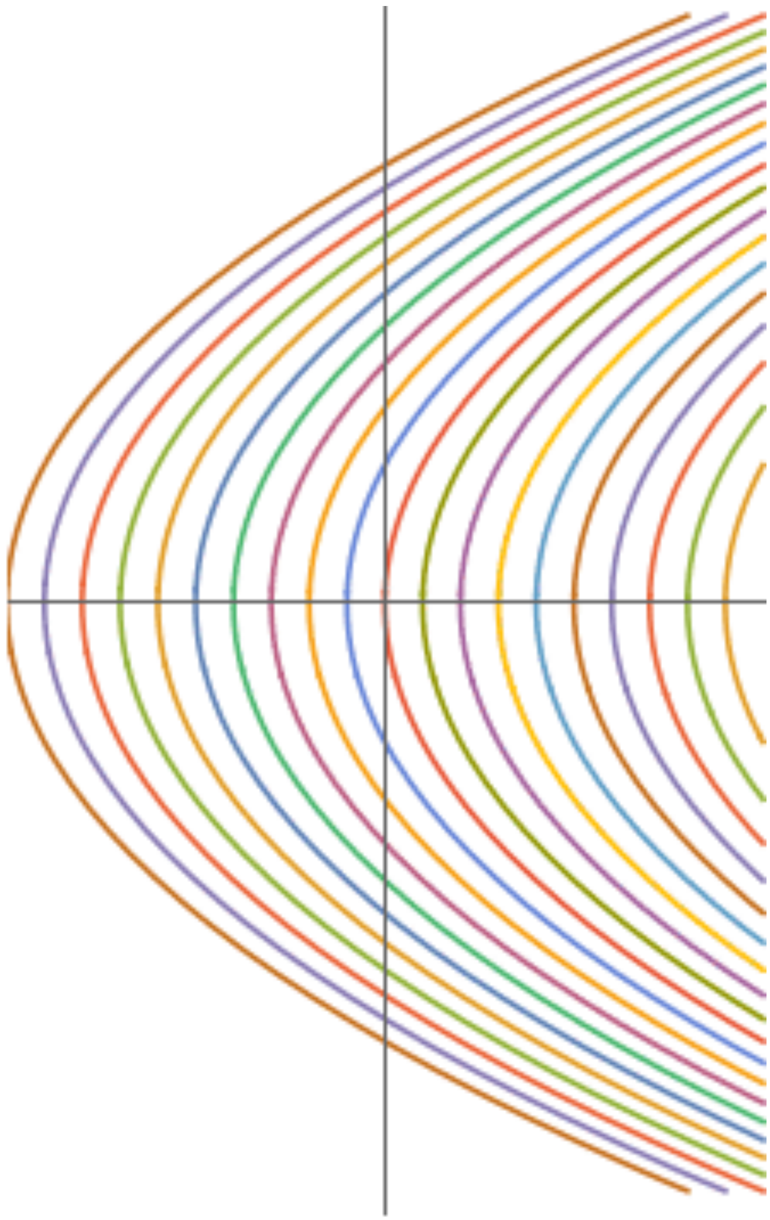

Figure 2: Example of quadratic curve characteristics.

so that rather than telling us that u is *constant along characteristics*, (12) tells us *how u changes along characteristics*. Recalling the integral curve equation $bdt = adx$ (3), we have

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}\frac{b}{a}dt = \frac{au_t + bu_x}{a}dt = \frac{S}{a}dt$$

so that

$$\frac{dt}{a} = \frac{dx}{b} = \frac{du}{S}. \tag{13}$$

Example 3. Consider the PDE

$$u_t + xu_x = x$$

which yields the integral curve equation

$$dt = \frac{dx}{x}.$$

This can be directly integrated to obtain the integral curves (Figure 3)

$$t = \log(x) + (\text{constant}) \quad \text{or equivalently} \quad x = Ce^t.$$

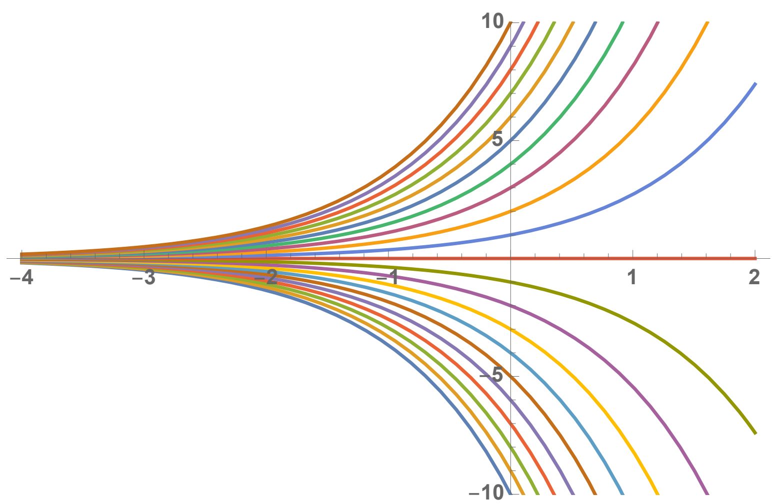

Figure 3: Example of exponential curve characteristics.

From (13) we have $du = dx$ along integral curves, so that $u = x + D$ where D is constant along integral curves. Hence the general solution to the PDE is

$$u(x,t) = x + \phi(xe^{-t})$$

where ϕ is an arbitrary function of a single variable.

Let's check that this really *does* solve our PDE. Letting $s = xe^{-t}$, we have

$$\frac{\partial u}{\partial t} = \frac{\partial s}{\partial t}\frac{d\phi}{ds} = -xe^{-t}\phi'(xe^{-t})$$
$$\frac{\partial u}{\partial x} = 1 + \frac{\partial s}{\partial x}\phi'(s) = 1 + e^{-t}\phi'(xe^{-t})$$

and so

$$u_t + xu_x = -xe^{-t}\phi'(xe^{-t}) + x + xe^{-t}\phi'(xe^{-t}) = x.$$

1.3.1 Linear and semilinear case

Recall from Lecture 1 that:

- If $a = a(x,t)$, $b = b(x,t)$, and S is a linear function of u, then (11) is *linear*.
- If $a = a(x,t)$, $b = b(x,t)$, and $S = S(x,t,u)$ is not linear in u, then (11) is *semilinear*.

Equation (13) still holds, and gives us an ODE for u along the integral curves that we can solve.

Example 4. Consider the PDE

$$u_t + xu_x = -u^2.$$

As in Example 3, the integral curves are given by $x = Ce^t$. Equation (13) gives us the equation

$$-\frac{du}{u^2} = dt$$

which we may solve to find

$$\frac{1}{u} = t + D$$

where D is contant along the integral curves $xe^{-t} = C$. The general solution is therefore given by

$$u(x,t) = \frac{1}{t + \phi(xe^{-t})}$$

where ϕ is an arbitrary function of one variable. See the Mathematica file (or visualisations page) for this lecture for animated solutions with initial conditions $u(x,0) = e^{-x}$ and $u(x,0) = \frac{1}{2\log|x|}$.

1.3.2 Quasilinear Transport Equation

Recall from Lecture 1 that if $a = a(x,t,u)$ or $b = b(x,t,u)$ then (13) is a *quasilinear* equation. Note that a and b here are functions of u, **not** of the derivatives of u.

A naive application of the method of characteristics to the solution of a quasilinear equation will usually fail: the characteristic curves can intersect, leading to regions where u appears multivalued, and regions with no characteristic curves where u appears undetermined.

It is possible to salvage the method of characteristics approach through a subtler analysis which incorporates expansion/rarefaction waves which can emanate from intersection points of characteristic curves, and shock waves which propagate a discontinuity between two solutions in regions where naively the solution would be multivalued.

These approaches are beyond the scope of this course: for more information see [IvrXX, §12.1] (available here) or [Str08, Ch.14.1].

1.4 An IBVP for the Transport Equation

So far we have only considered IVPs for the transport equation, where we look for a solution defined for all $x \in \mathbb{R}$. We could also consider IBVPs, where we look for a solution only for some subset of $x \in \mathbb{R}$ – e.g. traffic entering a tunnel beginning at $x = 0$. In such a situation it is often not enough to simply impose initial conditions – one must impose boundary conditions as well.

Example 5. Consider the IBVP for the transport equation

$$\begin{cases} u_t + cu_x &= 0, & x > 0, t > 0, & \text{PDE} \\ u(x,0) &= f(x), & x > 0, & \text{IC} \\ u(0,t) &= g(t), & t > 0, & \text{BC} \end{cases} \tag{14}$$

where $c > 0$, so we are considering a right-moving flow with general solution $u(x,t) = \phi(x - ct)$, for some function ϕ which we will now determine.

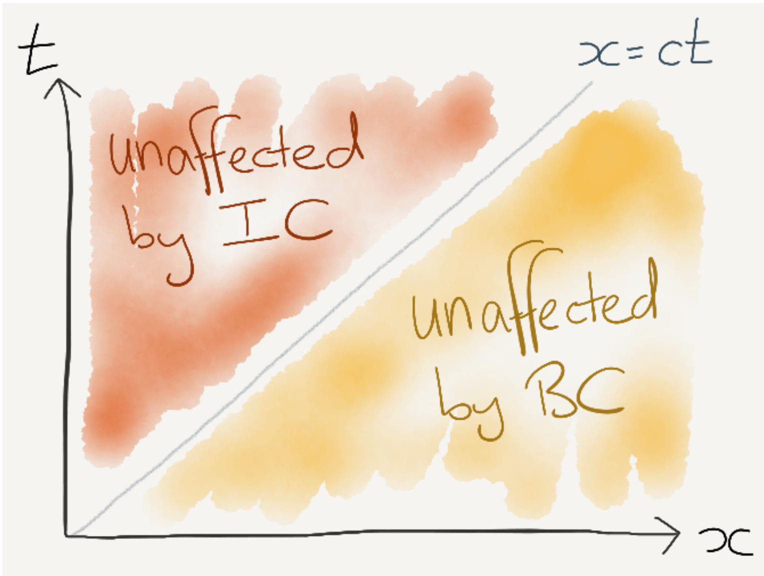

Figure 4: IC and BC dependent regions of the IBVP (14).

To apply the initial condition: we set $t = 0$ to obtain

$$\phi(x) = f(x) \quad \text{for } x > 0,$$

which (when we propagate forward in time) yields

$$u(x,t) = f(x - ct) \quad \text{for } x - ct > 0.$$

To solve for ϕ in the region $x < ct$, we apply the boundary condition at $x = 0$

$$\phi(-ct) = g(t) \quad \text{for } t > 0$$

and so as we allow this solution to propagate forward in *space* along the characteristics $t - \frac{x}{c} = C$ we obtain

$$u(x,t) = g\left(t - \frac{x}{c}\right).$$

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.