

APM 346 Lecture 2.

Richard Derryberry

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1 A menagerie of differential equations

This lecture is devoted to the derivation and explication of various partial differential equations and problems, especially those which we will be studying during this course. Further details and example derivations can be found in [IvrXX, §1.4] (link) and [Str08, §1.3].

1.1 The Transport Equation

For our first derivation, let us model the dynamics of a substance flowing through space (e.g. a pollutant suspended in flowing water, traffic on a road). Assume that the variations of density and flux are negligible in the y and z directions. Let

- $u(x, t)$ denote the density of the substance (units: quantity/volume)
- $\phi(x, t)$ is the flux of the substance (units: quantity/(time \times area))

1.1.1 The Continuity Equation

The quantity of substance in a cylindrical region determined by the interval $[a, b]$ and a constant cross-section A is

$$Q(t; a, b, A) := \int_a^b u(x, t) A dx$$

while the net flux into $[a, b]$ is

$$\Phi(t; a, b, A) := \phi(a, t)A - \phi(b, t)A.$$

Suppose we have an external function $S(x, t, u)$ which is increasing or decreasing the density of the substance (other than through flux) – e.g. a chemical reaction producing more pollutant, onramps and offramps on a highway. Then the rate of change of the total amount of substance in $[a, b]$ is

$$\frac{d}{dt} Q(t; a, b, A) = \Phi(t; a, b, A) + \int_a^b S(x, t, u) A dx$$

i.e.

$$\frac{d}{dt} \int_a^b u(x, t) A dx = \phi(a, t)A - \phi(b, t)A + \int_a^b S(x, t, u) A dx,$$

and so rearranging all terms to appear on the same side of the equation and under the integral, and dividing by the cross-sectional area A , we find

$$\int_a^b (u_t(x, t) + \phi_x(x, t) - S(x, t, u)) dx = 0.$$

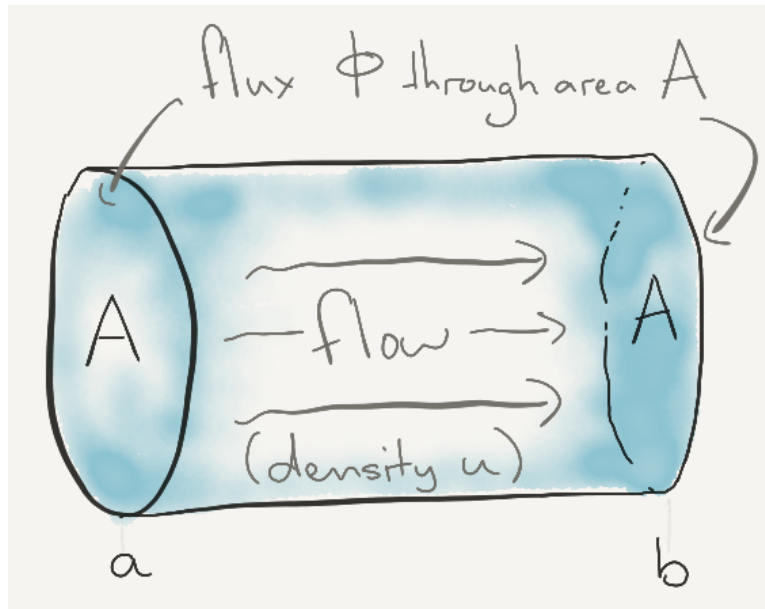


Figure 1: Flow of substance through cylindrical region

Since $[a, b]$ was arbitrary, we find that everywhere in our spatial domain the following equation holds:

$$u_t(x, t) + \phi_x(x, t) = S(x, t, u) \quad (1)$$

Remark 1.1. Equation (1) is known as the *continuity equation*.

Remark 1.2. Regions where $S > 0$ are *sources*, which increase the density of our substance (onramps); regions where $S < 0$ are *sinks* which decrease the density of our substance (offramps).

1.1.2 The Transport Equation

Now, suppose that the velocity of the flowing substance is given by the function $V(x, t)$. Then the flux of the substance is

$$\phi(x, t) = V(x, t)u(x, t)$$

and so substituting this expression into the continuity equation (and consolidating some terms on the right hand side) gives us the *transport equation*

$$u_t(x, t) + V(x, t)u_x(x, t) = S(x, t, u). \quad (2)$$

In an IVP for the transport equation, we are therefore trying to determine the function $u(x, t)$ that satisfies (2) subject to the constraint $u(x, 0) = u_0(x)$ for some initial density profile $u_0(x)$.

1.2 The Wave Equation

As our next example, we will derive the 1d wave equation by modelling the vibrations of a perfectly flexible string (i.e. one whose tension is directed tangentially along the string).

Consider a segment of string between two arbitrary points x_0 and x_1 as in Figure 2. Let T denote the magnitude of the tension vector, and let u denote the vertical displacement of the string (so the u_x is the

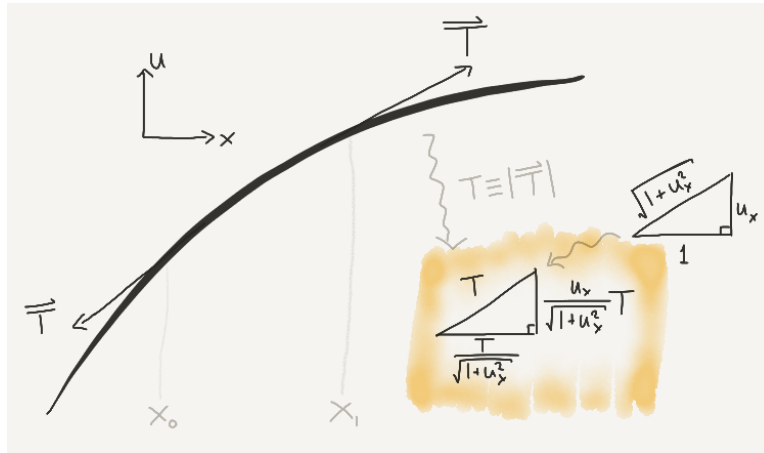


Figure 2: String segment with tension split into longitudinal and transverse components

slope of the string at any point). Then the longitudinal and transverse components of the tension between x_0 and x_1 are given by

$$\frac{T}{\sqrt{1 + u_x^2}} \Big|_{x_0}^{x_1} \quad \text{and} \quad \frac{T u_x}{\sqrt{1 + u_x^2}} \Big|_{x_0}^{x_1}$$

respectively. Splitting Newton's second law $\vec{F} = m\vec{a}$ into longitudinal and transverse components, we have

$$\frac{T}{\sqrt{1 + u_x^2}} \Big|_{x_0}^{x_1} = 0 \quad (3)$$

$$\frac{T u_x}{\sqrt{1 + u_x^2}} \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} f(x, t) dx = \int_{x_0}^{x_1} \rho u_{tt} dx \quad (4)$$

where

- f is the vertical force per unit length (we are assuming there are no forces acting longitudinally on the string), and
- ρ is the mass density of the string.

Now, assume that the displacements of the string are small,¹ ($|u_x|$ is small). Then we may drop the quadratic and higher terms in the Taylor expansion

$$\sqrt{1 + u_x^2} = 1 + \frac{1}{2} u_x^2 + \dots \simeq 1$$

By equation (3), T is constant in x (i.e. constant along the string); rearranging the terms in equation (4) we find

$$\int_{x_0}^{x_1} (\rho u_{tt} - (T u_x)_x - f) dx = 0 \quad (5)$$

Using the fact that $\rho \neq 0$ anywhere (physically sensible – the string cannot be massless anywhere), and setting $c := \sqrt{\frac{T}{\rho}}$ we derive the equation of motion:

$$u_{tt} - c^2 u_{xx} = \frac{f}{\rho} \quad (6)$$

Later, we will interpret c as the speed of wave propagation.

¹Relative to some relevant length scale in our system.

Remark 1.3. We will mostly study this equation in the absence of external forces, i.e. $f \equiv 0$,

$$u_{tt} = c^2 u_{xx} \quad (7)$$

Example 1. If the only external force acting on the string is Earth's gravity, then

$$f dx = -mg = -\rho(x)g dx$$

so that $f = -\rho g$, where $g \simeq 9.8 \text{m/sec}^2$. Then the equation of motion becomes

$$u_{tt} - c^2 u_{xx} = -g.$$

Example 2. Standing waves: see the associated Mathematica file for this example.

1.3 The Heat Equation

Finally, let's consider the *heat equation*. We will consider two derivations: the 1d diffusion equation, which models diffusion of particles dissolved in a solvent, and the 3d heat equation, which models the flow of heat through some medium.

1.3.1 The Diffusion Equation

First, let us consider a 1d model for diffusion of particles in a solvent (e.g. a coloured dye inserted into a tube of water). For simplicity, we will assume that there are no sources or sinks. The continuity equation (1) still models the evolution of the particle density u in terms of flux ϕ , so setting $S = 0$ in (1) we obtain

$$u_t = -\phi_x.$$

To derive the transport equation (2) we assumed that the flux was determined by the velocity (and density) of the solute particles. To instead model diffusion we will assume that the particles move from regions of higher concentration to regions of lower concentration, with rate of motion proportional to the concentration gradient (Fick's law), i.e.

$$\phi = -D u_x$$

where D is the *diffusion coefficient*, which we will assume is constant. Substituting this in to the continuity equation yields the *1d diffusion equation*:

$$u_t = D u_{xx} \quad (8)$$

Example 3. Fundamental solution: evolution of a point source of heat (delta function) – see associated Mathematica file for this example.

1.3.2 3d Heat Flow

Next, let us model heat flow in 3d – although the derivation and result will be similar to that of Section 1.3.1, it will be a good exercise in vector calculus, and we will see that the heat equation can be used to model two different physical processes (related in that they are both diffusive processes).

So, let $T(x, y, z, t)$ be the temperature, and let $H(t; \Omega)$ be the amount of heat (in some units of energy) contained in a region Ω . Then the heat energy in Ω is given by

$$H(t; \Omega) = \iiint_{\Omega} c \rho T \, dx \, dy \, dz \quad (9)$$

where c is the specific heat of the material (the amount of heat required to raise the temperature of 1 kg of mass by 1 K) and ρ is the mass density of the material. So,

$$\frac{dH}{dt}(t; \Omega) = \iiint_{\Omega} c\rho \frac{\partial T}{\partial t} dx dy dz.$$

By conservation of energy the only way for Ω to lose heat is for it to flow out through the boundary $\partial\Omega$, and Fourier's law of heat conduction states that heat flows from hotter to colder regions proportional to the temperature gradient; hence,

$$\frac{dH}{dt}(t; \Omega) = \iint_{\partial\Omega} \kappa \frac{\partial T}{\partial n} dS,$$

where κ is the *heat conductivity* and

$$\frac{\partial T}{\partial n} = \vec{n} \cdot \nabla T$$

where \vec{n} is the outward unit normal vector field along the boundary $\partial\Omega$. An application of the divergence theorem yields

$$\iint_{\partial\Omega} \kappa \frac{\partial T}{\partial n} dS = \iiint_{\Omega} \nabla \cdot (\kappa \nabla T) dx dy dz,$$

and comparing expressions for $\frac{dH}{dt}(t, \Omega)$ yields

$$\iiint_{\Omega} c\rho \frac{\partial T}{\partial t} dx dy dz = \iiint_{\Omega} \nabla \cdot (\kappa \nabla T) dx dy dz.$$

Since the region Ω was arbitrary, we obtain the *heat equation*

$$c\rho \frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T). \quad (10)$$

When we consider the heat equation we will usually consider the situation where c , ρ and κ are all constant, so that the equation reduces to

$$\frac{\partial T}{\partial t} = D\Delta T \quad (11)$$

where Δ is the Laplace operator, $\Delta T = T_{xx} + T_{yy} + T_{zz}$.

Note that the assumption that c , ρ and κ are constant is unrealistic – in reality we expect these to depend on the temperature T . Moreover, the integrand of (9) can be discontinuous in T when there is a *phase transition* – e.g. melting ice to water (both at 0 degrees Celsius) and boiling water to create steam (both at 100 degrees Celsius).

1.4 Laplace's Equation

The Laplace equation in n -dimensions is the differential equation

$$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0.$$

We will see later in the course that solutions to the Laplace equation – *harmonic functions* – have important applications in both mathematics and physic.

One application of the Laplace equation is as the *stationary equations* for the heat and wave equations, where we assume that the solutions are independent of t (and so all t -derivatives are set to zero).

Example 4. Suppose that we have a solid block of metal, where the distribution of heat may be modelled by the simple heat equation (11), and suppose that we heat the surface of the metal in a time independent way (though not necessarily in a spatially uniform way). If we wait long enough (take $t \rightarrow \infty$), we expect that the system will eventually reach an equilibrium state where the temperature distribution is constant in time. Since the equilibrium state will obey the heat equation **and** be constant in time, it must therefore obey the Laplace equation.

I.e.: Physically, we expect that if left long enough the temperature distribution will eventually reach a time-independent equilibrium state, and this equilibrium state will be a harmonic function.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Str08] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.