APM 346 Lecture 1.

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1 Course Information and Administration

The following are highlights from the syllabus – for details and for complete course information, you should read the syllabus.

1.1 Contact Information

This section is lectured by me (Richard Derryberry), and meets Tuesday 10-12 and Thursday 11-12 in LM 162. My office hours will be Thursday 10-11am and 1-2pm in PG 107. I can be reached by email at derryberry.teaching@gmail.com.

The other section is lectured by Prof. Victor Ivrii, and the TAs for this course are Justin Ko, Tristan Milne, and Ivan Telpukhovskiy. See the syllabus for contact details, and for tutorial times and office hours.

1.2 Tutorial Times

Two important announcements regarding tutorials:

1. They will start in week 2.
2. TUT5102 will probably be moved to a different time/day – those who are enrolled in this section should make sure to check the schedule for this section, and those who are not enrolled in this (or any!) tutorial section may wish to check whether it ends up at a more convenient time for them.

If you wish to make changes in your enrollment, the deadline to do so is **January 20**.

Note also that you must attend only the lecture and tutorial sections you are enrolled in. This is both for space purposes, and so that we have an accurate headcount for printouts. If there is a week in which you must attend a different section to the one you are enrolled in, you should petition to do so by 11am of the previous Thursday (so that we can print the correct number of quizzes). If you attend a section you are not enrolled in and have not petitioned to do so, you may not be able to sit the quiz – the students enrolled in that section will be given first priority.

1.3 References

The primary text for this course is the online textbook by Prof. Ivrii, [IvrXX], found at `http://www.math.toronto.edu/courses/apm346h1/20181/PDE-textbook/contents.html`.

A non-required secondary text is the textbook by Walter Strauss, [Str08]. Additional resources are available online and through the library.

1.4 Assessment

Your grade in this course is determined as follows:

- 15% from 7 in-class quizzes (3% each, worst 2 scores dropped)
- 40% from 2 midterm exams (20% each), held 14th Feb and 21st Mar, 1610-1800, room EX100
- 45% from the final exam, date/time/location TBD

The weeks of the in-class quizzes will be announced in advance, however the exact class will be left a surprise.

We will be using Crowdmark for grading.

1.5 Course Content

The course will discuss the following topics:

- **Foundational:** What is a PDE/BVP/IVP? What types of PDE are there? Existence/uniqueness of solutions?
- **Examples:** Transport equation, wave equation, heat equation, Laplace equation. Physical applications.
- **Techniques:** Includes method of characteristics, separation of variables, Fourier transform, Green's functions, variational methods.

Refer to the syllabus for a detailed course schedule and list of course objectives.

2 Introduction to Partial Differential Equations (PDEs)

References for the rest of the lecture are [IvrXX, §1.1-1.3]: §1.1; §1.2; §1.3.

2.1 What is a PDE?

Recall: An *ordinary differential equation (ODE)* is an equation for a one-variable function $u(t)$, which involves the independent variable t, the function, and the derivatives of the function:

$$F(t, u(t), u'(t), u^{(2)}(t), \ldots, u^{(m)}(t)) = 0. \tag{1}$$

Similarly, a *partial differential equation (PDE)* is an equation for a multi-variable function $u(\vec{x}) = u(x, y, \ldots)$, which involves the independent variables $x, y, \ldots$, the function, and the partial derivatives of the function:

$$F(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0. \tag{2}$$

If we replace the function u by a vector of functions $\vec{u} = (u_1, u_2, \ldots, u_N)$ we obtain a *system of ODEs/PDEs*.

2.2 First properties of PDEs

2.2.1 Order/Degree

The *order* or *degree* of a PDE is the order of the highest derivative in the equation.

Example 1. The PDE $u_{xyz} = uu_x^4$ is a PDE of degree 3.

2.2.2 Linearity

A PDE of the form

$$Lu = f(\vec{x}) \tag{3}$$

is called a *linear equation* if Lu is a partial differential expression linear with respect to u. It is called *homogeneous* if $f \equiv 0$, and *inhomogeneous* otherwise.

The coefficients of u and its derivatives in the expression Lu may be constants (*linear equation with constant coefficients*) or functions of the independent variables (*variable coefficients*).

Example 2. The PDE $Lu = au_{xx} + bu_y + cu = 0$ is a linear homogeneous equation with constant coefficients.

Example 3. The PDE $Lu = \sin(x-y)u_{yy} + u_x = \cos(y)$ is a linear inhomogeneous equation with variable coefficients.

If a PDE is not linear, we say it is...*nonlinear*. This is a rather brutal distinction – a more subtle classification is:

- If the RHS of (3) is a function of the lower order derivatives of u, we say the PDE is *semilinear*.
- If the coefficients of the highest order derivatives of u are also functions of the lower order derivatives of u, we say the PDE is *quasilinear*.

Example 4. The PDE $u_{xx} - u_{yy} = u_x^2$ is a second order semilinear equation.

Example 5. The PDE $u_t + uu_x = 0$ is a first order quasilinear equation (Burgers' equation).

2.3 Examples of PDEs

Let's look now at a variety of PDEs (only some of which will be studied or derived in this course).

Example 6. Transport equation: This is the first order linear equation $u_t + cu_x = 0$. It models the flow of a conserved continuous substance, where u represents the density of the substance.

Example 7. Cauchy-Riemann equations: Let $z = x + iy$ be coordinates on the complex plane, and let $f(x,y) = u(x,y) + iv(x,y)$ be the decomposition of a complex valued function into real and imaginary parts. Then f is *complex analytic* if and only if it satisfies the *Cauchy-Riemann equations:*

$$u_x = v_y, \qquad u_y = -v_x.$$

This is a system of first order linear equations.

Example 8. Laplace's Equation: This is the second order linear equation $\Delta u := \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0$. Solutions to Laplace's equation are called *harmonic functions*, and are extremely important in many branches of mathematics, including PDEs (as we will see later in the semester), differential geometry, representation theory (very roughly, the study of "linearized symmetries"), and more.

Example 9. Korteweg–de Vries Equation: The *KdV equation* is the nonlinear PDE

$$u_t + u_{xxx} - 6uu_x = 0$$

which models the dynamics of shallow water waves. The KdV equation is an *exactly solvable model* – a nonlinear PDE which is solvable (in some precise sense). A general nonlinear PDE does **not** have this property.

See the accompanying Mathematica file for an example "soliton solution" to thee KdV equation.

Example 10. Navier-Stokes Equations: This is another system of nonlinear equations, which describe the dynamics of an incompressible fluid in $\mathbb{R}^n$:

$$\vec{v}_t + (\vec{v} \cdot \nabla)\vec{v} = \nu\Delta\vec{v} - \nabla p + \vec{F}$$

where $\vec{v}$ is the fluid velocity, $\nu > 0$ is the viscosity of the fluid, p is the pressure, $\vec{F}$ is an externally applied force, and we take $t \geq 0$. If you can prove that smooth solutions of this equation exist (in $\mathbb{R}^3$, with $\vec{F} = 0$), you win one million dollars.

The primary examples we will study in this course are the Laplace equation, and the following:

Example 11. Heat Equation: This is a second order linear equation that describes the diffusion of heat in a given region over time:

$$u_t = k\Delta u.$$

Example 12. Wave Equation: This is a second order linear equation that describes the propagation of waves through space over time:

$$u_{tt} - c^2\Delta u = 0.$$

3 What does it mean to solve a PDE?

First: what does it mean to solve an ODE? Given an ODE

$$F(t, u(t), u'(t), \cdots) = 0,$$

a solution on the interval $[T_1, T_2]$ is a function $U : [T_1, T_2] \to \mathbb{R}$ such that $F(t, U(t), U'(t), \cdots) = 0$ for $t \in [T_1, T_2]$.

Similarly: given a PDE on $\mathbb{R}^n$

$$F(\vec{x}, u(\vec{x}), \nabla u(\vec{x}), \cdots) = 0,$$

a solution in the region $\Omega \subset \mathbb{R}^n$ is a function $U : \Omega \to \mathbb{R}$ such that $F(\vec{x}, U(\vec{x}), \nabla U(\vec{x}), \cdots) = 0$ for all $\vec{x} \in \Omega$.

3.1 Initial and boundary value problems

Recall that to specify a particular solution to an ODE it is not enough to consider just the ODE itself – one must provide extra data, for instance specifying the value of the function at a given point in time.

Example 13. Suppose we wish to solve the ODE $u''(t) - u(t) = 0$ on the domain $t \in [0, \infty)$. How many constraints do we need to give in order to specify a unique solution?

- We could give **one** constraint, such as $u(0) = 1$. But both e^t and e^{-t} satisfy the ODE with this constraint, so this fails to select a unique solution.

- We could give **two** constraints, such as $u(0) = 1$ and $u'(0) = 1$. Now, only $u(t) = e^t$ satisfies the ODE with constraints – we have found a unique solution.
- What if we tried to specify **three** constraints, such as the value of $u''(0)$? Then we run the risk of a solution not existing – indeed, from the ODE we immediately obtain the consistency condition $u''(0) = u(0)$.

The most general solution is the function $u(t) = Ae^t + Be^{-t}$ solves the ODE, where A and B are arbitrary constants. We have two unknowns in this general solution: hence, we need two constraints to uniquely determine a solution.

Note that our constraints did not need to be the values of u or its derivatives. For instance, one could specify that $u(0) = e$, and that u is bounded on $[0, \infty)$ ($|u(t)| \leq M$ for some constant $M > 0$). Since e^t is not bounded $A = 0$; hence $B = e$ and $u(t) = e^{1-t}$.

Similarly, to specify a particular solution to a PDE it is not enough to consider just the PDE itself – again, extra data must be specified. For a PDE, however, a general solution will contain not just unknown constants, but unknown *functions*.

Example 14. The most general solution to the simple PDE $\frac{\partial u}{\partial y}(x, y) = 0$ is $u(x, y) \equiv f(x)$, an arbitrary function of x. Why? If we hit the RHS with $\frac{\partial}{\partial y}$ it vanishes; conversely, if there were non-constant y-dependence in the function on the RHS then it would have nonzero partial y-derivative.

Example 15. What about the PDE $v_{xy} = 0$, where v is a function of two variables? Integrating first with respect to x, we obtain $v_y(x, y) = f(y)$ as above. Letting $F(y)$ be an antiderivative of $f(y)$, this can be rearranged to give

$$(v(x, y) - F(y))_y = 0$$

and so $v(x, y) = G(x) + F(y)$ for arbitrary functions of one variable G and F.

Upshot: Solutions to PDEs in n-dimensions typically depend on one or several arbitrary functions of $n - 1$ variables. To select a unique solution we must supply extra conditions.

We call a PDE together with a set of additional constraints a *problem.* Typical problems are:

- *Initial Value Problem (IVP):* One of the variables is interpreted as time, t, and conditions are imposed at some (initial) moment in time: e.g. $u(t, \vec{x})|_{t=t_0} = u_0(\vec{x})$, $u_t(t, \vec{x})|_{t=t_0} = v_0(\vec{x})$. Usually we will take $t_0 = 0$.

  You will have encountered IVPs before, in a course on ODEs.
- *Boundary Value Problem (BVP):* Constraints are imposed on the boundary of the spatial domain Ω: e.g. $u|_{\partial\Omega} = \phi$.
- *Mixed Problem (IBVP):* One of the variables is interpreted as time, and some conditions are imposed at a given moment in time while others are imposed on the boundary of the spatial domain.

In Example 13 the constraints $u(0) = u'(0) = 1$ describe an IVP. The second set of listed constraints, with the boundedness condition, do not describe an IVP, BVP, or IBVP.

Example 16. We can solve Laplace's equation $\Delta u = 0$ on $\Omega = \mathbb{D}$ the unit disk by prescribing the value of u on the boundary S^1, the unit circle. For instance, if we impose $u|_{S^1} = \cos(4\theta)$, the unique solution is given by $u(r, \theta) = r^4 \cos(4\theta)$.

Remark 3.1. It may not be immediately obvious what the correct number or type of constraints should be in order for a problem to have a unique solution. That said, usually we will have some sort of physical system in mind that we wish to model, and this will guide us in our choice of constraints.

3.2 Example boundary conditions

There are many different constraints that one could impose as boundary conditions. Some important ones, together with how they might be interpreted when modelling heat flow, are:

- *Dirichlet condition:* The restriction of the function u to the boundary of the domain is specified, i.e.

$$u|_{\partial\Omega} = \phi$$

  This would model the boundary being held at a particular temperature distribution, for example through contact with a large stable heat reservoir.

- *Neumann condition:* The normal derivative of the function u at the boundary of the domain is specified, i.e.

$$\frac{\partial u}{\partial n} = \psi$$

  If we set $\psi = 0$, this would model $\partial\Omega$ as a perfect insulator that allows no heat flow either in or out of Ω.

- *Robin condition:* A mixed condition,

$$\left(\frac{\partial u}{\partial n} + au\right)\bigg|_{\partial\Omega} = \chi$$

  This could model a hot body Ω radiating heat into a reservoir, with rate of radiation proportional to the temperature difference between the hot body and the reservoir.

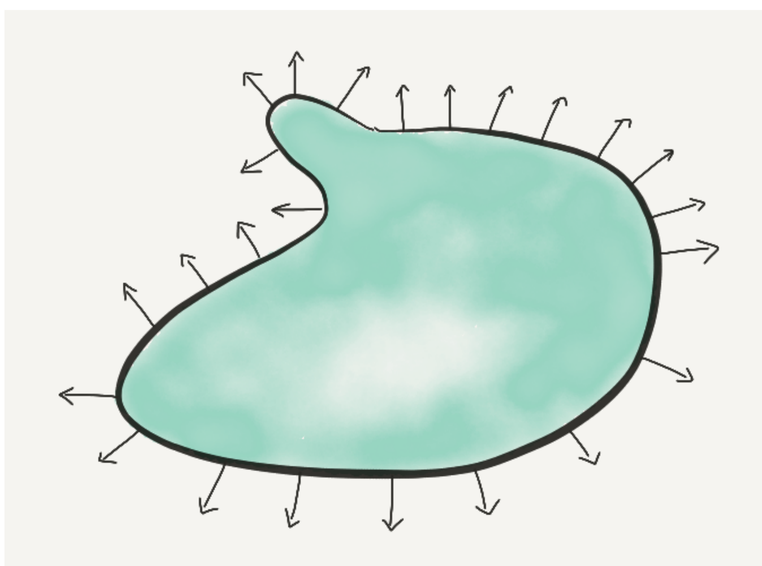

Figure 1: Domain (shaded) with outward pointing normal vectors along the boundary

If the domain Ω is unbounded, then instead of restricting the function to the boundary one prescribes "boundary conditions at infinity", for instance:

- *Normalisation condition:* In quantum mechanics one is interested in finding wavefunctions ψ (solutions to the Schrödinger equation), with the constraint that $\int \|\psi\|^2 d\vec{x} = 1$ (we say that ψ has "unit L^2-norm").

- *Sommerfield radiation condition:* This is a boundary condition in the scattering of acoustic or electromagnetic waves,
$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right) = 0$$
which physically corresponds to picking out the solution to the Helmholz equation which corresponds to waves radiating energy out *from* a source *to* infinity (with no energy allowed to unphysically radiate in from infinity) [Som12].

3.3 Well-posed problems

We want to consider *well-posed* problems, which are those systems of PDEs with auxilliary constraints that satisfy the following requirements:

- *Existence:* There exists at least one solution to the problem.
- *Uniqueness:* There exists at most one solution to the problem.
- *Stability:* The unique solution depends continuously on the data of the problem.

Existence and uniqueness are self-explanatory. Stability means intuitively that if the problem is changed "only a little" then the solution should also change "only a little". A problem that is not well-posed is called "ill-posed".

4 Elliptic, hyperbolic and parabolic equations

To finish this lecture, let us consider a general second order differential equation:
$$Lu = \sum_{1\le i,j\le n} a_{ij} u_{x_i x_j} + \text{l.o.t.} = f(\vec{x}) \tag{4}$$
where l.o.t. means *lower order terms*, and where we may assume that $a_{ij} = a_{ji}$. Form the *matrix of principal coefficients*
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$
Under a change of variables $\vec{x} = \vec{x}(\vec{x}')$ the matrix of principal coefficients changes by conjugation via the Jacobi matrix
$$J = \left(\frac{\partial x_i}{\partial x'_j}\right)_{1\le i,j\le n}$$
and so if the principal coefficients are real and constant, the matrix A can be diagonalised via a linear change of variables,
$$A \mapsto D = \operatorname{diag}(d_1, \cdots, d_n).$$

Definition 4.1. The PDE (4) is called

- *elliptic* if the $d_1, \ldots, d_n$ are all strictly positive or strictly negative;
- *hyperbolic* if the $d_1, \ldots, d_n$ are all nonzero, and one of the eigenvalues has the opposite sign to the other $n - 1$;

- *parabolic* if one of the eigenvalues is zero and the others all have the same sign.

Remark 4.1. The is also *ultrahyperbolic* where none of the d_i vanish but at least two of them are positive and at least two of them are negative. We will not be interested in ultrahyperbolic equations, however.

Remark 4.2. The terms elliptic/hyperbolic/parabolic are borrowed from the analysis of conic sections (here $a, b \neq 0$):

- The equation $ax^2 + by^2 = 1$ defines an *ellipse*.
- The equation $ax^2 - by^2 = 1$ defines a *hyperbola*.
- The equation $ax^2 + by = 1$ defines a *parabola*.

Just as the different conic section have distinct geometries, so do elliptic, hyperbolic, and parabolic equations have different properties.

Example 17. In 2d, with real constant coefficients, then in appropriate coordinates we have

- *Elliptic:* $u_{xx} + u_{yy} + \text{l.o.t.} = f$
- *Hyperbolic:* $u_{xx} - u_{yy} + \text{l.o.t.} = f$
- *Parabolic:* $u_{xx} - cu_y + \text{l.o.t.} = f$

For the parabolic case, the IVP $u(x, 0) = g(x)$ is well-posed for $y > 0$ if $c > 0$ and for $y < 0$ if $c < 0$.

Note that Definition 4.1 does not provide a complete classification of linear 2nd order PDEs:

Example 18. The 2d Schrödinger equation is

$$u_{xx} + icu_y = 0$$

for $c \in \mathbb{R}_{\neq 0}$. The IVP $u(x, 0) = g(x)$ is well-posed for both $y > 0$ and $y < 0$.

In fact, in higher dimensions a randomly chosen PDE of the form (4) will not fall into the classification of Definition 4.1. The equations we are primarily interested in for this course, however, do fall into this classification:

Example 19. Laplace's equation $\Delta u = 0$ is the prototypical elliptic equation.

Example 20. The wave equation $u_{tt} - c^2 \Delta u = 0$ is the prototypical hyperbolic equation.

Example 21. The heat equation $u_t - k\Delta u = 0$ is the prototypical parabolic equation.

4.1 Equations of variable type

Finally, if the coefficients of (4) are not constant, there can be more complicated situations where the PDE has different types in different regions of the domain Ω. For instance, consider the *Tricomi equation*

$$u_{xx} + xu_{yy} = 0$$

which is a toy model for transsonic flow of gas (gas that is moving above, at, and below the speed of sound at the same time in different regions around an object). The Tricomi equation is

- elliptic when $x > 0$,
- hyperbolic when $x < 0$, and
- parabolic when $x = 0$.

See the accompanying Mathematica file to play with an example solution to the Tricomi equation.

References

[IvrXX] Victor Ivrii. Partial Differential Equations. online textbook for APM346, 20XX.

[Som12] A. Sommerfeld. Die greensche funktion der schwingungslgleichung. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 21:309–352, 1912.

[Str08] Walter A. Strauss. *Partial differential equations.* John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.