APM 346 Deriving the Integral Curve Equation.

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The problem is as follows: given a vector field $\vec{l}(t,x) = (a(t,x), b(t,x))$ on $\mathbb{R}^2_{t,x}$, find a curve

$$\gamma : \mathbb{R}_s \to \mathbb{R}^2_{t,x}$$

such that $\vec{l}$ is the tangent vector field of γ, i.e. so that

$$\frac{d\gamma}{ds}(s) = \vec{l}(\gamma(s)). \tag{1}$$

Taking the s-derivative of $\gamma(s) = (t(s), x(s))$ and comparing with (1) gives

$$\frac{d\gamma}{ds}(s) = \left(\frac{dt}{ds}(s), \frac{dx}{ds}(s)\right) = (a(\gamma(s)), b(\gamma(s))) = \vec{l}(\gamma(s))$$

i.e.

$$\frac{dt}{ds} = a(\gamma(s)) \quad \text{and} \quad \frac{dx}{ds} = b(\gamma(s)).$$

But now the projections of γ to the t- and x-coordinates, $t(s)$ and $x(s)$, satisfy

$$\begin{aligned} dt &= \frac{dt}{ds}ds = a(\gamma(s))ds \\ dx &= \frac{dx}{ds}ds = b(\gamma(s))ds \end{aligned}$$

so that

$$\frac{dt}{a(\gamma(s))} = ds = \frac{dx}{b(\gamma(s))}$$

which is exactly the integral curve equation as presented in class.