TOPICS IN D-MODULES.

TYPED BY RICHARD HUGHES FROM LECTURES BY SAM GUNNINGHAM

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1. Introduction: Local Systems.

1.1. **Local Systems.** Consider the differential equation (DE)

$$\frac{d}{dz}u(z) = \frac{\lambda}{z}u(z), \qquad \lambda \in \mathbb{C} \text{ a constant.}$$

An undergraduate might say that a solution to this is z^λ – but what does this mean for λ non-integral?

$$u(z) = z^\lambda = e^{\lambda \log(z)} \text{ for a branch of log.}$$

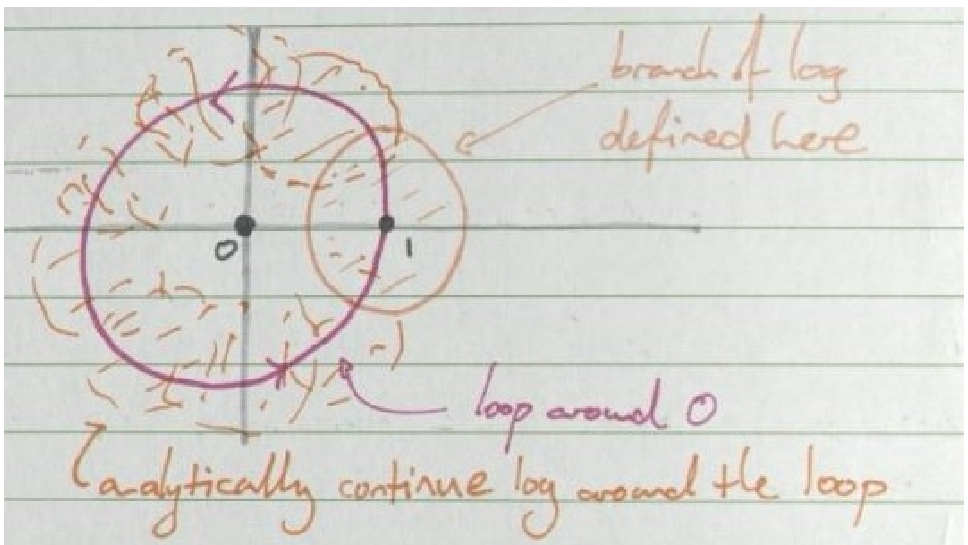


Figure 1. Analytic continuation of a solution of a DE around a loop.

The analytic continuation changes

$$\left.\begin{array}{rcl} \log(z) & \mapsto & \log(z) + 2\pi i \\ z^\lambda & \mapsto & e^{2\pi i\lambda} z^\lambda \end{array}\right\} \text{“monodromy”}$$

Could say that z^λ is a multivalued function. The space of *local* solutions form a *local system*. We will see various definitions of local systems shortly.

Let X be a (reasonable) topological space, and $\mathrm{Op}(X)$ the lattice of open sets; i.e. a category with a unique morphism $U \to V$ if $U \subset V$.

Definition 1. A *presheaf* (of vector spaces) $\mathscr{F}$ is a functor

$$\mathscr{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Vect} = \text{vector spaces over } \mathbb{C}.$$

I.e. for every $U \subset X$ open, $\mathscr{F}(U)$ is a vector space, and if $U \subset V$ there is a unique linear map (think “restriction”) $\mathscr{F}(V) \to \mathscr{F}(U)$. We call $\mathscr{F}(U) = \Gamma(U; \mathscr{F}) =$ “*sections of $\mathscr{F}$ of U*”.

Definition 2 (Non-precise)**.** A presheaf $\mathscr{F}$ is a *sheaf* if sections of $\mathscr{F}$ on U are "precisely determined by their restriction to any open cover."

Definition 3 (Formal)**.** If $\{U_i\}$ is an open cover of U,

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is an equaliser diagram.

Example 1. Various kinds of functions form sheaves:

- all functions;
- C_X the sheaf of continuous functions on X;
- C_X^∞ smooth functions (if X is a smooth manifold);
- $\mathscr{O}_X^{\mathrm{an}}$ holomorphic functions (if X is a complex manifold).

Example 2. The functor $\mathbb{C}_X^{\mathrm{pre}}$ of constant functions on X is *not* a sheaf (consider the two point space with the discrete topology).

Example 3. We can improve the above example so that it becomes a sheaf by instead considering the **locally** constant functions $\mathbb{C}_X$. We call this a *constant sheaf.*

Remark Presheaves form a category (functor category); sheaves are contained in this as a full subcategory.

Definition 4. A sheaf is *locally constant of rank r* if there is an open cover $\{U_i\}$ of X such that

$$\mathscr{F}|_{U_i} \cong \mathbb{C}_{U_i}^{\oplus r}.$$

Example 4. The solutions to the DE at the start of this section form a locally constant sheaf. On contractible sets not containing 0 the solutions form a 1d vector space, and there are no global sections.

Definition 5 (One definition of a local system)**.** A *local system* is a locally constant sheaf.

Definition 6. The *stalk* of a sheaf $\mathscr{F}$ at a point x is

$$\mathscr{F}_x = \xrightarrow[\text{open } U \ni x]{\text{colim}} \mathscr{F}(U).$$

We sometimes call elements of the stalk *germs of sections of $\mathscr{F}$ near x.*

I.e. the stalk is sections on $\mathscr{F}(U \ni x)$ where $s_U \in \mathscr{F}(U)$ and $s_V \in \mathscr{F}(V)$ are equivalent if there is $W \subset U \cap V$ containing x such that $s_U|_W = s_V|_W$.

If $\mathscr{F}$ is locally constant,

$$\mathscr{F}_x = \mathscr{F}(U) \cong \mathbb{C}^r \text{ for some small enough open set } U \ni x.$$

Since the rank of a locally constant sheaf is constant on components, if x and y are in the same component, $\mathscr{F}_x \cong \mathscr{F}_y$. How can we realize this isomorphism?

If $\gamma : [0,1] \to X$ is a path with $\gamma(0) = x$ and $\gamma(1) = y$, and $\mathscr{F}$ is a locally constant sheaf, we can make the following observation:

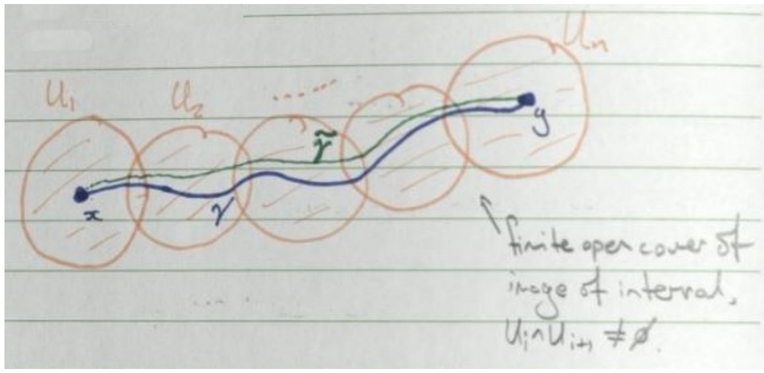


FIGURE 2. Parallel transport of a section along a path.

Proposition 1.1. *There is an isomorphism* $t_\gamma : \mathscr{F}_x \xrightarrow{\sim} \mathscr{F}_y$.

Proof. The isomorphism t_γ is given by the chain of isomorphisms

$$\mathscr{F}_x \xleftarrow{\sim} \mathscr{F}(U_1) \xrightarrow{\sim} \mathscr{F}(U_1 \cap U_2) \xleftarrow{\sim} \mathscr{F}(U_2) \to \cdots \to \mathscr{F}(U_{n-1} \cap U_n) \xleftarrow{\sim} \mathscr{F}(U_n) \xrightarrow{\sim} \mathscr{F}_y.$$

□

Proposition 1.2. *If* γ *is homotopic to* $\tilde{\gamma}$ *then* $t_\gamma = t_{\tilde{\gamma}}$.

Proof. Can contain both paths in a compact contractible set, then run a similar proof to above. □

Definition 7. The *fundamental groupoid* of X, denoted $\Pi_1(X)$, is a category with

- Objects: the points of X,
- Morphisms: $\{$paths from x to $y\}/$homotopy.

Observe that this really is a groupoid (not difficult).

Theorem 1.3. *There is an equivalence of categories*

$$\{\textit{Locally constant sheaves on } X\} \xrightarrow{\sim} \{\textit{Functors } \Pi_1(X) \to \textit{Vect}^{fd}\}$$

$$\mathscr{F} \longmapsto \left\{ \begin{array}{ccc} x & \mapsto & \mathscr{F}_x \\ t_\gamma : \mathscr{F}_x & \to & \mathscr{F}_y \end{array} \right\}$$

This gives us another perspective on local systems. In particular, they are a homotopic invariant, not a homeomorphism invariant.

1.1.1. *Another perspective: Where do local systems come from?* If X is a smooth manifold and $\pi : E \to X$ is a smooth vector bundle, we can functorially produce the *sheaf of sections* $\mathscr{E}$,

$$\mathscr{E}(U) = \Gamma(U; E) \ni (s : U \to E : \pi \circ s = \mathrm{id}).$$

How can we make sense of local constancy? **Connections.** Write $\Gamma(E) = \Gamma(X; E) = \mathscr{E}(X)$ for *global* sections, and let $\nabla : \Gamma(E) \to \Omega^1(E)$ be a connection.

Remark Can properly think of this as a map of sheaves $\mathscr{E} \to \Omega^1(\mathscr{E})$ – we get away with conflating bundles with sheaves because our sheaves have nice properties (in particular we have partitions of unity at our disposal).

What is the candidate for local constancy? *Horizontal sections:*

$$\ker(\nabla) = \{s \in \Gamma(E) | \nabla s = 0\}.$$

When does this behave nicely? This goes back to Frobenius: for concreteness let's work locally with local coordinates $x_1, \ldots, x_n$ on X and $s_1, \ldots, s_r$ a basis of (local) sections of E. Then

$$\nabla_{\frac{\partial}{\partial x_i}}(s_j) = \sum_k a_{ijk} s_k,$$

where a_{ijk} is the connection matrix. Write $s = \sum f_j s_j$. Then

$$\nabla s = 0 \quad \iff \quad \left\{ \frac{\partial f_j}{\partial x_i} + \sum_k a_{ijk} f_k = 0 \right\} \quad \text{system of PDEs.}$$

What then does it mean to have a locally constant sheaf? Morally: *"Given an initial condition at a point, there is a unique solution to this system of PDEs on some contractible neighbourhood."* Let's phrase this in a more precise and familiar way. Write $\nabla_i := \nabla_{\frac{\partial}{\partial x_i}}$.

Theorem 1.4 (Frobenius Theorem)**.** *If $\nabla_i \nabla_j = \nabla_j \nabla_i$ for all i, j, then the sheaf* $\ker(\nabla)$ *is locally constant. We then say that the connection is* flat*, or* integrable*.*

Remark Globally this is phrased as $\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}$ where X, Y are vector fields.

Thus we can expand on the previous theorem.

Theorem 1.5. *There is an equivalence of categories*

$$\{\textit{Locally constant sheaves on } X\} \xleftrightarrow{\sim} \{\textit{Functors } \Pi_1(X) \to \textit{Vect}^{fd}\} \xleftrightarrow{\sim} \{\textit{Integrable/flat connections on } X\}$$

2. Sheaves.

Fix X a topological space. Recall that a presheaf on X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Vect}$; or more generally we could take

$$\mathscr{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \left\{ \begin{array}{c} \mathrm{Set} \\ \mathrm{Ab} \\ \mathrm{Rings} \\ \mathrm{etc.} \end{array} \right.$$

$\mathscr{F}$ is a *sheaf* if "sections can be defined locally".

Example 5. Given $\pi : Y \to X$ a continuous map of spaces, *sections of Y over X*, $\mathscr{S}_{Y/X}$ is a presheaf defined by

$$\mathscr{S}_{Y/X}(U) = \{s : U \to Y \mid \pi s(x) = x\}.$$

This is a presheaf of *sets*.

Claim: $\mathscr{S}_{Y/X}$ is a sheaf.

Proof. Suppose $\{U_i\}$ is an open cover of $U \subset X$, U open.

(1) If $s, s' \in \mathscr{S}_{Y/X}(U)$ such that $s|_{U_i} = s'|_{U_i}$ for all i, then it is clear that $s = s'$ (functions are defined pointwise).
(2) If $s_i \in \mathscr{S}_{Y/X}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, we can define

$$s \in \mathscr{S}_{Y/X}(U) \quad \text{by} \quad s(x) = s_i(x) \text{ if } x \in U_i.$$

$\square$

This example is very important – it gives us a huge variety of examples, and in some sense it gives us *all* sheaves (we will make this precise soon).

Example 6. If $Y = X \times Z \xrightarrow{\pi_1} X$,

$$\mathscr{S}_{X\times Z/X}(U) = C(U, Z) = \{\text{cts. functions } U \to Z\}.$$

Example 7. If $Z = \mathbb{C}$ with the Euclidean topology, $\mathscr{S}_{X\times\mathbb{C}/X} = C_X$, continuous complex valued functions on X. Observe that we can consider this as a sheaf of sets, vector spaces, rings, etc.

Example 8. If $Z = \mathbb{C}^{\mathrm{disc}}$, $\mathbb{C}$ with the discrete topology, then

$$\mathscr{S}_{X\times\mathbb{C}^{\mathrm{disc}}/X} = \mathbb{C}_X,$$

locally constant functions on X (constant sheaf).

Example 9. If $E \xrightarrow{\pi} X$ is a complex vector bundle, i.e. there is an open cover $\{U_i\}$ of X such that

$$E|_{U_i} = \pi^{-1}(U_i) \cong U_i \times \mathbb{C}^r,$$

commuting with projection

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, leftrightarrow, "\cong"] \arrow[dr] & & U_i \times \mathbb{C}^r \arrow[dl] \\ & U_i & \end{tikzcd}$$

such that

$$\pi^{-1}(x) = E_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^r.$$

Then $\mathscr{S}_{E/X} = \mathscr{E}$ is a sheaf of vector spaces – but in fact it is even more than that:

$$\underline{\mathscr{E} \text{ is a sheaf of modules for the sheaf of rings } C_X.}$$

Definition 8. A *bundle of groups with fibre/structure group* G is a map $E \to X$ of spaces such that there is an open cover U_i such that $E|_{U_i} \cong U_i \times G$.

Example 10. What is a bundle of sets? A set is a discrete topological space, so a bundle of sets is a convering space.

Example 11. A bundle of topological spaces is a fibre bundle.

Example 12. This is slightly subtle: a bundle of discrete vector spaces is **not** a vector bundle! Sections here are somehow 'locally constant'. This will give us another way to think about local systems.

Claim: If $Y \to X$ is a covering space, then $\mathscr{S}_{Y/X}$ is a locally constant sheaf of sets. I.e., locally it looks like the sheaf of locally constant functions.

We would like a converse to this: given a locally constant sheaf, produce a covering space. We will actually consider a more general construction.

2.1. **Étalé spaces.** Given a sheaf $\mathscr{F}$ on X, define the *étalé space* $\text{Ét}(\mathscr{F})$ as follows:

- As a set, it is the collection of germs of sections of $\mathscr{F}$,
$$\text{Ét}(\mathscr{F}) = \coprod_{x\in X} \mathscr{F}_x.$$
- Topology: If $U \subset X$ is open, $s \in \mathscr{F}(U)$, we have $s_x \in \mathscr{F}_x$ for all $x \in U$. Then declare the sets $\{s_x \mid x \in U\} \subset \text{Ét}(\mathscr{F})$ to be open, and $\{s_x \mid x \in U\} \leftrightarrow U$ a homeomorphism.

We have a map $\acute{\mathrm{E}}\mathrm{t}(\mathscr{F}) \xrightarrow{\pi} X$ with $\mathscr{F}_x = \pi^{-1}(x)$. In fact, π is a local homeomorphism. I.e. if $s_x \in \acute{\mathrm{E}}\mathrm{t}(\mathscr{F})$ then there exists $U \ni s_x$ open in $\acute{\mathrm{E}}\mathrm{t}(\mathscr{F})$ such that $\pi|_U$ is a homeomorphism onto its image.

Claim: $\mathscr{S}_{\acute{\mathrm{E}}\mathrm{t}(\mathscr{F})/X} \cong \mathscr{F}$.

For each $x \in X$, need to give an element of $\mathscr{F}_x$ (to define a section $\acute{\mathrm{E}}\mathrm{t}(\mathscr{F}) \to X$). Then we want to show that given the defined topology, the stalks glue to a legitimate section of $\mathscr{F}$.

A little clearer: check that the assignment

$$\begin{array}{rcl} \mathscr{F} & \to & \mathscr{S}_{\acute{\mathrm{E}}\mathrm{t}(\mathscr{F})/X} \\ s \in \mathscr{F}(U) & \mapsto & \{x \mapsto s_x\} \end{array}$$

is continuous. So: this gives an equivalence of categories

$$\{\text{Local homeomorphisms over } X\} \xleftrightarrow{\sim} \{\text{Sheaves of sets on } X\}$$

$$\acute{\mathrm{E}}\mathrm{t}(\mathscr{F}) \longleftrightarrow \mathscr{F}$$

Inside of this, we have the equivalence

$$\begin{array}{ccc} \{\text{Local homeomorphisms over } X\} & \longleftrightarrow & \{\text{Sheaves of sets on } X\} \\ \cup & & \cup \\ \{\text{Covering spaces}\} & \longleftrightarrow & \{\text{Locally constant sheaves}\} \end{array}$$

Remark If $\mathscr{F}$ is a presheaf, $\acute{\mathrm{E}}\mathrm{t}(\mathscr{F}) \to X$ is still a local homeomorphism, so we can still take its sheaf of sections

$$\mathscr{F}^{\mathrm{sh}} = \mathscr{F}^{+} = \mathrm{sh}(\mathscr{F}) := \mathscr{S}_{\acute{\mathrm{E}}\mathrm{t}(\mathscr{F})/X}$$

which we call the *sheafification* of $\mathscr{F}$. Sheafification is left adjoint to the inclusion functor $i\colon \mathrm{Sheaves}(X) \to \mathrm{Presheaves}(X)$,

$$\mathrm{Hom}_{\mathrm{Presheaves}(X)}(\mathscr{F}, i(\mathscr{G})) \cong \mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}).$$

2.2. **Functors on sheaves.** Given $f : X \to Y$, what can we do with sheaves? We would like to be able to push them forward and pull them back:

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, bend left, "f_*"] & \mathrm{Sh}(Y) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Given a sheaf $\mathscr{F}$ on X, define

$$f_*(\mathscr{F})(U) = \mathscr{F}(f \in (U))$$

where $U \subset Y$ is open. If $\mathscr{G}$ is a sheaf on Y, define

$$(f^*\mathscr{G})^{\mathrm{pre}}(V) = \xrightarrow[U \subset f(V)]{\mathrm{colim}} \mathscr{G}(U),$$

where V runs over the open sets containing U, and define

$$f^*\mathscr{G} = \mathrm{sh}(f^*\mathscr{G}^{\mathrm{pre}}).$$

In terms of local homeomorphisms (étalé spaces), the pullback sheaf/inverse image sheaf should be the sections of the pullback

$$\begin{tikzcd} X \times_Y E \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Example 13. Let $f : X \to \mathrm{pt}$, $\mathscr{F}$ a sheaf on X. Then $f_*(\mathscr{F}) = \Gamma(\mathscr{F}) = \mathscr{F}(X)$.

So f_* is a generalization of global sections. Think of f_* as being sections along the fibres (at least when f looks like a fibration).

If A is a set (i.e. a sheaf on a point), then

$$f^*(A) = A_X, \quad \text{the constant sheaf.}$$

Example 14. Let $i : \{x\} \hookrightarrow X$. Then $i^*\mathscr{F} = \mathscr{F}_x$.

So we can think of pullback as a generalization of the stalk, at least when we are looking at inclusion of a subspace.

Definition 9. If A is a set, i_*A is called the *skyscraper sheaf* at x.

2.3. **Homological properties of sheaves.** From now on we will turn away from sheaves of sets. Today, $\mathrm{Sh}(X)$ means sheaves of *abelian groups* on X. We are interested in the following fact:

"Sh(X) is an abelian category."

Definition 10. A category $\mathcal{C}$ is *abelian* if

(1) it contains a zero object (initial and terminal),
(2) is contains all binary products and coproducts,
(3) it contains all kernels and cokernels,
(4) every monomorphism is a kernel and every epimorphism is a cokernel.

Example 15. The category of all groups is a non-example – if $H \subset G$ is non-normal then G/H is not a group.

Properties: In an abelian category,

- $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group.
- Finite products = finite coproducts.
- The *first isomorphism theorem* holds:

$$\begin{tikzcd}
\ker(f) \arrow[r, hook] & A \arrow[r, "f"] \arrow[d, two heads] & B \arrow[r, two heads] & \operatorname{coker}(f) \\
& \operatorname{coim}(f) \arrow[r, "\sim"] & \operatorname{im}(f) \arrow[u, hook] &
\end{tikzcd}$$

Example 16. For a ring R, R-modules form an abelian category.

Definition 11. In an abelian category $\mathcal{C}$,

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called *exact* if $\mathrm{im}\,(f) = \ker(g)$.

Observe that this implies that $g \circ f = 0$.

Definition 12. A *short exact sequence* (SES) is an exact sequence of the form

$$0 \to A \to B \to C \to 0.$$

Example 17. $0 \to A \to A \oplus C \to C \to 0$; such a SES is called *split*.

Proposition 2.1 (Splitting lemma)**.** *A SES* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is* split *iff either*

(1) there exists some $s : B \to A$ such that $s \circ f = id_A$; or,
(2) there exists some $t : C \to B$ such that $g \circ t = id_C$.

Warning! This is a property of *abelian* categories. A t-splitting in the category of all groups would only tell us that B is a *semi*-direct product.

Definition 13. A *complex* in $\mathcal{C}$ is a sequence of objects and morphisms

$$\cdots \to A^{i-1} \xrightarrow{d_{i-1}} A^i \xrightarrow{d_i} A^{i+1} \to \cdots = A^\bullet$$

such that $d_i \circ d_{i-1} = 0$ for any i. We often write $d^2 = 0$. Then the *cohomology* of this complex is

$$H^i(A^\bullet) = \frac{\ker(d_i)}{\mathrm{im}\ (d_{i-1})}.$$

Definition 14. A *morphism* of complexes $A^\bullet \to B^\bullet$ is a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & A^i \arrow[r] \arrow[d] & A^{i+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

A morphism $A^\bullet \to B^\bullet$ is a *quasi-isomorphism* if it induces an isomorphism of cohomology $H^i(A^\bullet) \xrightarrow{\sim} H^i(B^\bullet)$.

If $\mathcal{C}$ and $\mathcal{D}$ are abelian categories and $F : \mathcal{C} \to \mathcal{D}$ is a functor, we say

- F is *additive* if it preserves finite coproducts.
- F is *left exact* if it is additive and preserves kernels.
- F is *right exact* if it is additive and preserves cokernels.

If $0 \to A \to B \to C \to 0$ is a SES in $\mathcal{C}$, then

- If F is left exact, $0 \to F(A) \to F(B) \to F(C)$ is exact.
- If F is right exact, $F(A) \to F(B) \to F(C) \to 0$ is exact.

In abelian groups, we have the functors (for fixed $A \in \mathrm{Ab}$)

$$\mathrm{Hom}(A, -) : \mathrm{Ab} \to \mathrm{Ab} \qquad \text{(left exact)}$$
$$A \otimes_{\mathbb{Z}} (-) : \mathrm{Ab} \to \mathrm{Ab} \qquad \text{(right exact)}$$

Example 18. Let $A = \mathbb{Z}/2$ and take the SES $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$.

- Apply $\mathrm{Hom}(A, -)$: $0 \to 0 \to 0 \to \mathbb{Z}/2$.
- Apply $A \otimes_{\mathbb{Z}} (-)$: $\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$.

Claim: $\mathrm{Sh}(X)$ is an abelian category.

Sketch of proof. Consider $\mathscr{F}, \mathscr{G} \in \mathrm{Sh}(X)$, $\phi : \mathscr{F} \to \mathscr{G}$.

$$\ker(\phi)(U) = \ker(\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)),$$

so $\ker(\phi) \in \mathrm{Sh}(X)$. We can define

$$\mathrm{coker}(\phi) := \mathrm{sh}(\mathrm{coker}(\phi)^{\mathrm{pre}})$$

where

$$\mathrm{coker}(\phi)^{\mathrm{pre}}(U) = \mathrm{coker}(\phi_U) = \mathscr{G}/\phi_U(\mathscr{F}(U)).$$

□

Example 19 (Cokernel presheaf is not a sheaf.)**.** Let $X = \mathbb{R}$, $\mathscr{F} = \mathbb{Z}_{\mathbb{R}}$, $\mathscr{G} = \mathbb{Z}_x \oplus \mathbb{Z}_y$, $x \neq y$ in $\mathbb{R}$. How can we define a map $\phi : \mathbb{Z}_{\mathbb{R}} \to \mathbb{Z}_x \oplus \mathbb{Z}_y$? Such a map is equivalent to a global section $s \in \Gamma(\mathbb{Z}_x \oplus \mathbb{Z}_y) \cong \mathbb{Z} \oplus \mathbb{Z}$. Choose $(1, 1) \in \mathbb{Z} \oplus \mathbb{Z}$. What is the cokernel of this map?

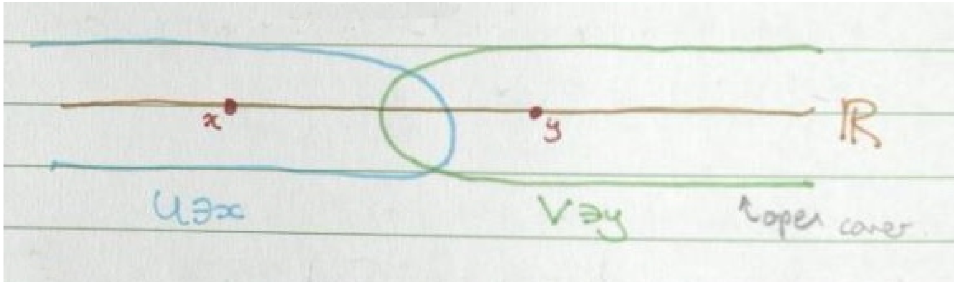


Figure 3. Cover of $\mathbb{R}$ by two open sets U and V.

Then

$$\phi_U : \mathbb{Z} \xrightarrow{\sim} (\mathbb{Z}_x \oplus \mathbb{Z}_y)(U) = \mathbb{Z},$$
$$\phi_V : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}.$$

So

$$\operatorname{coker}(\phi)^{\mathrm{pre}}(U) = 0 \quad \text{and} \quad \operatorname{coker}(\phi)^{\mathrm{pre}}(V) = 0.$$

But!

$$\operatorname{coker}(\phi)^{\mathrm{pre}}(\mathbb{R}) = \operatorname{coker}(\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}.$$

Recall that sheafification is global sections of the étalé space $\text{Ét}(\mathscr{F}) = \coprod \mathscr{F}_z$. From the above, $\mathscr{F}_z = 0$ for all $z \in \mathbb{R}$. Thus,

$$\operatorname{coker}(\phi) = 0.$$

Example 20. Let $X = \mathbb{R}_t$, $\mathscr{F} = \mathscr{G} = \mathscr{C}^\infty_{\mathbb{R}}$ (complex valued smooth functions). We have a map

$$\frac{d}{dt} : \mathscr{C}^\infty_{\mathbb{R}} \to \mathscr{C}^\infty_{\mathbb{R}}, \quad \text{with} \quad \ker\left(\frac{d}{dt}\right) = \mathbb{C}_{\mathbb{R}} \quad \text{(locally constant functions)}.$$

What about the cokernel? Let $U \subset \mathbb{R}$ be open, $f \in \mathscr{C}^\infty(U)$. Want to construct a function F such that $\frac{d}{dt}F = f$; we can do this (fundamental theorem of calculus), e.g. let $F(t) = \int_{t_0}^t f(x)dx$. So this map of sheaves is surjective.

Example 21. Let $X = S^1 = \mathbb{R}/\mathbb{Z}$, $\frac{d}{dt} : \mathscr{C}^\infty_{S^1} \to \mathscr{C}^\infty_{S^1}$. Then $\ker\left(\frac{d}{dt}\right) = \mathbb{C}_{S^1}$ again. Since any open interval $U \subset S^1$ is diffeomorphic to $\mathbb{R}$, $\frac{d}{dt}|_U : \mathscr{C}^\infty_U \to \mathscr{C}^\infty_U$ is surjective. So the cokernel sheaf is $\operatorname{coker}\left(\frac{d}{dt}\right) = 0$.

But: That the constant function $1_{S^1} \in \mathscr{C}^\infty_{S^1}$. This is not in the image of $\frac{d}{dt}|_{S^1} : \mathscr{C}^\infty(S^1) \to \mathscr{C}^\infty(S^1)$.

Example 22. Phrased differently, we have a SES of sheaves on S^1,

$$0 \to \mathbb{C}_{S^1} \to \mathscr{C}^\infty_{S^1} \xrightarrow{\frac{d}{dt}} \mathscr{C}^\infty_{S^1} \to 0.$$

But if we take global sections Γ,

$$0 \to \mathbb{C} \hookrightarrow C^\infty(S^1) \xrightarrow{\frac{d}{dt}} C^\infty(S^1),$$

i.e. the final map is not surjective. This is a manifestation of the fact the Γ is left exact (but not exact). If we do take the cokernel we have

$$0 \to \mathbb{C} \hookrightarrow C^\infty(S^1) \xrightarrow{\frac{d}{dt}} C^\infty(S^1) \to H^1_{\mathrm{dR}}(S^1) = \mathbb{C}.$$

Proposition 2.2. *In general,* $\Gamma : Sh(X) \to Ab$ *is* left *exact.*

On the other hand, exactness can be checked locally.

Proposition 2.3. *A sequence of sheaves* $\mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$ *is exact if and only if* $\mathscr{F}_x \xrightarrow{\phi_x} \mathscr{G}_x \xrightarrow{\psi_x} \mathscr{H}_x$ *is exact for all* $x \in X$.

If $f : X \to Y$ is a map of topological spaces, $\mathscr{F} \in \mathrm{Sh}(X)$ and $\mathscr{G} \in \mathrm{Sh}(Y)$, recall we have

$$f_*(\mathscr{F}) \in \mathrm{Sh}(Y) \quad \text{and} \quad f^*(\mathscr{G}) \in \mathrm{Sh}(X).$$

Recall that

$$f^*(\mathscr{G})(U) = \mathrm{sh}\left(U \mapsto \xrightarrow[V \supset f(U)]{\mathrm{colim}} \mathscr{G}(V)\right).$$

Proposition 2.4. *f^* is left adjoint to f_*.*

I.e. $\mathrm{Hom}_{\mathrm{Sh}(X)}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F})$ is a natural bijection.

Proof. Want to construct natural transformations

$$f^*f_* \xrightarrow{c} \mathrm{id}_{\mathrm{Sh}(X)} \quad \text{and} \quad \mathrm{id}_{\mathrm{Sh}(Y)} \xrightarrow{u} f_*f^*.$$

Why? Given u, c as above,

$$\begin{tikzcd}
\mathrm{Hom}(f^*\mathscr{G}, \mathscr{F} \arrow[r, "f_*"] \arrow[dr] & \mathrm{Hom}(f_*f^*\mathscr{G}, f_*\mathscr{F}) \arrow[d] \\
& \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F})
\end{tikzcd}$$

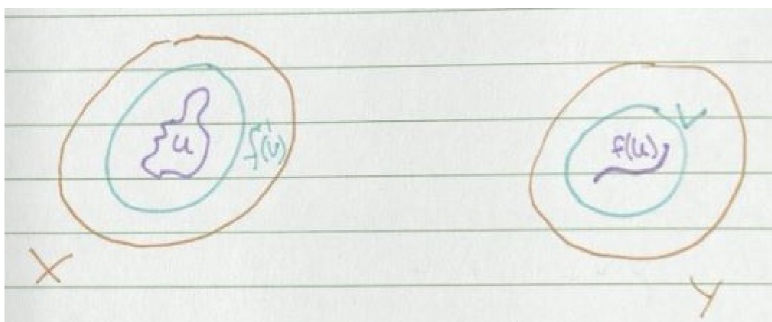


FIGURE 4. Since the map f may not be open, a colimit is required.

Now,

$$(f^*)^{\mathrm{pre}} f_*(\mathscr{F})(U) = \xrightarrow[V \supset f(U)]{\mathrm{colim}} \mathscr{F}(f^{-1}(V)),$$

and we have restrictions $\mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$ for each such V, and thus a map from the colimit. This defines a map $(f^*)^{\mathrm{pre}} f_* \to \mathrm{id}_{\mathrm{PSh}(X)}$; then we use the universal property of sheafification to obtain the counit $f^*f_* \to \mathrm{id}_{\mathrm{Sh}(X)}$. □

2.4. **Simplicial homology. Simplices:**

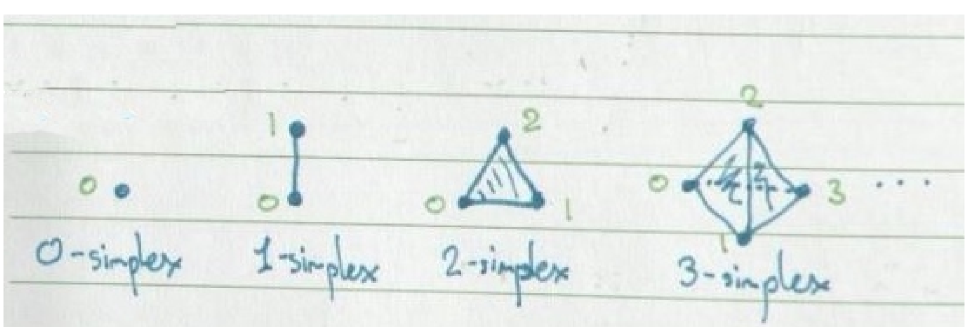


FIGURE 5. Low dimensional simplices.

So,

$$n\text{-simplex} \leftrightarrow (0 \to 1 \to \cdots \to n) = [n].$$

The *faces* of an n-simplex are the ordered subsets $S \subset \{0, \ldots, n\}$.

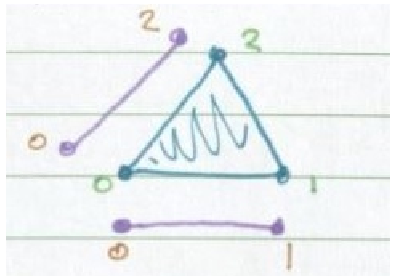


Figure 6. Faces of a 2-simplex.

Define the *simplex category* Δ:

Objects: $[0], [1], [2], \ldots$.
Morphisms: order preserving maps $[n] \to [m]$.

Definition 15. A *simplicial set* is a functor $X : \Delta^{\text{op}} \to \text{Set}$ (i.e. a presheaf on Δ).

So we have the category sSet, and there is an embedding

$$\begin{aligned} \Delta &\xhookrightarrow{\text{Yoneda}} \text{sSet} \\ [n] &\longmapsto \text{Hom}(-, [n]) =: [n] \end{aligned}$$

The fully faithful category given by this is the one with all simplices and all colimits of such (things glued together from simplices).

A simplicial set gives a recipe for building a simplicial topological space.

$$\begin{aligned} \text{sSet} &\xrightarrow{|\cdot|} \text{Top} \\ [n] &\longmapsto \Delta^n \end{aligned}$$

where $|\cdot|$ is *geometric realization* and we extend this definition to all sSet preserving colimits. If $X : \Delta^{\text{op}} \to \text{Set}$ is a simplicial set we write

$$X([n]) = X_n$$

for the *set of n-simplices*, and if we have a map $f : [n] \to [m]$ in Δ this induces $X(f) : X_m \to X_n$ and $f_\Delta : \Delta^m \to \Delta^n$ in Set. So now take concretely

$$|X| = \coprod_n (X_n \times \Delta^n)/\sim$$

where

$$(\sigma, f_\Delta(t)) \sim (X(f)(\sigma), t) \quad \text{for } \sigma \in X_n, t \in \Delta^n.$$

Consider the following diagram in Δ:

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & {[1]} \arrow[l, "s^0" description] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & {[2]} \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & {[3] \cdots} \end{tikzcd}$$

The arrows in this diagram give all possible order preserving maps between adjacent simplices. The maps shown are distinguished maps which we call *coface* (d^i) and *codegeneracy* (s^i) maps. If X is a sSet we have a diagram with *face* and *degeneracy* maps

$$\begin{tikzcd} X_0 \arrow[r, shift left=1.5] & X_1 \arrow[l, shift left=3] \arrow[l] \arrow[r, shift left=3] \arrow[r] & X_2 \cdots \arrow[l, shift left=4.5] \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \end{tikzcd}$$

A simplex in X_n is called *degenerate* if it is in the image of a degeneracy map. Think:

"Degenerate simplices are secretly lower-dimensional simplices."

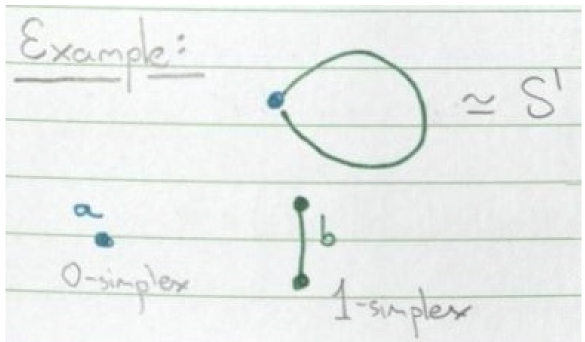


FIGURE 7. Simplicial description of S^1 for example 23.

Example 23. How to prescribe the complex from Figure 7? As a *semi*-simplicial set (use only face maps),

$$X_0 = \{a\} \leftleftarrows \{b\} = X_1,$$

so that we send both faces to the point a. As a simplicial set we would need to keep track of the degeneracies, e.g. $X_1 = \{b, s_0(a)\}$.

If Y is a topological space, we can produce a simplicial set $S(Y)$ by talking

$$S(Y)_n = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, Y),$$

the *singular simplicial set*. This is a sSet, since

$$f : [m] \to [n] \quad \text{induces} \quad f_\Delta : \Delta^m \to \Delta^n \quad \text{induces} \quad S(Y)_n \to S(Y)_m.$$

Theorem 2.5. *If Y is a CW-complex, then $|S(Y)| \simeq Y$ (homotopy equivalence).*

We can also talk about *simplicial abelian groups*,

$$\Delta^{\mathrm{op}} \to \mathrm{Ab},$$

and more generally *simplicial objects in a category $\mathscr{C}$* are $\Delta^{\mathrm{op}} \to \mathscr{C}$.

Example 24. If X is a simplicial set we can define a "free" simplicial abelian group $\mathbb{Z}X$ by

$$(\mathbb{Z}X)_n = \mathbb{Z} \cdot X_n.$$

Given a simplicial abelian group A we can make a chain complex $C(A)_\bullet$:

$$C(A)_n = A_n, \qquad \delta_n : \begin{array}{ccc} A_n & \to & A_{n-1} \\ a & \mapsto & \sum_i (-1)^i d_i(a) \end{array},$$

and one can prove that $\delta_{n-1}\delta_n = 0$ (left as an exercise).

Remark This construction did **not** use the degeneracy maps – so this makes sense for semi-simplicial sets.

We have a subcomplex (check!) of *degenerate simplices* $D(A) \subset C(A)$.

Proposition 2.6. *$D(A)$ is chain homotopic to 0.*

In fact,

$$C(A) = D(A) \oplus N(A)$$

where $N(A)$ is the *normalised chain complex* and $N(A) \sim C(A)$ (chain homotopic).

If X is a simplicial set we can define

$$C_\bullet(X;\mathbb{Z}) = C(\mathbb{Z}X),$$

and the singular homology is

$$H_i(X;\mathbb{Z}) = H_i(C(\mathbb{Z}X)).$$

If Y is a topological space then $S(Y)$ produces

$$C_i^{\text{sing}}(Y;\mathbb{Z}) \quad \text{(singular chains on } Y\text{)}, \quad \text{and} \quad H_i^{\text{sing}}(Y;\mathbb{Z}) \quad \text{(singular homology)}.$$

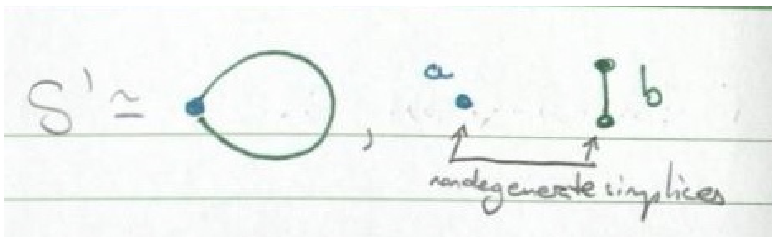


FIGURE 8. Nondegenerate simplices of S^1.

Example 25. Claim that

$$\begin{array}{lccc} N(S^1;\mathbb{Z}) = & \mathbb{Z}a & \xleftarrow[0]{} & \mathbb{Z}b \\ \text{degree:} & 0 & & 1 \end{array}$$

Of course,

$$D(\mathbb{Z}S^1) = \mathbb{Z}a \leftarrow \mathbb{Z}b \oplus \mathbb{Z}s_0(a) \leftarrow \cdots$$

plus higher degenerate simplices which make no contribution to homology.

We can also define more generally $H_i(X;A)$ for A an abelian group.

2.4.1. *Homology with coefficients in a local system.* Suppose $\mathscr{E}$ is a local system on $|X|$ (i.e. $\mathscr{E}$ is a locally constant sheaf of abelian groups). Want to define $H_i(X;\mathscr{E})$. Take (provisionally)

$$C_n(X;\mathscr{E}) = \{(x,s) \,|\, x \in X_n,\ s \in \Gamma(x^*\mathscr{E})\}.$$

What does this mean? $x \in X_n$, so think of this as $x : \Delta^n \to |X|$. But we can't add simplices, so we actually want our n-chains to be:

$$C_n(X;\mathscr{E}) = \langle (x,s) \,|\, x \in X_n,\ s \in \Gamma(x^*\mathscr{E}) \rangle$$

(i.e. the abelian group generated by the terms in the angle brackets.) Now, the n-simplex is contractible, thus $x^*\mathscr{E}$ is trivializable (so we can take sections):

$$\begin{tikzcd} x^*\mathscr{E} \arrow[d] & \mathscr{E} \arrow[d] \\ \Delta^n \arrow[u, bend left, "s"] \arrow[r, "x"] & {|X|} \end{tikzcd}$$

Now stalkwise, $A = \mathscr{E}_y$ (for some $y \in |X|$), and so on Δ^n, $\Gamma(x^*\mathscr{E}) = A$.

If we took the trivial local system, we would see that we recover our previous notion of singular homology.

We define face maps $d_i : C_n(X;\mathscr{E}) \to C_{n-1}(X;\mathscr{E})$ as before by restricting sections to faces.

Example 26. See Figure 9.

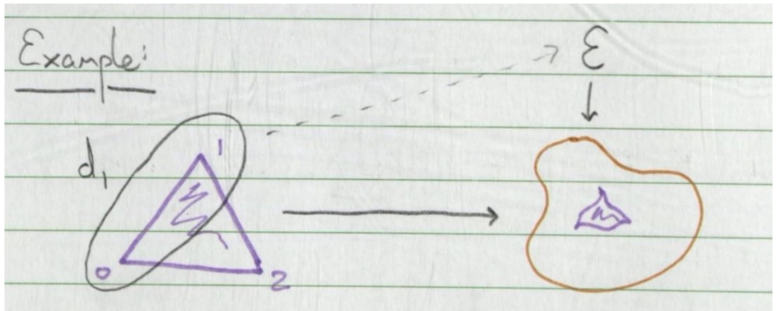


FIGURE 9. Visual representation of the "simplices" in local cohomology.

Then

$$\partial := \sum (-1)^i d_i : C_n(X; \mathscr{E}) \to C_{n-1}(X; \mathscr{E}).$$

Example 27. $X = S^1$, so $\pi_1(S^1) = \mathbb{Z}$. Then

$$\text{Local system on } S^1 \iff \text{Rep of } \pi_1(S^1) \iff (A, t \in \mathrm{Aut}(A))$$

where A is the stalk of $\mathscr{E}$ at some chosen basepoint.

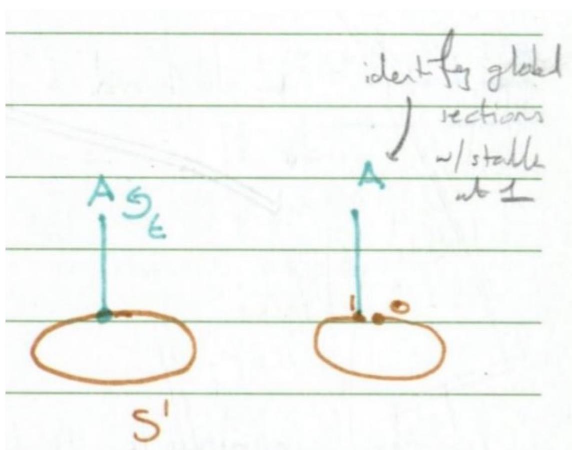


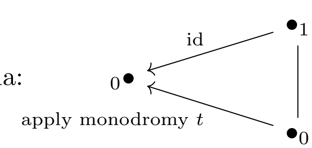
FIGURE 10. A local system on S^1.

What is our complex?

$$A \xleftarrow[\mathrm{id}-t]{} A$$

degree: 0 1

which we can see graphically via:



Example 28. If $\mathscr{E} = \mathbb{Z}_{S^1}$, then we have $\mathbb{Z} \xleftarrow{0} \mathbb{Z}$.

Example 29. If $A = \mathbb{C}$, $t = \lambda \in \mathbb{C}^\times$, then we have

$$\mathbb{C} \xleftarrow{\mathrm{id}-\lambda} \mathbb{C}.$$

So if $\lambda \neq 1$, $\mathrm{id} - \lambda : \mathbb{C} \cong \mathbb{C}$, so $H_0 = 1, H_1 = 0$, etc...

2.4.2. *Cohomology with coefficients in a local system.* Above, we have defined for a sSet/topological space X and local system $\mathscr{E}$

$$(\Delta^i \to X) \in C_i(X;\mathscr{E}) \overset{H_*}{\rightsquigarrow} H_i(X;\mathscr{E})$$

We can give this a slightly more down to earth description: an i-simplex with coefficients in $\mathscr{E}$ is an i-simplex with a lift

$$\begin{tikzcd} & \text{Ét}(\mathscr{E}) \arrow[d] \\ \Delta^i \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

and the differential is induced by the face inclusions $\Delta^{i-1} \hookrightarrow \Delta^i$. We can also define *cohomology with coefficients in a local system,* $H^i(X;\mathscr{E})$, by taking the cochains to be

$$C^i(X;\mathscr{E}) := \left\{ \phi \;\middle|\; \phi \text{ assigns to each simplex a lift, } \begin{tikzcd} & \text{Ét}(\mathscr{E}) \arrow[d] \\ \Delta^i \arrow[r, "\sigma"] \arrow[ur, dashed, "\phi(\sigma)"] & X \end{tikzcd} \right\},$$

and defining the differential $d : C^i \to C^{i+1}$ to be the alternating sum of coface maps $d^r : C^i \to C^{i+1}$, where

$$d^r(\phi)(\sigma^{i+1}) = \phi(d_r(\sigma^{i+1})).$$

3. Sheaf cohomology as a derived functor.

3.1. **Idea and motivation. Where are we going?** Our next goal is to prove *Poincaré Duality*: If M is a closed n-manifold, then

$$H_i(M;\mathbb{Z}_M) \cong H^{n-i}(M;\mathscr{O}r_M),$$

where $\mathscr{O}r_M$ is the orientation local system.

Remark If M is orientable, $\mathscr{O}r_M$ is the constant sheaf.

Example 30. If $M = S^1$,

$$\begin{tikzcd} H_0, H_1 = \mathbb{Z} \arrow[d, leftrightarrow, bend left, "\text{Poincaré Duality}"] \\ H^1, H^0 = \mathbb{Z} \end{tikzcd}$$

We will actually recover Poincaré duality as a special case of *Verdier duality.* So, in order to continue, we need to define sheaf cohomology.

Motivating sheaf cohomology. If $\mathscr{E}$ is a local system, what is $H^0(X;\mathscr{E})$? A 0-cochain assigns to each 0-simplex $x \in X$ an element of the stalk at x. I.e. this is a *discontinuous* section of Ét$(\mathscr{E})$.

What does it mean to be closed? If I have two points and a path between them, the elements of the stalk have to be compatible. I.e. we must have a *continuous* function:

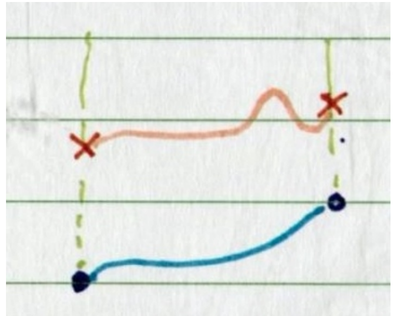


FIGURE 11. H^0 gives continuous sections.

I.e. $H^0(X;\mathscr{E}) = \Gamma(\mathscr{E})$.

In general, given a sheaf $\mathscr{F}$ we want to define functors $H^i(X;\mathscr{F})$ such that $H^0(X;\mathscr{F}) = \Gamma(\mathscr{F})$.

Idea: These functors should measure the *non-exactness* of Γ.

Given a SES of sheaves $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, taking Γ gives

$$0 \to \Gamma(\mathscr{F}') \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{F}'') \to ?$$

In order to measure the failure of exactness, the we will define a next term in this sequence called $H^1(\mathscr{F}')$ – it turns out that this will only depend on $H^1(\mathscr{F}')$.

Definition 16. A sheaf $\mathscr{F}$ is called *flabby* (or *flasque*) if for each $V \subseteq U \subseteq X$ of open sets, $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

Proposition 3.1. *Suppose* $0 \to \mathscr{F}' \xrightarrow{f} \mathscr{F} \xrightarrow{g} \mathscr{F}'' \to 0$ *is a SES, and* $\mathscr{F}'$ *is flabby. Then*

$$0 \to \Gamma(\mathscr{F}') \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{F}'') \to 0$$

is exact.

Proof. Let $s'' \in \Gamma(\mathscr{F}'') = \mathscr{F}''(X)$. Let's define a set

$$\mathcal{S} := \{(U, s) \mid s \in \mathscr{F}(U) \text{ such that } g(s) = s''|_U\}.$$

We want to show that there is an element of this set with $U = X$. $\mathcal{S}$ has a partial order

$$(U_1, s_1) \leq (U_2, s_2) \text{ if } U_1 \subseteq U_2 \text{ and } s_2|_{U_1} = s_1.$$

Zorn's lemma implies that there is a maximal (U, s) in $\mathcal{S}$.

Suppose that $x \in X - U$. We can find $V \ni x$ open in X, and $t \in \mathscr{F}(V)$ such that $g(t) = s''|_V$.

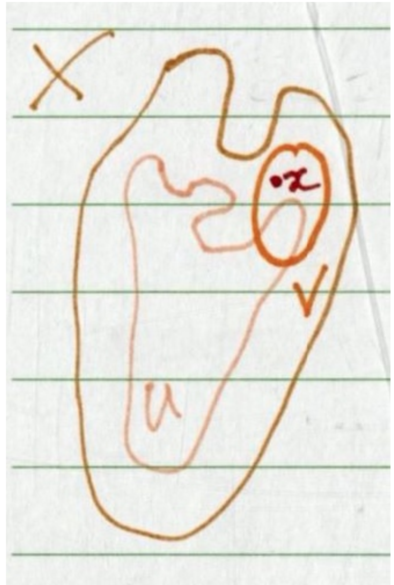


FIGURE 12. Proving that the maximal element is X.

On $U \cap V$, $q := s|_{U\cap V} - t|_{U\cap V} \in \mathscr{F}(U \cap V)$ has the property that $g(q) = 0$. By left exactness, $g(q) = 0$ implies that there exists $w \in \mathscr{F}'(U \cap V)$ such that $f(w) = q$.

Now, since $\mathscr{F}$ is flabby, there exists $r \in \mathscr{F}'(X)$ such that $r|_{U\cap V} = w$. Let $t' = t + f(r)|_V \in \mathscr{F}(V)$ Note that

$$g(t') = s''|_V = g(t)$$

by exactness of $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, and that $s|_{U\cap V} = t'|_{U\cap V}$. Thus there exists a section $\tilde{s} \in \mathscr{F}(U \cup V)$ such that $g(\tilde{s}) = s''|_{U\cup V}$. But this contradicts maximality of (U, s). Thus, $U = X$. □

3.2. **Homological Algebra.** **Idea:** If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a SES, we want a (functorial) LES

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(\mathscr{F}') \arrow[r] & \Gamma(\mathscr{F}) \arrow[r] & \Gamma(\mathscr{F}'') \arrow[dll, hook] \\ & H^1(\mathscr{F}') \arrow[r] & H^1(\mathscr{F}) \arrow[r] & H^1(\mathscr{F}'') \arrow[dll, hook] \\ & H^2(\mathscr{F}') \arrow[r] & H^2(\mathscr{F}) \arrow[r] & H^2(\mathscr{F}'') \arrow[r] & \cdots \end{tikzcd}$$

The collection of functors $\{H^i\}_{i\in\mathbb{Z}_{\geq 0}}$ is called a δ-functor.

There are also naturality conditions. Given a map of SES,

$$\begin{tikzcd} 0 & 0 \\ \mathscr{F}_1'' \arrow[u] \arrow[r] & \mathscr{F}_2'' \arrow[u] \\ \mathscr{F}_1 \arrow[u] \arrow[r] & \mathscr{F}_2 \arrow[u] \\ \mathscr{F}_1' \arrow[u] \arrow[r] & \mathscr{F}_2' \arrow[u] \\ 0 \arrow[u] & 0 \arrow[u] \end{tikzcd} \quad \overset{\text{require that}}{\rightsquigarrow} \quad \begin{tikzcd} H^{i+1}(\mathscr{F}_1') \arrow[r] & H^{i+1}(\mathscr{F}_2') \\ H^{i+1}(\mathscr{F}_1'') \arrow[u, "\delta"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & H^{i+1}(\mathscr{F}_2'') \arrow[u, "\delta"'] \end{tikzcd}$$

Where does this come from? If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a SES of cochain complexes (concentrated in nonnegative degrees), there exists a LES

$$\begin{tikzcd} 0 \cdots \arrow[r] & H^i(A) \arrow[r] & H^i(B) \arrow[r] & H^i(C) \arrow[dll, rounded corners, "\delta"] \\ & H^{i+1}(A) \arrow[r] & H^{i+1}(B) \arrow[r] & H^{i+1}(C) \arrow[r] & \cdots \end{tikzcd}$$

I.e. $\{H^i\}$ is a δ-functor.

Suppose we have

$$\begin{tikzcd} 0 \arrow[r] & A^i \arrow[r, "f"] \arrow[d, "d"] & B^i \arrow[r, "g"] \arrow[d, "d"] & C^i \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A^{i+1} \arrow[r, "f"] & B^{i+1} \arrow[r, "g"] & C^{i+1} \arrow[r] & 0 \end{tikzcd}$$

We want to define a map $H^i(C) \xrightarrow{f} H^{i+1}(A)$. Let $c \in H^i(C)$, and choose a representative $\tilde{c} \in C^i$ such that $d\tilde{c} = 0$. There exists $\tilde{b} \in B^i$ such that $g(\tilde{b}) = \tilde{c}$; $g(d\tilde{b}) = 0$, so there exists $\tilde{a} \in A^{i+1}$ such that $f(\tilde{a}) = d\tilde{b}$. But now, $f(d\tilde{a}) = 0$, so $d\tilde{a} = 0$ and thus $\tilde{a}$ represents a class $a \in H^{i+1}(A)$. Thus we define

$$h(c) = a.$$

We can see the argument diagrammatically as follows:

$$\begin{tikzcd} & & \tilde{b} \arrow[r, mapsto] & \tilde{c} \\ 0 \arrow[r] & A^i \arrow[r] \arrow[d] & B^i \arrow[r] \arrow[d] & C^i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A^{i+1} \arrow[r] & B^{i+1} \arrow[r] & C^{i+1} \arrow[r] & 0 \\ & \tilde{a} \arrow[r, mapsto] & d\tilde{b} \end{tikzcd}$$

and so

$$\begin{tikzcd} 0 \arrow[r] & A^{i+2} \arrow[r] & B^{i+2} \\ & d\tilde{a} \arrow[r, mapsto] & 0 \end{tikzcd}$$

Computing cohomology. Let's assume that we've already constructed

$$H^i(X; -) : \mathrm{Sh}(X) \to \mathrm{Ab}$$

as a δ-functor. How would we compute this?

Remark There is a category of cohomological δ-functors, and there is a notion of a *universal* δ-functor: a terminal object in this category. The $H^i(X; -)$ will be universal in this sense.

Suppose also that there exists a collection of sheaves $\mathbb{A} := \{\mathscr{A}\}$, such that $H^i(\mathscr{A}) = 0$ for $i > 0$ when $\mathscr{A} \in \mathbb{A}$ (*acyclics*), and for each $\mathscr{F}$ there is some $\mathscr{A} \in \mathbb{A}$ such that $\mathscr{F} \hookrightarrow \mathscr{A}$.

It turns out that flabby sheaves are such an example:

$$\mathscr{F} \hookrightarrow G(\mathscr{F}) := \prod_{x \in X} i_x(\mathscr{F}_x).$$

$G(\mathscr{F})$ is called the *sheaf of discontinuous sections of* $\mathscr{F}$.

Now, let's compute H^1. There is a SES $0 \to \mathscr{F} \to \mathscr{A}^0 \to \mathscr{K}^1 \to 0$ which we can continue into an exact sequence by finding an acyclic $\mathscr{A}^1$ such that $\mathscr{K}^1 \hookrightarrow \mathscr{A}^1$, splicing in the result, and then repeating the procedure for $\mathscr{K}^2$ and etc.:

$$\begin{tikzcd}
& & & & 0 \arrow[dr] & & 0 & & \\
& & & & & \mathscr{K}^2 \arrow[ur] \arrow[dr, hook] & & & \\
0 \arrow[r] & \mathscr{F} \arrow[r, hook] & \mathscr{A}^0 \arrow[rr, dashed] \arrow[dr] & & \mathscr{A}^1 \arrow[rr, dashed] \arrow[ur] & & \mathscr{A}^2 \arrow[rr, dashed] \arrow[dr] & & \mathscr{A}^3 \\
& & & \mathscr{K}^1 \arrow[ur, hook] \arrow[dr] & & & & \mathscr{K}^3 \arrow[ur, hook] & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & &
\end{tikzcd}$$

Taking the cohomology LES gives

$$H^0(\mathscr{F}) \to H^0(\mathscr{A}^0) \to H^0(\mathscr{K}^1) \to H^1(\mathscr{F}) \to H^1(\mathscr{A}^0) = 0,$$

since $\mathscr{A}^0$ is acyclic. So we can express

$$H^1(\mathscr{F}) = H^0(\mathscr{A}^0)/\mathrm{im}\ (H^0(\mathscr{A}^0)).$$

Continuing the LES,

$$H^1(\mathscr{A}^0) = 0 \to H^1(\mathscr{K}^1) \xrightarrow{\sim} H^2(\mathscr{F}) \to 0.$$

Now as in the above splicing picture, we can play the same game for $\mathscr{K}^1 \hookrightarrow \mathscr{A}^1$:

$$H^2(\mathscr{F}) \cong H^1(\mathscr{K}^1) = H^0(\mathscr{K}^2)/\mathrm{im}\ (H^0(\mathscr{A}^1)).$$

How can we splice this information together? $\mathscr{A}^\bullet$ is a complex, and we have a *cohomology sheaf* $\mathscr{H}^i(\mathscr{A}^\bullet)$. We also (importantly!) have the

$$H^i(\Gamma(\mathscr{A}^\bullet)) = H^i(\mathscr{F}).$$

Why? Exactness gives that

$$\ker(\mathscr{A}^1 \to \mathscr{A}^2) = \mathscr{K},$$

so

$$H^1(\mathscr{F}) = \ker(H^0(\mathscr{A}^1) \to H^0(\mathscr{A}^2))/\mathrm{im}\ (H^0(\mathscr{A}^0) \to H^0(\mathscr{A}^1)).$$

Summary: If for all $\mathscr{F} \in \mathrm{Sh}(X)$ there exists $\mathscr{A}^\bullet$ such that $\mathscr{F} \hookrightarrow \mathscr{A}^\bullet$ and $H^i(\mathscr{A}^\bullet) = 0$ for all $i > 0$, then we can compute $H^i(\mathscr{F})$ as $H^i(\Gamma(\mathscr{A}^\bullet))$. We call such an $\mathscr{A}^\bullet$ an *acyclic resolution.*

Definition 17. A δ-functor which has the above property (i.e. existence of acyclic resolutions) is called *effacable.*

Theorem 3.2. *An effacable δ-functor is universal.*

3.2.1. *Injective resolutions.* Let $\mathcal{C}$ be an abelian category. An object $I \in \mathcal{C}$ is *injective* if $\mathrm{Hom}(-, I)$ is exact. I.e.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed, "\exists"] \\ & I & \end{tikzcd}$$

Remark If $0 \to I \to B \to C \to 0$ is a SES it is split, since

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[dl, dashed, "\text{splitting map}"] & C \arrow[r] & 0 \\ & I & & & \end{tikzcd}$$

Remark If F is an additive functor, it preserves split exact sequences.

Example 31. In Ab, $\mathbb{Z}$ is not injective; e.g.

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

but $\mathbb{Z} \ncong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. On the other hand, $\mathbb{Q}$ *is* injective.

Definition 18. A category $\mathcal{C}$ is said to have *enough injectives* if for all $A \in \mathcal{C}$ there exists I injective such that $A \hookrightarrow I$. Having enough injectives implies the existence of *injective resolutions* $A \to I^\bullet$.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is left exact, and $\mathcal{C}$ has enough injectives. Then we can define a (universal) δ-functor R^iF as follows:

$$R^iF(A) := H^i(F(I^\bullet)),$$

where $A \to I^\bullet$ is an injective resolution.

Why is this well defined, and why is it a δ-functor? This boils down to the *comparison lemma*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ I^\bullet \arrow[r, dashed, "\exists"] & J^\bullet \end{tikzcd}$$

and the *Horseshoe lemma*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dashed] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^\bullet \arrow[r, dashed] & J^\bullet := I^\bullet \oplus K^\bullet \arrow[r, dashed] & K^\bullet \arrow[r] & 0 \end{tikzcd}$$

In particular:

$$H^i(X; -) = R^i\Gamma(X; -).$$

Remark The LES sequence of the δ-functor is exactly the cohomology LES of

$$0 \to F(I^\bullet) \to F(J^\bullet) \to F(K^\bullet) \to 0.$$

We can compute cohomology using injective resolutions.

3.3. **Why does $\mathbf{Sh}(X)$ have enough injectives?** Ab has enough injectives, since $\mathbb{Q}/\mathbb{Z}$ is injective and

$$A \xrightarrow{\text{embeds}} \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{embeds}} \prod \mathbb{Q}/\mathbb{Z}.$$

Then for $\mathscr{F} \in \mathrm{Sh}(X)$, we can construct injectives $I(-)$ using the above procedure (see [W]) to obtain

$$\mathscr{F} \hookrightarrow G(\mathscr{F}) = \prod_{x \in X} (i_x)_*(\mathscr{F}_x) = \prod_{x \in X} (i_x)_* I(\mathscr{F}_x).$$

3.4. **Computing sheaf cohomology.** Recall that we can compute cohomology using acyclic resolutions.

Proposition 3.3. *Flabby sheaves are acyclic.*

Proof. If $\mathscr{F}$ is flabby, take an injective resolution $0 \to \mathscr{F} \to I^\bullet$. Now, injective sheaves are flabby (exercise), and in the SES

$$0 \to \mathscr{F} \to I^\bullet \to \mathscr{K} \to 0$$

$\mathscr{K}$ is also flabby (exercise). So the LES of a δ-functor gives

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathscr{F}) \arrow[r] & H^0(I) \arrow[r, two heads] & H^0(\mathscr{K}) \arrow[dll, rounded corners, "0"] \\ & H^1(\mathscr{F}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, rounded corners] \\ & H^2(\mathscr{F}) \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

□

We have $\mathscr{F} \hookrightarrow G(\mathscr{F})$, so we have from this the *Godement resolution* $\mathscr{F} \hookrightarrow G^\bullet(\mathscr{F})$, and so

$$H^i(X; \mathscr{F}) = H^i(\Gamma(G^\bullet(\mathscr{F}))).$$

Computing cohomology with the Godement resolution is, however, bloody stupid. Thankfully we have already seen that all we need are acyclic resolutions.

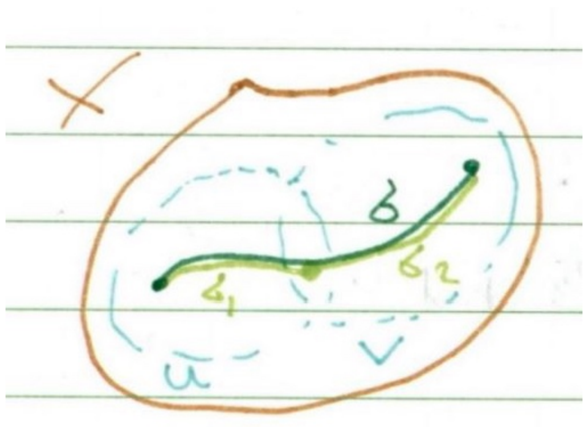


Figure 13. Sum of two singular chains.

3.4.1. *Singular cohomology.* Let $\mathscr{C}_X^{\mathrm{sing,pre}}$ be the presheaf of singular cochains on X with coefficients in $\mathbb{Z}$. This is **not** a sheaf!

Let $\sigma \in C_i(X)$ be as pictured in Figure 13, and let $\varphi \in C^i(X)$ be defined by $\varphi(\sigma) = 1$ and $\varphi(\tilde{\sigma}) = 0$ if $\tilde{\sigma} \neq \lambda\sigma$. Then in particular, $\varphi|_U \equiv 0$.

So $\mathscr{C}_X^{\mathrm{sing,pre}}$ is not a sheaf. But for sort of a silly reason: if we define σ_1 and σ_2 as in Figure 13 then $\varphi(\sigma_i) = 0$, but $\varphi(\sigma_1 + \sigma_2) = \varphi(\sigma) = 1$. We really should have that $\sigma_1 + \sigma_2$ and σ represent the same object.

So, define $\mathscr{C}_X^{\mathrm{sing}} := \mathrm{sh}(\mathscr{C}_X^{\mathrm{sing,pre}})$.

Claims:

(1) There is a quasi-isomorphism $\mathscr{C}_X^{\mathrm{sing}}(X)^\bullet \simeq \mathscr{C}_X^{\mathrm{sing,pre}}(X)^\bullet = C^{\mathrm{sing}}(X)^\bullet$ (this uses "the lemma of small chains").
(2) $\mathscr{C}_X^{\mathrm{sing}}$ is flabby.
(3) If X is locally contractible, then
$$\mathbb{Z}_X \to \mathscr{C}_X^{\mathrm{sing}}$$
is a resolution. For this, exactness on small enough contractible opens is sufficient; then $H^0 = \mathbb{Z}$ and $H^i = 0$ for $i > 0$.

So if X is locally contractible, then for a locally constant sheaf $\mathscr{E}$,
$$H^i(X; \mathscr{E}) = H^i_{\mathrm{sing}}(X; \mathscr{E}).$$

3.4.2. *De Rham cohomology.* If M is a smooth manifold, then we have the *sheaf of smooth i-forms on M*, $\mathscr{A}_M^i$. Using the de Rham differential we get a complex
$$\mathscr{A}_M^\bullet := \mathscr{C}_M^\infty = \mathscr{A}_M^0 \xrightarrow{d} \mathscr{A}_M^1 \xrightarrow{d} \mathscr{A}_M^2 \xrightarrow{d} \cdots$$
Then the Poincaré lemma says that $\mathbb{C}_M \to \mathscr{A}_M^\bullet$ is a resolution. I.e.,
$$0 \to \mathbb{C} \to A^0(\mathbb{R}^n) \to A^1(\mathbb{R}^n) \to \cdots \to A^n(\mathbb{R}^n) \to 0$$
is exact ("any closed form on $\mathbb{R}^n$ is exact").

Remark Really this is just an application of the fundamental theorem of calculus.

Now: the $\mathscr{A}_M^i$ are not flabby, but they are *fine* (and *soft*).

Exercise 3.1. Prove that the $\mathscr{A}_M^i$ are acyclic (hint: partitions of unity). Thus it will follow that
$$H^i(M; \mathbb{C}_M) \cong H^i(\mathscr{A}_M^\bullet(M)) = H^i_{\mathrm{dR}}(M; \mathbb{C}).$$

3.4.3. *Dolbeault resolution.* If X is a complex manifold, we have sheaves $\mathscr{O}_X$ of holomorphic functions and Ω_X^i of holomorphic i-forms. We have that
$$\mathbb{C}_X \to \Omega_X^0 \xrightarrow{d} \cdots \to \Omega_X^n$$
is a resolution. But we can't compute cohomology with this: the Ω_X^i are not acyclic!

What we can do is the following. Begin by embedding $\Omega_X^0 \hookrightarrow \mathscr{A}_X^0$ (smooth functions). Then we have a resolution
$$\Omega_X^0 \to \mathscr{A}_X^0 \xrightarrow{\bar\partial} \mathscr{A}^{0,1} \xrightarrow{\bar\partial} \mathscr{A}^{0,2} \to \cdots \to \mathscr{A}^{0,n},$$
and this is an acyclic resolution. In fact, we can form a double complex:
$$\begin{tikzcd}
\mathscr{A}^{0,n} & & & \\
\vdots \arrow[u, "\bar\partial"] & \vdots & & \\
\mathscr{A}^{0,1} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathscr{A}^{1,1} \arrow[u, "\bar\partial"'] & & \\
\mathscr{A}_X^0 \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathscr{A}^{1,0} \arrow[u, "\bar\partial"'] \arrow[r] & \cdots & \\
\Omega_X^0 \arrow[u] \arrow[r] & \Omega_X^1 \arrow[u] \arrow[r] & \cdots \arrow[r] & \Omega_X^n
\end{tikzcd}$$
Then we have that in fact
$$H^p(\Omega_X^q) = H^{q,p}_{\mathrm{Dol}}(X) =: H^p(\Gamma(\mathscr{A}^{q,\bullet}), \bar\partial).$$

4. More derived functors.

If $f : X \to Y$ we have the functor

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y);$$

observe that $f_* = \Gamma$ if $f : X \to \mathrm{pt}$. f_* is a left exact functor between abelian categories, so we can define the right derived functors of f_*

$$R^i f_*(\mathscr{F}) = \mathscr{H}^i(f_*(\mathscr{I}^\bullet))$$

where $\mathscr{I}$ is an injective/flabby/acyclic resolution of $\mathscr{F}$, and so $f_*(\mathscr{I})$ is a complex of sheaves on Y. Thus $R^i f_*(\mathscr{F}) \in \mathrm{Sh}(Y)$.

Example 32. If $f : X \to Y$ is a fibration (or fibre bundle, or submersion), then for the constant sheaf $R^i f_*(\mathbb{Z}_X) \in \mathrm{Sh}(Y)$, and on stalks

$$R^i f_*(\mathbb{Z}_X)_y = H^i(f^{-1}(y); \mathbb{Z}).$$

Warning! Take $i : \mathbb{C}^\times \hookrightarrow \mathbb{Z}$ and pushforward the constant sheaf (or sheaf of singular cochains). Then since locally around $0 \in \mathbb{C}$ we have punctured open balls ($\cong \mathbb{C}^\times$) our stalk at zero picks up

$$R^1 i_*(\mathbb{Z}_{\mathbb{C}^\times})_0 \cong H^1(\mathbb{C}^\times; \mathbb{Z}) = \mathbb{Z}.$$

Now, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ it is easy to show that $(g \circ f)_* = g_* f_*$. What about $R^i g_* \circ R^j f_*$?

Example 33. Using $Z = \mathrm{pt}$ we might wish to try and compute cohomology on a fibration by understanding how the cohomology of the fibres varies and taking the sheaf cohomology of that. This is computed using the *Leray spectral sequence*, which is a potential topic for another day.

For the moment we take a different tack. We have

$$R^i f_*(\mathscr{F}) = \mathscr{H}^i(f_*(\mathscr{I}^0) \to f_*(\mathscr{I}^1) \to \cdots),$$

so we define the *total derived functor* to be

$$Rf_*(\mathscr{F}) = f_*(\mathscr{I}^0) \to f_*(\mathscr{I}^1) \to \cdots$$

We will worry about the dependence upon a choice of $\mathscr{I}$ in a second. Observe that

$$Rg_* \circ Rf_* = R(g \circ f)_* = g_* f_*(\mathscr{I}^\bullet).$$

This makes sense, as the pushforward of an injective resolution is still an injective resolution.

Remark The equation $Rg_* \circ Rf_* = R(g \circ f)_*$ secretly encodes the Leray-Serre spectral sequence.

Remark If $\mathscr{F}^\bullet$ is a *complex* of sheaves (bounded below) we can find an injective resolution

$$\mathscr{F}^\bullet \xrightarrow{\text{quasi-isomorphism}} \mathscr{I}^\bullet,$$

and $Rf_*(\mathscr{F}^\bullet) = f_*(\mathscr{I}^\bullet)$.

What is going on here? We would like to say that we have a functor

$$Rf_* : \{\text{Complexes of sheaves on } X.\} \to \{\text{Complexes of sheaves on } Y.\},$$

but this doesn't make sense – there are sheaves which are quasi-isomorphic but **not** isomorphic. We can fix this by taking (roughly)

$$Rf_* : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{Sh}(Y)),$$

where $D^+(\mathrm{Sh}(X))$ is the *derived category*,

$$D^+(\mathrm{Sh}(X)) = \{\text{bounded below complexes of sheaves on } X\}[\text{quasi-isomorphisms}]^{-1},$$

a category whose objects are complexes, whose morphisms are morphisms of complexes, but where all quasi-isomorphisms have been inverted.

Warning! This is not rigorous definition – there are problems we will tackle later.

Remark If $\mathscr{F}$ and $\mathscr{G}$ are objects in $D^+(\mathrm{Sh}(X))$ and $\mathscr{H}^i(\mathscr{F}) \cong \mathscr{H}^i(\mathscr{G})$ for all i, it is not necessarily true that $\mathscr{F} \simeq \mathscr{G}$ (quasi-isomorphism).

Example 34. Consider the Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{f} S^2$. We want to consider $Rf_*(\mathbb{Z}_{S^3}) \in D^+\mathrm{Sh}(S^2)$. Let's look at the cohomology objects $R^if_*(\mathbb{Z}_{S^3}) \in \mathrm{Sh}(S^2)$. f is a fibration, so for $U \subset S^2$ a small disk,

$$\Gamma(U; R^if_*(\mathbb{Z}_{S^3})) = H^i(\underbrace{f^{-1}(U)}_{S^1\times U}; \mathbb{Z}) \cong H^i(S^1; \mathbb{Z}).$$

So the $R^if_*(\mathbb{Z}_{S^3})$ are locally constant, and since S^2 is simply connected, locally constant sheaves are constant. Hence,

$$R^0f_*(\mathbb{Z}_{S^3}) = \mathbb{Z}_{S^2} \qquad \text{(measuring } H^0 \text{ of fibres)}$$
$$R^1f_*(\mathbb{Z}_{S^3}) = \mathbb{Z}_{S^2} \qquad \text{(measuring } H^1 \text{ of fibres)}$$

What is the total derived functor $Rf_*(\mathbb{Z}_{S^3})$? There is always an obvious guess:

$$\mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1].$$

Consider

$$S^3 \xrightarrow{f} S^2 \xrightarrow{p} \mathrm{pt},$$

which gives

$$\underbrace{Rp_*}_{R\Gamma(S^2;-)} \; Rf_* = R(p \circ f)_* = R\Gamma(S^3; -).$$

Now if this were the total derived functor, we would have

$$Rp_*(\mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1]) = \underbrace{Rp_*(\mathbb{Z}_{S^2})}_{C^*(S^2;\mathbb{Z})} \oplus \underbrace{Rp_*(\mathbb{Z}_{S^2})[-1]}_{C^*(S^2;\mathbb{Z})[-1]},$$

and upon taking cohomology of this complex, we get

$$H^*(Rp_*(\mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1])) = \mathbb{Z} \oplus \mathbb{Z}[-1] \oplus \mathbb{Z}[-2] \oplus \mathbb{Z}[-3].$$

But $H^*(S^3) = \mathbb{Z} \oplus \mathbb{Z}[-3]$, and so

$$Rf_*(\mathbb{Z}_{S^3}) \not\simeq \mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1].$$

Remark A spectral sequence calculation relates $H^*(S^3)$ and $H^*(Rp_*(\mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1]))$. The calculation makes transparent how the degree 1 and 2 $\mathbb{Z}$ terms are killed off.

4.1. **Compactly supported sections.** Define the *compactly supported sections* functor by

$$\Gamma_c(X; -) : \mathrm{Sh}(X) \to \mathrm{Ab}$$
$$\Gamma_c(X; \mathscr{F}) = \{s \in \Gamma(X; \mathscr{F}) \,|\, \mathrm{supp}(s) \subseteq K \subseteq X \text{ for some compact } K\}$$

where

$$\mathrm{supp}(s) := \{x \in X \,|\, s_x \in \mathscr{F}_x \text{ is nonzero}\}.$$

Exercise 4.1. Show that $\mathrm{supp}(s)$ is closed.

Γ_c is left exact, so we can consider its right derived functors which we call the *compactly supported cohomology* of $\mathscr{F}$:

$$R^i\Gamma_c(X; \mathscr{F}) = H^i_c(X; \mathscr{F}).$$

Example 35. If X is compact, then $\Gamma_c(X; -) = \Gamma(X; -)$.

Example 36. If $X = \mathbb{R}^n$, $\Gamma_c(X; \mathbb{Z}_{\mathbb{R}^n}) = 0$.

Given $f : X \to Y$ we can define

$$f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

by

$$f_!(\mathscr{F})(V) = \left\{ s \in \mathscr{F}(f^{-1}(V)) \;\middle|\; \begin{array}{c} \mathrm{supp}(s) \subseteq f^{-1}(V), \text{ and} \\ \mathrm{supp}(s) \xrightarrow{f|_{\mathrm{supp}(s)}} V \text{ is proper} \end{array} \right\}.$$

(Recall that a map is *proper* if the preimage of a compact set is compact.)

We should assume some 'niceness' properties – e.g. Hausdorff, etc. We would need to change our notion of properness to work with, e.g. the Zariski topology for a variety.

Example 37. If Y is a point, $f_! = \Gamma_c$

Observe that

a) If $i : Z \hookrightarrow X$ is a closed embedding, then i is proper.
b) If $j : U \hookrightarrow X$ is an open embedding, then j is **not** proper.

Example 38.

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, hook, "\text{open emb.}"] & \mathbb{C} \\ D^\times \arrow[r, hook, "\text{open emb.}"] & \overline{D} \arrow[u, hook, "\text{closed emb.}"'] \end{tikzcd}$$

4.2. **Summary of induced functors so far.** A map $f : X \to Y$ induces:

$$\begin{tikzcd} f_* : \mathrm{Sh}(X) \arrow[r, shift left, leftarrow] \arrow[r, shift right] & \mathrm{Sh}(Y) : f^* \\ \text{right adjoint} \arrow[u, squiggly] & \text{left adjoint (and exact)} \arrow[u, squiggly] \end{tikzcd}$$

$$\begin{tikzcd} Rf_* : D^+\mathrm{Sh}(X) \arrow[r, shift left, leftarrow] \arrow[r, shift right] & D^+\mathrm{Sh}(Y) : f^* \\ \text{right adjoint} \arrow[u, squiggly] & \text{left adjoint (no need to derive)} \arrow[u, squiggly] \end{tikzcd}$$

$$\begin{tikzcd} f_! : \mathrm{Sh}(X) \arrow[r] \arrow[d, squiggly] & \mathrm{Sh}(Y) \\ Rf_! : D^+\mathrm{Sh}(X) \arrow[r] & D^+(\mathrm{Sh}(Y)) \end{tikzcd}$$

We call $f_!$ the *pushforward/direct image with proper supports.*

Example 39. If $Y = \mathrm{pt}$, then

- $Rf_* = R\Gamma(X; -)$.
- $R^i f_* = H^i(X; -)$.
- $R^i f_! = H^i_c(X; -)$.

4.3. **Computing derived functors.** Let $j : U \hookrightarrow X$ be an open embedding, and let $\mathscr{G} \in \mathrm{Sh}(U)$. Then

$$j_!(\mathscr{G})(V) = \{ s \in \Gamma(U \cap V; \mathscr{G}) \,|\, \mathrm{supp}(s) \hookrightarrow V \text{ is proper} \}.$$

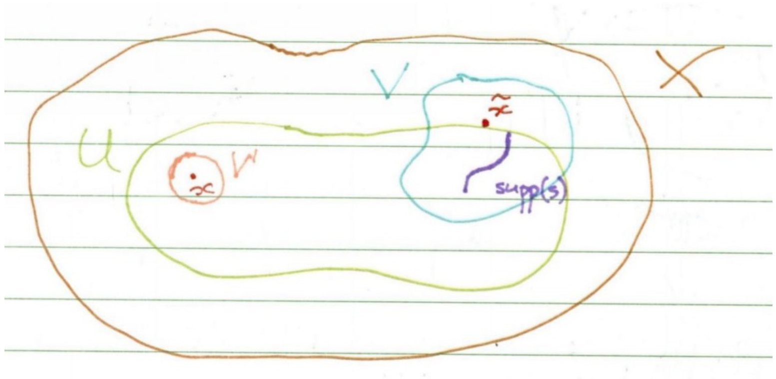


Figure 14. Defining and computing !-pushforward.

Observe that $\mathrm{supp}(s) \hookrightarrow V$ is proper iff $\mathrm{supp}(s)$ is closed in V.

Let's try and compute the stalk of this sheaf. For $x \in U$ we can always find $V' \subset U$ containing x; thus the condition *supp*$(s) \hookrightarrow V'$ *is proper* is vacuous (since $s \in \Gamma(U \cap V'; \mathscr{G}) = \Gamma(V'; \mathscr{G})$. Thus the stalk at x is just $\mathscr{G}_x$, the stalk of $\mathscr{G}$ at x.

Now consider $\tilde{x} \in U$, $X \ni \tilde{x}$.

$$\Gamma(V; j_! \mathscr{G}) = \{s \in \mathscr{G}(V \cap U) \,|\, \mathrm{supp}(s) \hookrightarrow V \text{ is closed}\}.$$

Given such an s we can take $\tilde{x} \in V' \subseteq V$ such that $V' \cap \mathrm{supp}(s) = \emptyset$ (this probably requires our space to be Hausdorff).

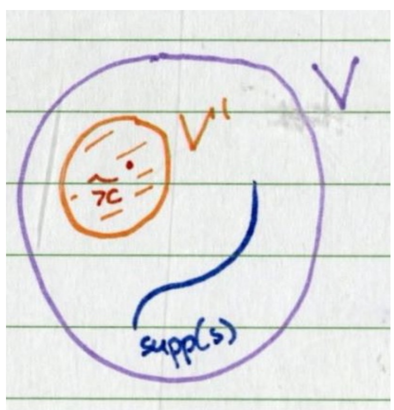


Figure 15. Separating V' and $\mathrm{supp}(s)$.

Then

$$s \mapsto 0 \in (j_! \mathscr{G})_{\tilde{x}}.$$

We summarize:

$$(j_! \mathscr{G})_x = \begin{cases} \mathscr{G}_x & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

We call this *extension of* $\mathscr{G}$ *by 0*.

Now, it is clear that $j^* j_! \mathscr{G} \simeq \mathscr{G}$. What about $j_! j^* \mathscr{F}$? There is a map $j_! j^* \mathscr{F} \to \mathscr{F}$ given on open sets V by taking a section on $\mathscr{F}(U \cap V)$ and extending by 0 to all of V.

Proposition 4.1. *$j_!$ is left adjoint to j^*.*

The counit of the adjunction is $j_!j^*\mathscr{F} \to \mathscr{F}$, and the unit of the adjunction is $\mathscr{G} \to j^*j_!\mathscr{G}$. I.e. we have

$$\operatorname{Hom}(j_!\mathscr{G}, \mathscr{F}) \simeq \operatorname{Hom}(\mathscr{G}, j^*\mathscr{F}).$$

To move between the unit/counit and Hom description of the adjunction, take e.g.

$$\begin{tikzcd}
\operatorname{Hom}(j_!\mathscr{G}, j_!\mathscr{G}) \arrow[r, leftrightarrow, "\sim"] & \operatorname{Hom}(\mathscr{G}, j^*j_!\mathscr{G} \\
\text{unit} \arrow[r, mapsto] & \text{id}
\end{tikzcd}$$

Remark The functor $j_!$ is exact (since it is exact on stalks by our earlier calculation).

Remark This adjunction and exactness is *only* for open embeddings!

4.4. **Functoriality.** Consider

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr] & Y \arrow[d, "p"] \\
& \text{pt}
\end{tikzcd}$$

We want to compute $H^*(X; f^*\mathscr{G})$. We have

$$\begin{tikzcd}
R\Gamma(X; f^*\mathscr{G}) \arrow[r, equal] & Rp_* \circ Rf_*(f^*\mathscr{G}) & Rp_*(\mathscr{G}) \arrow[l, "\text{use adj.}"'] \arrow[d, equal] \\
& & R\Gamma(Y; \mathscr{G}) \arrow[ull, dashed]
\end{tikzcd}$$

So for example, if $\mathscr{G} = \mathbb{Z}_Y$, then we have a map

$$H^*(Y; \mathbb{Z}) \to H^*(X; \mathbb{Z}).$$

What about for compactly supported cohomology? We can try and do the same trick. **Assume f is proper.** Then $Rf_* \simeq Rf_!$, so we can run the same argument as above to get maps

$$H^*_c(X; \mathscr{G}) \to H^*_c(X; f^*\mathscr{G}),$$

which are given by

$$\begin{tikzcd}
R\Gamma_c(X; f^*\mathscr{G}) \arrow[r, equal] & Rp_! \circ Rf_!(f^*\mathscr{G}) \arrow[r, leftrightarrow, "f \text{ proper}", "\sim"'] & Rp_! \circ Rf_*(f^*\mathscr{G}) & Rp_*(\mathscr{G}) \arrow[l] \arrow[d, equal] \\
& & & R\Gamma(Y; \mathscr{G}) \arrow[ulll, dashed]
\end{tikzcd}$$

Example 40. Consider the open embedding

$$\begin{tikzcd}
\mathbb{C}^\times \arrow[r, hook, "j"] \arrow[d, equal] & \mathbb{C} \arrow[d, equal] \\
\mathbb{R}^2 - \{0\} & \mathbb{R}^2
\end{tikzcd}$$

We understand $j_!(\mathbb{Z}_{\mathbb{C}^\times})$ – it is constant away from 0, and has no sections on sets containing 0. Now we claim

$$j_*\mathbb{Z}_{\mathbb{C}^\times} = \mathbb{Z}_{\mathbb{C}}.$$

Letting U be a small ball around 0. Then this follows from

$$(j_*\mathbb{Z}_{\mathbb{C}^\times})(U) = \mathbb{Z}_{C\mathbb{C}^\times}(U \cap \mathbb{C}^\times) \cong \mathbb{Z}.$$

j_* is not exact, so let's compute its derived functor:

$$Rj_*(\mathbb{Z}_{\mathbb{C}^\times}) = j_*(\mathscr{C}^{\bullet,\text{sing}}_{\mathbb{C}^\times})(U) = C^{\bullet,\text{sing}}(U \cap \mathbb{C}^\times).$$

$U \cap \mathbb{C}^\times \simeq S^1$, so taking cohomology gives $\mathbb{Z}$ in degrees 0 and 1. Now, we have the cohomology sheaves

$$R^0j_*(\mathbb{Z}_{\mathbb{C}^\times}) = \mathbb{Z}_{\mathbb{C}}, \quad R^1j_*(\mathbb{Z}_{\mathbb{C}^\times}) = \mathbb{Z}_0,$$

where we observe that we have a skyscraper sheaf at 0 since there is no first cohomology on contractible sets not containing 0.

4.5. **Open-closed decomposition.** Now, consider $j : U \hookrightarrow X \hookleftarrow Z : i$ where $Z = X - U$.

Exercise 4.2. The sequence
$$0 \to j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0$$
is exact. (Hint: It is easy to check exactness on stalks.)

Think: "The sheaf $\mathscr{F}$ is built up from its restriction to open and closed complementary subsets."

Apply $R\Gamma_c(X; -)$ to the above SES:

$$\begin{tikzcd}
 & R\Gamma_c(X; j_! j^* \mathscr{F}) \arrow[r] \arrow[d, equal] & R\Gamma_c(X; \mathscr{F}) \arrow[r] & R\Gamma_c(X; i_* i^* \mathscr{F}) \arrow[d, equal] & \\
R(p_U)_!(j^* \mathscr{F}) \arrow[r, equal] & R(p_X)_! j_! j^* \mathscr{F} & & R(p_X)_! i_* i^* \mathscr{F} \arrow[d, equal] & \\
 & & & R(p_X)_! i_! i^* \mathscr{F} \arrow[r, equal] & R(p_Z)_!(i^* \mathscr{F})
\end{tikzcd}$$

where

$$\begin{tikzcd}
U \arrow[r, hook, "j"] \arrow[dr, "p_U"'] & X \arrow[d, "p_X"] \\
 & \mathrm{pt}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "p_X"'] & Z \arrow[l, hook', "i"'] \arrow[dl, "p_Z"] \\
\mathrm{pt} &
\end{tikzcd}$$

Then we obtain a LES
$$H_c^*(U; j^* \mathscr{F}) \to H_c^*(X; \mathscr{F}) \to H_c^*(Z; i^* \mathscr{F}),$$
which we call the *LES in compactly supported (CS) cohomology.*

Example 41. $j : \mathbb{R}^n \hookrightarrow S^n \hookleftarrow \mathrm{pt} : i$, with $\mathscr{F} = \mathbb{Z}_{S^n}$. Since pt and S^n are compact (and we assume $n \geq 1$), we find that the LES in CS cohomology for $\mathbb{Z}_{S^n}$ is:

$$\begin{tikzcd}
0 & 0 & 0 \\
H_c^n(\mathbb{R}^n; \mathbb{Z}) \arrow[r, "\sim"] & H^n(S^n; \mathbb{Z}) \cong \mathbb{Z} & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
H_c^0(\mathbb{R}^n; \mathbb{Z}) \arrow[r] & H^0(S^n; \mathbb{Z}) \cong \mathbb{Z} \arrow[r, "\sim"] & H^0(\mathrm{pt}\ \mathbb{Z}) \cong \mathbb{Z}
\end{tikzcd}$$

So
$$H_c^*(\mathbb{R}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } * = n, \\ 0 & \text{else.} \end{cases}$$

In particular, observe that H_c^* is not a homotopy invariant – it can distinguish between $\mathbb{R}^n$ of different dimensions.

Notation: If $j : U \to X$, write $\mathscr{F}_U$ for $j_! j^* \mathscr{F}$.

If $X = U_1 \cup U_2$ with both U_i open, we have a SES of sheaves
$$0 \to \mathscr{F}_{U_1 \cap U_2} \xrightarrow{(+,+)} \mathscr{F}_{U_1} \oplus \mathscr{F}_{U_2} \xrightarrow{(+,-)} \to 0.$$
This gives rise to another LES, the *Mayer-Vietoris sequence in CS cohomology.*

Next, given $j : U \hookrightarrow X \hookleftarrow Z : j$, consider the exact sequence

$$0 \to \Gamma_Z \mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F} =: \Gamma_U \mathscr{F},$$

where

$$(\Gamma_Z \mathscr{F})(V) = \{s \in \mathscr{F}(V) \,|\, \mathrm{supp}(s) \subseteq Z\}.$$

Γ_Z is always left exact, but is not exact in general. So we can derive

$$R\Gamma_Z : D^+\mathrm{Sh}(X) \to D^+\mathrm{Sh}(X).$$

There is another functor

$$R\Gamma_Z(X, -) : D^+\mathrm{Sh}(X) \to D^+(\mathrm{Ab}),$$

which we call the *local cohomology*.

5. Verdier Duality.

Start with a map of topological spaces $f : X \to Y$. So far we have seen the following solid arrows:

$$\begin{tikzcd}
D^+\mathrm{Sh}(X) \arrow[r, bend left=20, "Rf_*"] \arrow[r, bend right=10, "Rf_!"'] & D^+\mathrm{Sh}(Y) \arrow[l, bend right=40, "f^*"'] \arrow[l, dashed, bend left=40, "f_!"]
\end{tikzcd}$$

Natural question: Does the dashed adjoint exist?

Example 42. If $U \hookrightarrow X$ is an *open* embedding, we have seen that $j_! \dashv J^* \dashv Rj_*$, ($\dashv$ means "is left adjoint to"). So in this case $j^! = j^*$.

Example 43. If $i : Z \hookrightarrow X$ is a *closed* embedding,

$$i^* \dashv i_* \simeq i_! \dashv i^* \Gamma_Z,$$

where $i_* \simeq i_!$ since i is proper, i^* is exact, and Γ_Z is left exact. Thus we can right derive to get

$$i^! = i^* R\Gamma_Z.$$

Remark We have seen that previously the functors f^*, $Rf_!$, etc on $D^+(\mathcal{A})$ are induced from functors on $\mathcal{A}$ by (right) deriving. In general, $f^!$ is *only* defined in the derived category – it is not derived from the original categories.

For now, let us simplify by considering sheaves of rational vector spaces, $D^+\mathrm{Sh}_{\mathbb{Q}}(-)$. (This is a simplification since all $\mathbb{Q}$ vector spaces are injective.)

In general: We want $Rf_! \dashv f^!$.

Remark If f is proper, $Rf_* \simeq Rf_!$.

5.1. **Dualizing complex.** Consider $p : X \to \mathrm{pt}$. What is the "dualizing complex" $p^!(\mathbb{Q}) = \omega_X^\bullet$?

We haven't constructed it yet, but let us deduce some of its properties. Unless otherwise stated, $\mathrm{Hom}_X = \mathrm{Hom}_{D^+(\mathrm{Sh}(Z))}$. We will have

$$R\mathrm{Hom}_X(\mathscr{F}, \omega_X^\bullet) = R\mathrm{Hom}_X(\mathscr{F}, p^!\mathbb{Q}) \simeq R\mathrm{Hom}_{\mathbb{Q}}(Rp_!(\mathscr{F}), \mathbb{Q}) = R\Gamma_c(X; \mathscr{F})^\vee,$$

where we are thinking of $\mathbb{Q}$ as a complex concentrated in degree 0, and $-^\vee$ denotes the dual complex of vector spaces.

Remark $R\mathrm{Hom}(A^\bullet, B^\bullet)$ means the *cochain complex* $\mathrm{Hom}^\bullet(A^\bullet, I^\bullet)$ where $I^\bullet$ is an injective resolution of $B^\bullet$. I.e.

$$\mathrm{Hom}^n(A^\bullet, I^\bullet) = \bigoplus_i \mathrm{Hom}(A^i, I^{n+i}), \text{ which we can visualize as }$$

$$\begin{tikzcd}
\vdots & \vdots \\
A^2 \arrow[u] \arrow[r] & B^2 \arrow[u] \\
A^1 \arrow[u] \arrow[r] & B^1 \arrow[u] \\
A^0 \arrow[u] \arrow[r] \arrow[uur] & B^0 \arrow[u] \\
A^{-1} \arrow[u] \arrow[r] \arrow[uur] & B^{-1} \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

In the above, orange arrows are degree 0 homs, blue arrows are degree 2 homs, and the black arrows denote the differential we can put on this graded vector space such that

$$H^0(\mathrm{Hom}^\bullet(A^\bullet, I^\bullet)) = \mathrm{Hom}(A^\bullet, I^\bullet).$$

Remark The $R\mathrm{Hom}$ adjunction should follow from the Hom adjunction by the universal property of R.

Example 44. Let $\mathscr{F} = \mathbb{Q}_X = p^*\mathbb{Q}$ where $p : X \to \mathrm{pt}$. Then

$$R\mathrm{Hom}_X(\mathbb{Q}_X; \omega_X^\bullet) = R\mathrm{Hom}_{\mathbb{Q}}(Q, R\Gamma(X; \omega_X^\bullet)) = R\Gamma(X; \omega_X^\bullet)$$

by the p^* adjunction, and

$$R\mathrm{Hom}_X(\mathbb{Q}_X; \omega_X^\bullet) = R\Gamma_c(X; \mathbb{Q})^\vee = C_c^*(X; \mathbb{Q})^\vee$$

by the (desired) $p^!$ adjunction. Be aware that C_c^* is a cochain complex that is not necessarily bounded below.

Remark If $A^\bullet$ is a cochain complex of vector spaces, i.e.

$$\cdots \to A^i \to A^{i+1} \to A^{i+2} \to \cdots,$$

when we dualize we can either consider this as a chain complex, or we can relabel,

$$((A^\bullet)^\vee)^i = (A^{-i})^\vee,$$

so that $(A^\bullet)^\vee)$ is a cochain complex. In this class, unless explicitly stated, all complexes are cochain complex.

Remark For X finite dimensional, $R\Gamma(X; \omega_X^\bullet)$ is bounded above. Thus, in general, if X is infinite dimensional the !-pushforward does not exist (since we used this to get $C_c^*(X; \mathbb{Q})^\vee = R\Gamma(X; \omega_X^\bullet)$).

We define *Borel-Moore chains on* X to be

$$C_*^{\mathrm{BM}}(X; \mathbb{Q}) := C_c^*(X; \mathbb{Q})^\vee.$$

Example 45. Let $\mathscr{F} = \mathbb{Q}_U = j_!\mathbb{Q}_U$ for $j : U \hookrightarrow X$ an open subset. Then

$$R\Gamma(U; \omega_U^\bullet) = R\mathrm{Hom}(\mathbb{Q}_U; \omega_X^\bullet) = C_c^*(U; \mathbb{Q})^\vee,$$

where $\omega_U^\bullet = \omega_X^\bullet|_U$.

We have an assignment

$$\begin{pmatrix} \text{Open set } U \\ \text{in } X \end{pmatrix} \mapsto \begin{pmatrix} \text{Cochain complex} \\ C_c^*(U; \mathbb{Q})^\vee \end{pmatrix}.$$

Think of this as a presheaf in cochain complexes:

$$\begin{tikzcd} U \hookrightarrow V \arrow[r, squiggly] & C_c^*(U;\mathbb{Q}) \hookrightarrow C_c^*(V;\mathbb{Q}) \arrow[d, squiggly] \\ & C_c^*(U;\mathbb{Q})^\vee \leftarrow C_c^*(V;\mathbb{Q})^\vee \end{tikzcd}$$

If we were working fully homotopically, we could take this as a definition.

But, as written we have a problem – an object in the derived category is an *equivalence class* of complexes.To make sense of this, we would need a homotopical version of a sheaf (an ∞-stack).

5.1.1. *What is* $\omega_X^\bullet$*?* Let $\mathscr{I}^\bullet$ be the Godement resolution of $\mathbb{Q}_X$,

$$\mathscr{I}^0 = \prod_{x\in X} \mathbb{Q}_X.$$

Definition 19 (Tentative)**.** $\omega_X^i(U) = \Gamma_c(U;\mathscr{I}^{-i})^\vee$ (so the dualizing sheaf is concentrated in negative degrees – it is a not necessarily bounded below complex).

We now have the meaning/interpretation

$$R\mathrm{Hom}_X(\mathbb{Q}_X,\omega_X^\bullet) \cong R\Gamma(\omega_X^\bullet) = \Gamma_c(X;\mathscr{I}^{-\bullet})^\vee \cong R\Gamma_c(X;\mathbb{Q})^\vee$$

where the final isomorphism is because $\mathscr{I}$ resolves the constant sheaf.

We need the following assumption: X is *finite dimensional*.

$$\dim(X) := \max\{n \mid \exists \mathscr{F} \in \mathrm{Sh}(X),\, H_c^n(X;\mathscr{F}) = 0\}.$$

Fact: If $X = \mathbb{R}^n$, $\dim X = n$.

Another fact: Define $\mathscr{K}^\bullet \in D^+\mathrm{Sh}(X)$ by

$$\mathscr{K}^i := \begin{cases} \mathscr{I}^\bullet & \text{if } i < n, \\ \text{im } (\mathscr{I}^{n-1} \to \mathscr{I}^n) & \text{if } i = n, \\ 0 & \text{if } i > n. \end{cases}$$

If $\dim(X) = n$, then $\mathscr{K}^\bullet$ is a *soft* resolution of $\mathbb{Q}_X$.

Recall: $\mathscr{F}$ is soft if for every $Z \subseteq X$ closed, $\Gamma(X;\mathscr{F}) \to \Gamma(Z;\mathscr{F})$ is surjective. Soft sheaves are acyclic for Γ_c.

Upshot: If X is finite dimensional, we can find a *finite* soft resolution of $\mathbb{Q}_X$.

Definition 20. $\omega_X^i \in \mathrm{Sh}(X)$ is given by

$$\omega_X^i(U) = \Gamma_c(U;\mathscr{K}^{-i})^\vee.$$

With the above definition and the finite dimensionality assumption,

$$\omega_X^\bullet \in D^+(\mathrm{Sh}(X)).$$

Now, let's look at homology complexes.

$$\begin{aligned} {}^{\mathrm{pre}}\mathscr{H}^i(\omega_X^\bullet)(U) &= H^i(\Gamma_c(U;\mathscr{K}^\bullet)^\vee) \\ &= H^i(C_c^*(U;\mathbb{Q})^\vee) \\ &= H_c^{-i}(U;\mathbb{Q})^\vee \end{aligned}$$

where the final equality uses the rationality assumption (i.e. that we are dealing with vector spaces). In particular, on stalks we have

$$\mathscr{H}^i(\omega_X^\bullet)_x = \xrightarrow[U\ni x]{\text{colim}} H_c^{-i}(U;\mathbb{Q})^\vee = H_c^{-i}(U;\mathbb{Q})$$

where the final equality holds for small enough U and nice enough X.

Main point: If $X = M^n$ is a topological n-manifold, then

$$\mathscr{H}^{-n}(\omega_X^\bullet) \text{ is a locally constant sheaf.}$$

Why? For charts $U \cong \mathbb{R}^n$,

$$\mathscr{H}^{-n}(\omega_X^\bullet)(U) = H^n(U;\mathbb{Q})^\vee = \mathbb{Q}.$$

Definition 21 (Orientation)**.** $\mathscr{H}^{-n}(\omega_X^\bullet) = \mathscr{O}r_X$, is called the *orientation sheaf* of X. We say that X is *orientable* if $\mathscr{O}r_X \cong \mathbb{Q}_X$.

So: If $X = M^n$ is a compact, oriented manifold, we claim that

$$H^i(X;\mathbb{Q}) \cong H_{n-i}(X;\mathbb{Q}),$$

and in particular

$$b^i(X) = b_{n-i}(X) \qquad \text{(equality of Betti numbers)}.$$

Summary of properties (so far):

(1) $\mathscr{H}^{-i}(\omega_X^\bullet)$ is the sheafification of

$$U \mapsto H_c^i(U;\mathbb{Q})^\vee = H_i^{\mathrm{BM}}(U;\mathbb{Q}) := H^{-i}(U;\omega_U^\bullet).$$

The constant sheaf represents cochains; we can think of the dualizing complex $\omega_X^\bullet$ as representing Borel-Moore chains, i.e. "locally finite" (e.g. singular) chains.

(2) $R\mathrm{Hom}(\mathscr{F}^\bullet, \omega_X^\bullet) \simeq R\Gamma_c(X;\mathscr{F})^\vee$.

Example 46.

$$H_c^i(\mathbb{R}^n;\mathbb{Q}) = \left\{ \begin{array}{ll} \mathbb{Q}, & i = n \\ 0 & \text{otherwise} \end{array} \right\} = H_i^{\mathrm{BM}}(\mathbb{R}^n;\mathbb{Q}).$$

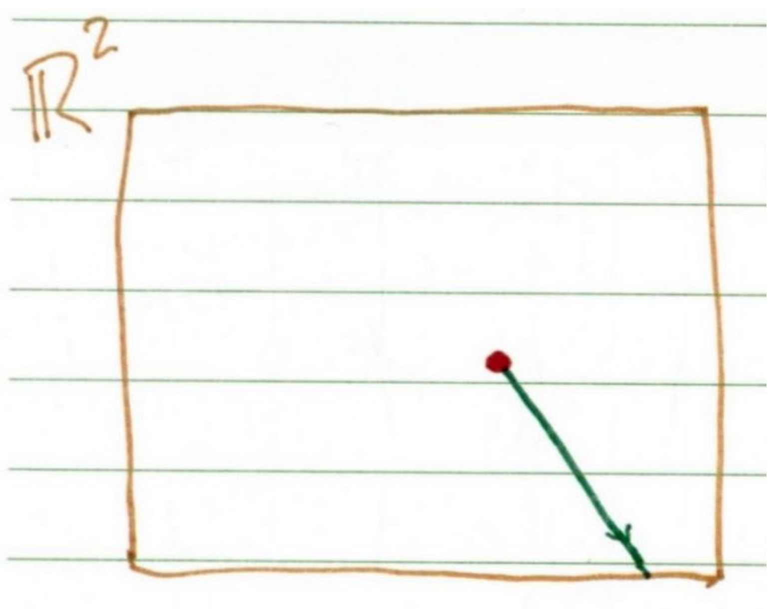


FIGURE 16. A Borel-Moore 1-cycle in $\mathbb{R}^2$.

For instance:

- $H_2^{\mathrm{BM}}(\mathbb{R}^2;\mathbb{Q}) \cong \mathbb{Q}$, generated by all of $\mathbb{R}^2$, a *locally* finite sum of 2-simplices.
- $H_0^{\mathrm{BM}}(\mathbb{R}^2;\mathbb{Q}) = 0$, since any point is the boundary of a ray headed out to ∞ (as per Figure 16).

If X is a topological n-manifold, then

$$\mathscr{H}^{-i}(\omega_X^\bullet) = 0 \quad \text{if } i \neq n,$$

and

$$\mathscr{H}^{-n}(\omega_X^\bullet) \quad \text{is locally constant.}$$

Define the *orientation sheaf* to be

$$\mathscr{O}r_X := \mathscr{H}^{-n}(\omega_X^\bullet).$$

We say that X is *orientable* if $\mathscr{O}r_X \simeq \mathbb{Q}_X$.

5.2. **Poincaré Duality.** If X is orientable,

$$H_i^{\mathrm{BM}}(X;\mathbb{Q}) := H^{-i}(X;\omega_X^\bullet) \cong H^{-i}(X;\mathbb{Q}_X[n]) \cong H^{n-i}(X;\mathbb{Q}).$$

The isomorphism $H_i^{\mathrm{BM}}(X;\mathbb{Q}) \cong H^{n-i}(X;\mathbb{Q})$ is called the *Poincaré Duality isomorphism* (PD). If X is compact this recovers the (potentially) more familiar statement of Poincaré Duality, since then $H_*^{\mathrm{BM}} = H_*$.

Remark $H_i^{\mathrm{BM}}(U;\mathbb{Q}) = H^{-i}(U;\omega_U^\bullet) = H^{-i}(\omega_X^\bullet(U))$.

5.2.1. *Why is PD true?* There are two types of "sheaves" (up to homotopy),

$$\begin{aligned} U \mapsto & C^*(U;\mathbb{Q}) \sim \mathbb{Q}_X \\ U \mapsto & C_c^*(U;\mathbb{Q})^\vee = C_*^{\mathrm{BM}}(U;\mathbb{Q}) \sim \omega_X^\bullet. \end{aligned}$$

A priori, these are two different (pre)sheaves. But if they agree on a basis of open sets, they must agree everywhere.

So on a manifold we can check these agree (up to a shift) locally on X, via the computation on $\mathbb{R}^n$.

Example 47. $\mathbb{RP}^2$ and the Möbius strip are non-orientable. $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, and $\mathscr{O}r_{\mathbb{RP}^2}$ is the local system corresponding to the non-trivial representation of $\mathbb{Z}/2\mathbb{Z}$.

5.3. **Borel-Moore homology.**

Example 48. Consider the singular space shown in Figure 17.

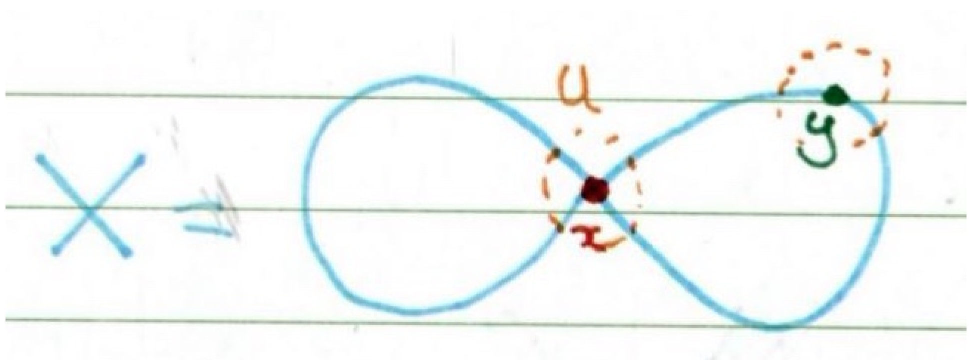


FIGURE 17. The 'figure 8' singular space.

Away from the singular point, we have

$$\mathscr{H}^{-i}(\omega_X^\bullet)_y = \begin{cases} \mathbb{Q}, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

Locally around the singular point x, the space looks like the set U of Figure 18.

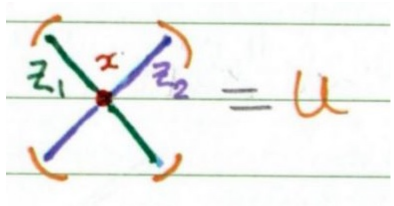


FIGURE 18. Local model around the singular point of the figure 8.

So,

$$\mathscr{H}^{-i}(\omega_X^\bullet)_x = H_i^{\mathrm{BM}}(U;\mathbb{Q}).$$

How can we compute this Borel-Moore homology? In general there is a Mayer-Vietoris sheaf SES,

$$0 \to \mathscr{F}_{Z_1\cup Z_2} \to \mathscr{F}_{Z_1} \oplus \mathscr{F}_{Z_2} \to \mathscr{F}_{Z_1\cap Z_2} \to 0.$$

So we have a SES of sheaves

$$0 \to \mathbb{Q}_{Z_1\cup Z_2} \to \mathbb{Q}_{Z_1} \oplus \mathbb{Q}_{Z_2} \to \mathbb{Q}_{Z_1\cap Z_2} \to 0.$$

Apply the functor $R\Gamma_c$:

$$\begin{tikzcd} H^0_c(Z_1\cup Z_2) \arrow[r] & H^0_c(Z_1)\oplus H^0_c(Z_2) \arrow[r] & H^0_c(Z_1\cap Z_2) \arrow[dll, rounded corners] \\ H^1_c(Z_1\cup Z_2) \arrow[r] & H^1_c(Z_1)\oplus H^1_c(Z_2) \arrow[r] & 0 \end{tikzcd}$$

Since $Z_i \cong \mathbb{R}$ and $Z_1 \cap Z_2 = \mathrm{pt}$, we have $H^1_c(Z_i) \cong \mathbb{Q}$, $H^0_c(Z_1\cap Z_2) \cong \mathbb{Q}$, and $H^0_c(Z_i) = 0$. Hence we get a SES

$$0 \to \mathbb{Q} \to H^1_c(Z_1\cup Z_2) \to \mathbb{Q}^2 \to 0,$$

and so

$$H^1_c(U) \cong \mathbb{Q}^3.$$

We can see this H_1^{BM} as generated by the three 'V' shaped cycles in Figure 19.

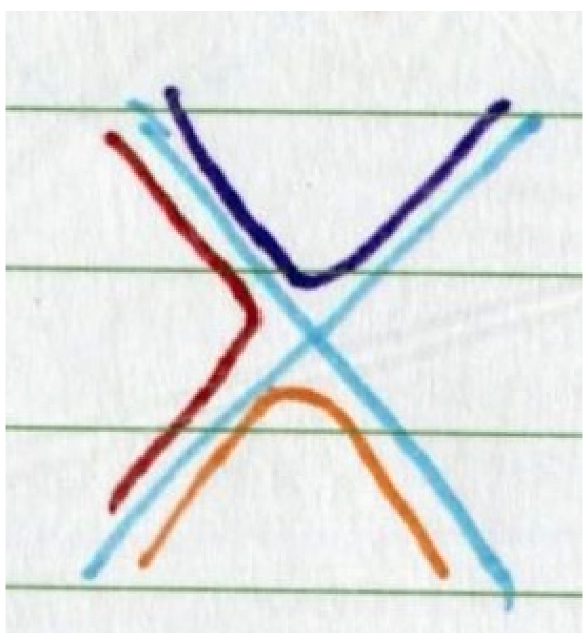


FIGURE 19. Borel-Moore generating 1-cycles for U.

So we can now see in Figure 20 that $\mathscr{H}^i(\omega_X^\bullet)$ is somehow measuring singularities in our space.

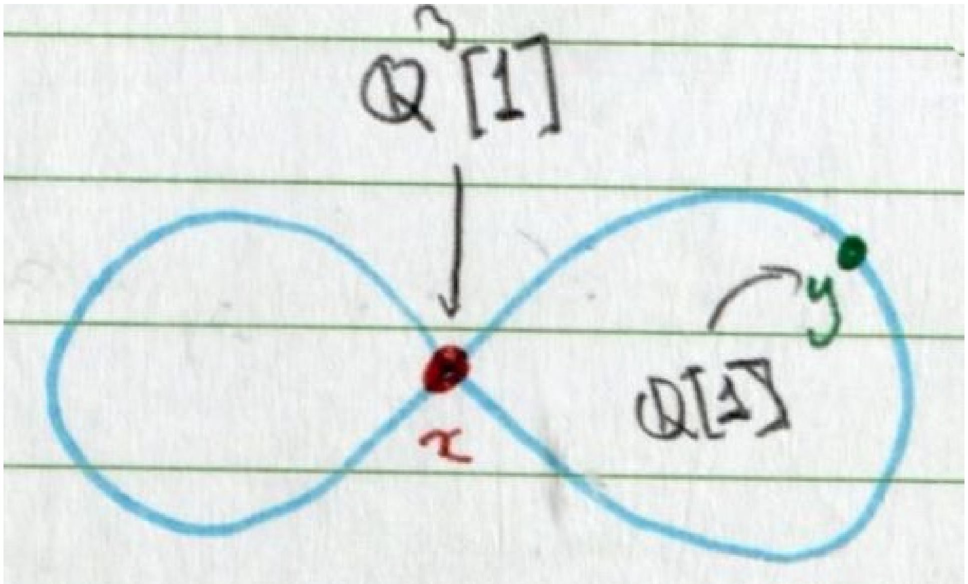


FIGURE 20. Stalks of the cohomology sheaf of $\omega_X^\bullet$.

Exercise 5.1. Think about the restriction maps for this example.

Example 49. Think of the cone with open ends, as in Figure 21.

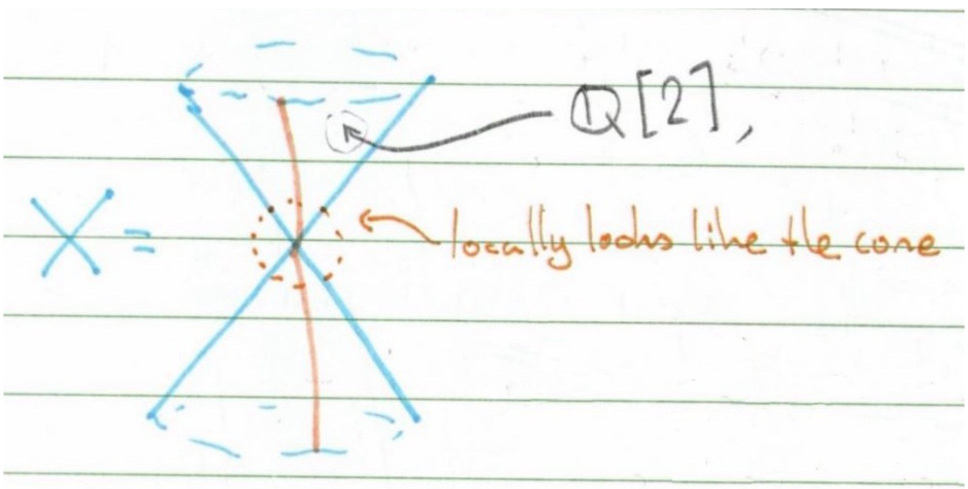


FIGURE 21. Borel-Moore homology of the cone with open ends.

Away from the singular point, the stalk of $\mathscr{H}^*(\omega_X^\bullet)$ is $\mathbb{Q}[2]$, since there the cone is locally a 2-manifold. Around the singular point, we have that locally the manifold looks again like all of X – so here we need to calculate $H_*^{\mathrm{BM}}(X;\mathbb{Q})$. We have

$$H_*^{\mathrm{BM}}(X;\mathbb{Q}) = \begin{cases} \mathbb{Q}^2, & * = 2 \text{ (generated by the upper and lower cones),} \\ \mathbb{Q}, & * = 1 \text{ (generated by the line in Figure 21).} \end{cases}$$

This is a rough geometric argument – we could also decompose and use a LES argument as we did in the previous example.

More generally, if $f : X \to Y$, we want to define
$$f^! : D^+\mathrm{Sh}(Y) \to D^+\mathrm{Sh}(X).$$
If $\mathscr{K} \in \mathrm{Sh}(X)$ is a soft sheaf, and $\mathscr{F} \in \mathrm{Sh}(Y)$ is injective,
$$f^!_{\mathscr{K}}(\mathscr{F}) = \mathrm{Hom}(f_!(\mathscr{K}, \mathscr{F}).$$
Then define
$$f^!(\mathscr{F}) = (f^!_{\mathscr{K}^n}(\mathscr{F}) \to f^!_{\mathscr{K}^{n-1}}(\mathscr{F}) \to \cdots \to f^!_{\mathscr{K}^0}(\mathscr{F})),$$
where the first term is in degree $-n$, the final term is in degree 0, and $\mathbb{Q}_X \to \mathscr{K}$ is the finite (length n) soft resolution as given before. By definition,
$$\omega_X^\bullet = f^!(\mathbb{Q}) \quad \text{when } Y = \mathrm{pt}.$$

Proposition 5.1. *$RHom(Rf_!(\mathscr{G}), \mathscr{F}) \simeq RHom(\mathscr{G}, f^!(\mathscr{F}))$.*

Proof. See [KS]. □

Warning! There is a condition on defining $f^!$: X and Y must both be finite dimensional.

Definition 22. $\omega_{X/Y} := f^!(\mathbb{Q}_Y)$.

5.3.1. *Topological submersions.* We saw (Poincaré Duality) that something nice happened with ω_X for X a manifold. What is a similarly nice situation here?

Definition 23. $f : X \to Y$ is a *topological submersion of relative dimension d* if for all $x \in X$ there exists open $U \subset X$ with $x \in U$, such that

$$\begin{tikzcd} U \arrow[rr, "\simeq"] \arrow[dr] & & f(U) \times \mathbb{R}^d \arrow[dl] \\ & f(U) & \end{tikzcd}$$

and $f(U)$ is open in Y.

Remark This is strictly stronger that just having manifolds for fibres.

Example 50. $\mathbb{R} \xrightarrow{x^2} \mathbb{R}$ is **not** a topological submersion (it fails at 0).

Proposition 5.2. *If f is a topological submersion of relative dimension d, then*
$$\mathscr{H}^{-d}(\omega_{X/Y}) =: \mathscr{O}r_{X/Y} \text{ is a local system,}$$
$$\mathscr{H}^{-i}(\omega_{X/Y}) = 0 \text{ for } i \neq d.$$

We say that X/Y is *orientable* iff $\mathscr{O}r_{X/Y} \simeq \mathbb{Q}_X$. In that case,
$$f^!(-) \simeq f^*(-)[d].$$
In general: for any f,
$$f^!(-) \simeq f^*(-) \otimes \omega_{X/Y}.$$

6. The 6-functor formalism.

We have functors $Rf_*, Rf_!, f^*, f^!, - \otimes_{\mathbb{Q}} -, R\mathscr{H}\mathrm{om}(-,-)$. These give rise to 4 kinds of (co)homology.

(1) $\underbrace{H^*(X;\mathbb{Q})}_{H^*(X)} = Rp_*p^*(\mathbb{Q})$ (ordinary cohomology)
(2) $H^*_c(X) = Rp_!p^*(\mathbb{Q})$ (compactly supported cohomology)

(3) $H^{\mathrm{BM}}_{-*}(X) = Rp_* \underbrace{p^!(\mathbb{Q})}_{\omega_X}$ (Borel-Moore homology)

(4) $H_{-*}(X) = Rp_! p^!(\mathbb{Q})$ (ordinary homology)

BM homology and ordinary homology use *Verdier duality.* Once way of thinking about this – Verdier duality gives a way to build a *cosheaf* from a sheaf, and we take *homology* of a cosheaf.

Unless explicitly stated we assume X is locally compact and finite dimensional.

6.1. **Functoriality from adjunctions.** $f : X \to Y$ gives a unit map

$$1_{D^+(Y)} \longrightarrow Rf_*f^*,$$

$$\mathbb{Q}_Y \longmapsto Rf_*(\mathbb{Q}(X))$$

which gives a map

$$R\Gamma(\mathbb{Q}_Y) = H^*(Y) \to H^*(X) = \underbrace{R(p_Y)_* Rf_*}_{R(p_X)_*}(\mathbb{Q}_X)$$

where

$$\begin{tikzcd} X \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & \mathrm{pt} & \end{tikzcd}$$

Similarly, there is a counit map

$$Rf_!f^! \longrightarrow 1_{D^+(Y)}$$

$$Rf_! \underbrace{f^!(\omega_Y)}_{\omega_X} \longmapsto \omega_y$$

Applying $R(p_Y)_!$ gives

$$H_{-*}(X) = R(p_Y)_!(Rf_!(\omega_X)) \to R(p_Y)_!(\omega_Y) = H_{-*}(Y).$$

I.e.:

- H^* is contravariant.
- H_* is covariant.
- H^*_c:
  - For an open embedding $j : U \hookrightarrow X$, $j^! \simeq j^*$, i.e. j^* is right adjoint to $j_!$, and so H^*_c is covariant.
  - For a proper map $f : X \to Y$, $f_! = f_*$, and so H^*_c is contravariant.
- H^{BM}_* is essentially opposite to H^*_c.

If $f : X \to Y$ is a topological submersion of relative dimension d, and

$$\mathscr{O}r_{X/Y}(= \mathscr{H}^{-d}(\omega_{X/Y})) \simeq \mathbb{Q}_X,$$

then

$$f^!(-) \simeq f^*(-)[d] \quad \text{and so} \quad f^![-d] \simeq f^*.$$

Thus,

$$1_{D^+(Y)} \to Rf_*f^* \simeq Rf_*f^![-d].$$

Apply to ω_Y:

$$\omega_Y \to Rf_* \underbrace{f^!(\omega_Y)}_{\omega_X}[-d] = Rf_*(\omega_X)[-d],$$

so applying $R(p_Y)_*$, we have

$$R(p_Y)_*(\omega_Y) \to \underbrace{R(p_Y)_* Rf_*}_{R(p_X)_*}(\omega_X)[-d],$$

and so taking cohomology gives
$$H^{\mathrm{BM}}_{-*}(Y) \to H^{\mathrm{BM}}_{-(*+d)}(X).$$

Example 51. If $Y = \mathrm{pt}$ (i.e. X is an orientable manifold),

$$\begin{array}{rclc} \mathbb{Q} = H^{\mathrm{BM}}_0(\mathrm{pt}) & \longrightarrow & H^{\mathrm{BM}}_d(X) & \text{(we are using \textit{homological} grading here)} \\ 1 & \longmapsto & [X] & \text{(the fundamental class)} \end{array}$$

So we should think of this map as relating to some *relative* fundamental class.

Note that for and open-closed decomposition $j : U \hookrightarrow X \hookleftarrow Z = X - U : i$, U open, we have an exact sequence of functors
$$0 \to j_! j^* \to 1_{\mathrm{Sh}(X)} \to i_! i^* \to 0$$
on the level of abelian categories; so exact sequences of sheaves for any $\mathscr{F} \in \mathrm{Sh}(X)$
$$0 \to \underbrace{j_! j^*}_{j_! j^!}(\mathscr{F}) \xrightarrow{\text{counit}} \mathscr{F} \xrightarrow{\text{unit}} \underbrace{i_! i^*}_{i_* i^*}(\mathscr{F}) \to 0.$$
Now apply $R(p_X)_!(-)$ to get a SES of complexes, thus a LES in compactly supported cohomology.

Example 52. If $\mathscr{F} = \mathbb{Q}_X$ we have

$$\begin{tikzcd} H^*_c(U) \arrow[r] & H^*_c(X) \arrow[r] & H^*_c(Z) \arrow[dll, rounded corners] \\ H^{*+1}_c(U) \arrow[r] & \cdots \end{tikzcd}$$

This gives rise to a *distinguished triangle* in $D(X)$,
$$Rj_! j^* \to 1_{D(X)} \to Ri_! i^* \xrightarrow{+1} \cdots$$
Note that if we take $R(p_X)_*$ we have a different interpretation,
$$\underbrace{H^*(X, Z)}_{\textit{relative}\text{ cohomology}} \longrightarrow H^*(X) \longrightarrow H^*(Z).$$

So:

- H^* of the decomposition deals with relative cohomology.
- $H^*_c(X)$ is built up out of $H^*_c(U)$ and $H^*_c(Z)$.

We also have another distinguished triangle,
$$Ri_* i^!(\mathscr{F}) \to \mathscr{F} \to Rj_* j^!(\mathscr{F}) \xrightarrow{+1} \cdots$$
which gives rise to a LES in Borel-Moore homology
$$H^{\mathrm{BM}}_*(Z) \to H^{\mathrm{BM}}_*(X) \to H^{\mathrm{BM}}_*(U).$$

Definition 24. The *Euler characteristic* of X is
$$\chi(X) = \sum (-1)^i \dim(H^i_c(X)).$$

Note that using H^*_c means that by the H^*_c LES,
$$\chi(X) = \chi(U) + \chi(Z),$$
i.e. additivity of the Euler characteristic under the open-closed decomposition $U \hookrightarrow X \hookleftarrow Z$.

Proposition 6.1. *If $i : Z \hookrightarrow X$ is a closed submanifold of codimension d, and the normal bundle $N_{Z/X}$ is orientable, then*
$$i^!(\mathbb{Q}_X) \simeq \mathbb{Q}_Z[-d].$$

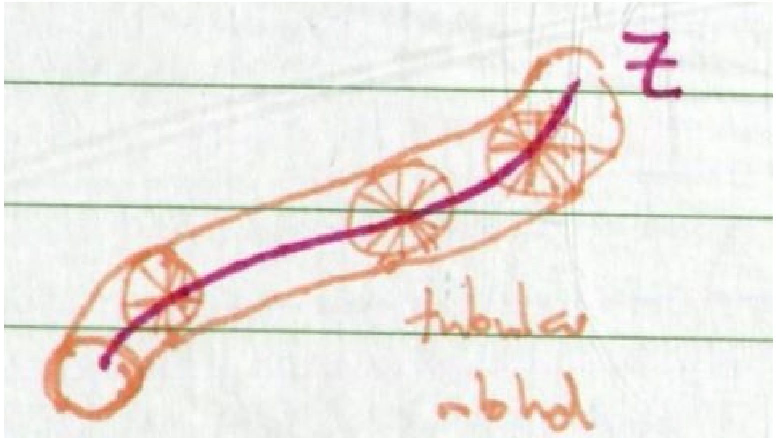


FIGURE 22. Tubular neighbourhood N of Z in X.

Proof. Let N be a tubular neighbourhood of Z in X as in Figure 22. Then

$$\begin{tikzcd} Z \arrow[r, hook, "i_0"] \arrow[rr, hook, bend left, "i"] & N \arrow[r, hook] \arrow[d, "\cong"] & X \\ & N_{Z/X} \arrow[ul, "\pi"] & \end{tikzcd}$$

π is a topological submersion which in orientable. So $i^!(\mathbb{Q}_X) = i_0^!(\mathbb{Q}_N)$, and we have reduced to working on a vector bundle.

$$i_0^!(\mathbb{Q}_N) = i_0^!(\pi_*\mathbb{Q}_Z) = i_0^!\pi^!\mathbb{Q}_Z[-d] = \mathbb{Q}_Z[-d]$$

since $i_0^!\pi^! = \mathrm{id}$, as i_0 is a section of π. Thus,

$$i^!(\mathbb{Q}_X) = \mathbb{Q}_Z[-d].$$

□

This leads to the following:

$$\begin{tikzcd} i_!i^!\mathbb{Q}_X \arrow[r] \arrow[d, equal] & \mathbb{Q}_X \\ i_*\mathbb{Q}_Z[-d] \arrow[r] & \mathbb{Q}_X \end{tikzcd}$$

Now apply Rp_* to this to get a *Gysin map* (or *wrong way map*)

$$H^{*-d}(Z) \to H^*(X).$$

But we have the distinguished triangle given by

$$i_!i^!\mathbb{Q}_X \to \mathbb{Q}_X \to j_*j^*\mathbb{Q}_X,$$

which gives rise to a Gysin LES

$$H^{*-d}(Z) \to H^*(X) \to H^*(U) \xrightarrow{+1} \cdots.$$

6.2. **Base change.** Suppose we have the cartesian square of a fibre product

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Then:

(1) $g^* f_! \simeq \tilde{f}_! \tilde{g}^* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ on the level of abelian categories; there is thus also the derived version $g^* R f_! \simeq R\tilde{f}_! \tilde{g}^*$.
(2) $g^! R f_* \simeq R\tilde{f}_* \tilde{g}^!$.

Example 53. $Y = \mathrm{pt} \xrightarrow{x} X$ gives

$$Rf_!(\mathscr{F})_x = R\Gamma_c(f^{-1}(x); \mathscr{F}).$$

Note:

- If f is proper, $f_* = f_!$.
- If f is an open embedding, $f^! = f^*$.
- If f is a topological submersion, $f^! \simeq f^*[d]$.

So in these (and other) cases we get interesting base change results.

7. Nearby and vanishing cycles.

Want to study the topology of singular varieties, e.g.,

$$X_0 = f^{-1}(0), \text{ where } f : X \to \mathbb{C} \text{ is a proper holomorphic map.}$$

Here X is a (smooth) complex manifold. Think of X as a parametrized collection of varieties (the fibres), all compact since f is proper. Some fibres, such as X_0, may be singular – see Figure 23 for an example.

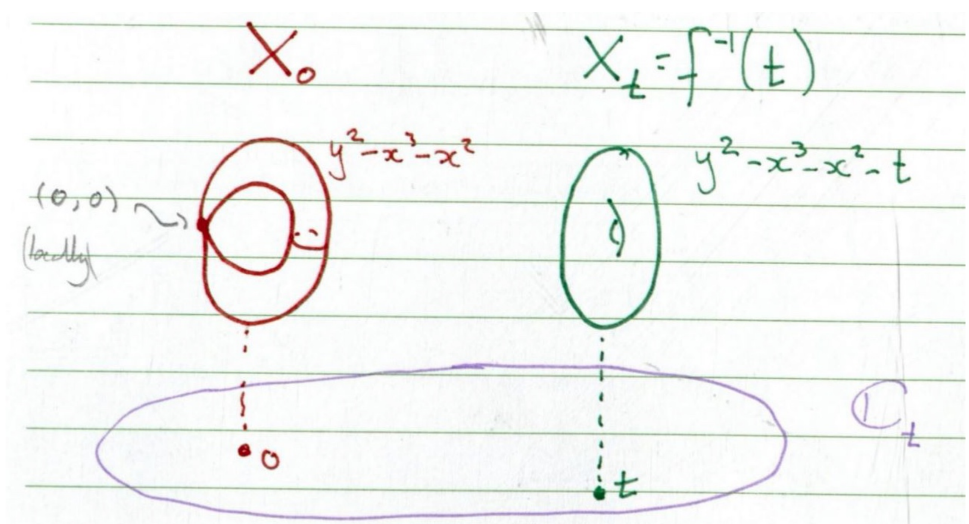


Figure 23. Family over $\mathbb{C}$ with singular fibre at 0.

Assume $0 \in \mathbb{C}$ is an (isolated) singular value, i.e. X_0 is singular. Then $f|_{f^{-1}(\Delta^*)}$ is a submersion, where $\Delta \subset \mathbb{C}$ is a small disk and $\Delta^* = \Delta - \{0\}$.

Example 54. $f(x, y) = y^2 - x^3 - x^2$ has partial derivatives

$$\frac{\partial f}{\partial x} = 3x^2 - 2x, \quad \frac{\partial f}{\partial y} = 2y,$$

so df is onto *except* for at $(x, y) = (0, 0)$; this is the pinch point.

For f proper, Ehresmann's theorem implies that $f|_{f^{-1}(\Delta^*)}$ is a locally trivial fibre bundle, i.e. for all $t \in \Delta^*$, there exists a neighbourhood U of t such that there is a diffeomorphism

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\cong"] \arrow[dr] & & X_t \times U \arrow[dl] \\ & U & \end{tikzcd}$$

Let's think about $\mathscr{H}^i := R^i f_*(\mathbb{Q}_X)$. Recall that the stalk

$$\mathscr{H}^i_t = H^i(X_t) \qquad \text{(this equality uses } \textit{proper} \text{ base change).}$$

$\mathscr{H}^i|_{\Delta^*}$ is a locally constant sheaf, so there is an automorphism

$$T : \mathscr{H}^i_{t_1} \xrightarrow{\cong} \mathscr{H}^i_{t_1},$$

where $t_1 \in \Delta^*$ is a choice of basepoint. T comes from a geometric monodromy (from Ehresmann's theorem)

$$X_{t_1} \xrightarrow{\cong} X_{t_1}.$$

Choosing a path from t to 0 and flowing along it, we also obtain a *specialisation map*

$$r_t : X_t \to X_0.$$

This induces

$$H^*(X_0) \to H^*(X_t)$$

via the restriction map

$$\mathscr{H}^i_0 = \mathscr{H}^i(U) \to \mathscr{H}^i(V) = \mathscr{H}^i_t$$

where U, V are as in Figure 24.

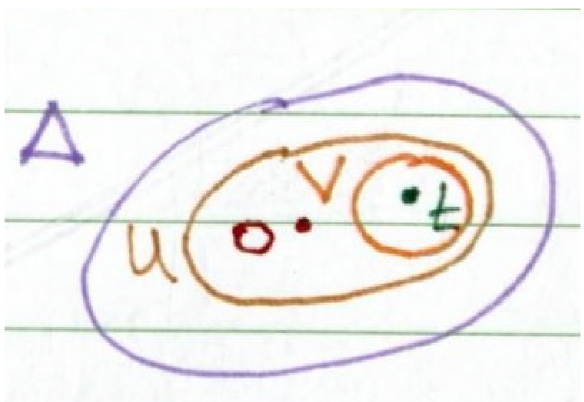


FIGURE 24. Restriction to an open away from the singular point.

Write $r \equiv r_t$ and define

$$\psi_f := Rr_*(\mathbb{Q}_{X_t}),$$

which, up to a shift, is the same as

$$\psi^!_f := Rr_*(\omega_{X_t}).$$

So we have a complex of sheaves $\psi_f \in D^+(X_0)$, and

$$H^*(X_0; \psi_f) = H^*(X_t).$$

Similarly,

$$H^*(X_0; \psi^!_f) = H^{\mathrm{BM}}_*(X_t).$$

We can also look at stalks. Consider the zoomed in picture around the singular fibre (Figure 25).

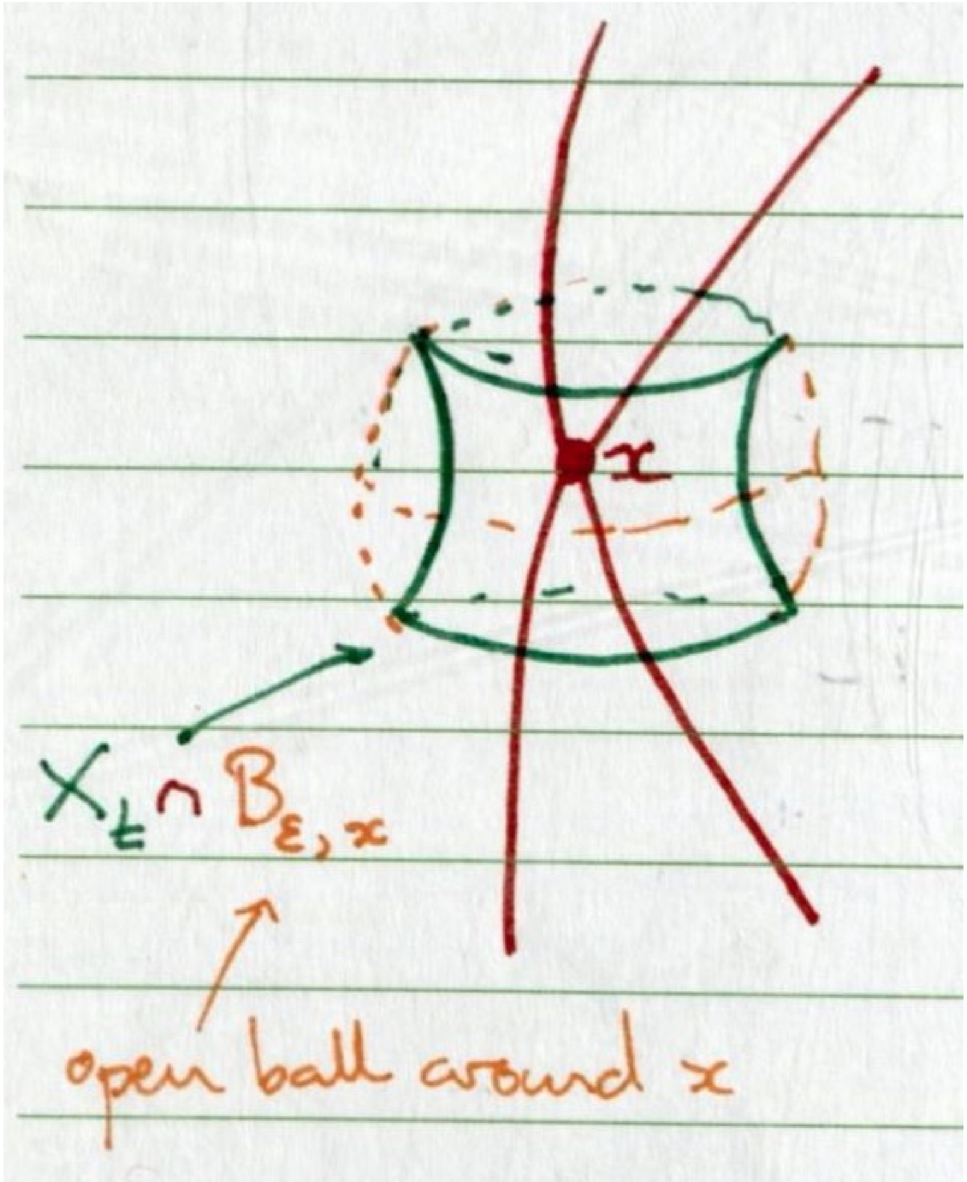


FIGURE 25. Milnor fibre around a singular point.

Then

$$(\psi_f)_x = H^*(\mathrm{Mil}_{f,x}),$$

where the *Milnor fibre* is

$$\mathrm{Mil}_{f,x} = B_{x,\epsilon} \cap X_t$$

where $\epsilon > |t| > 0$ are small enough, and $B_{x,\epsilon}$ is a ball of radius ϵ around x.

Why? Unpacking definitions, using that r is proper, and using base change,

$$(\psi_f)_x = H^*(r^{-1}(x));$$

then finally observe that $r^{-1}(x)$ is the Milnor fibre up to homotopy (compact core of Milnor fibre).

So ψ_f is a sheaf on X_0 that tells us about the cohomology of nearby fibres.

If $x \in X_0$ is nonsingular, $r^{-1}(x)$ is a single point, so that

$$(\psi_f)_x \cong H^*(r^{-1}(x)) \cong \mathbb{Q}.$$

I.e. the Milnor fibre $\mathrm{Mil}_{f,x}$ is contractible if x is nonsingular.

The monodromy map gives us a map of sheaves

$$T : \psi_f \to \psi_f.$$

If we take

$$X \cap B_{x,\epsilon} \to \Delta_\epsilon := f(X \cap B_{x,\epsilon}),$$

then over Δ_ϵ^* we have a fibre bundle with generic fibre $\mathrm{Mil}_{f,x}$. We call this the *Milnor fibration*. See [M] for an original reference.

Facts: If $x \in X_0$ is an *isolated* singularity (no matter how terrible),

$$\mathrm{Mil}_{f,x} \simeq \bigvee_{\mu(x)} S^n,$$

the wedge of $\mu(x)$ n-spheres where $n = \dim_{\mathbb{C}}(X_0)$. We call $\mu(x)$ the *Milnor number.*

Define ϕ_f as follows: there is a unit map

$$\mathbb{Q}_{X_0} \to \psi_f = R(r_t)_*(r_t^*\mathbb{Q}_{X_0})$$

and this corresponds to the specialisation map

$$H^*(X_0) \xrightarrow{\mathrm{sp}} H^*(X_t).$$

Define $\phi_f = \mathrm{cone(sp)}$. I.e. we have an exact triangle in $D^+(X_0)$

$$\mathbb{Q}_{X_0} \to \psi_f \to \phi_f \xrightarrow{+1} \cdots$$

So there is a LES in cohomology

$$H^*(X_0) \to H^*(X_t) \to H^*(X_0; \phi_f),$$

as well as a local version

$$H^*(X_0 \cap B_{\epsilon,x}) \to H^*(\mathrm{Mil}_{f,x}) \to H^*(\phi_{f,x}).$$

Alternatively, we could use $\psi^!$ to get

$$\begin{tikzcd}
\phi^!_f \arrow[r] & \psi^!_f \arrow[r] & \omega_{X_0} \\
\text{cone shifted by 1} \arrow[u] & \text{Borel-Moore chains on nearby fibre} \arrow[u] & \text{Borel-Moore chains on singular fibre} \arrow[u]
\end{tikzcd}$$

so $(\phi^!_f)_x$ are the cycles in $\mathrm{Mil}_{f,x}$ which go to 0 under specialisation. We call this the *vanishing cycles sheaf.*

Example 55. If X has isolated singularities, $\phi^!_f$ must be a sum of skyscraper sheaves supported at the singular points,

$$\phi^!_f = \bigoplus_{x \in X_0} \mathbb{Q}_x^{\mu(x)}[+n],$$

where this is implicitly 0 for nonsingular x.

Example 56. Want to compute the Milnor fibration for our example $y^2 - x^3 - x^2$ at $0 \in \mathbb{C}^2$ (i.e. studying the collapse of the cylinders to a cone near 0). The Hessian is non-degenerate, so the Morse lemma (holomorphic version) tells us we can change coordinates to

$$f(u, v) = uv : \mathbb{C}^2 \to \mathbb{C}.$$

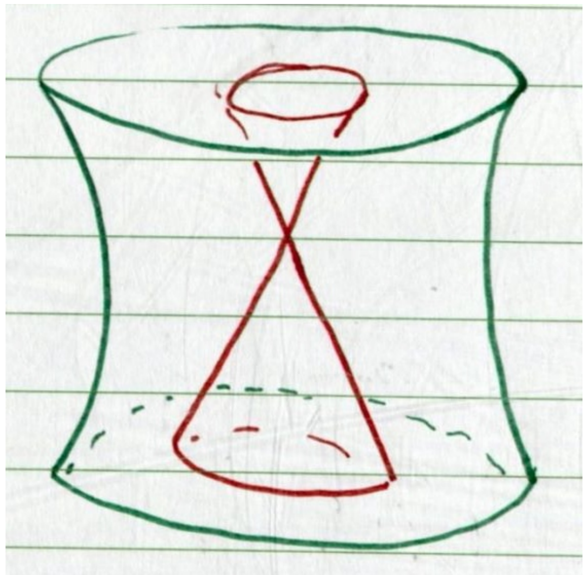


FIGURE 26. Picture of $f^{-1}(z)$ $(f(u,v) = uv)$ around the singular point 0.

We want to compute the monodromy for $X_t \cong \mathbb{C}^\times$ for $t \neq 0$ – we make this identification via the map

$$z \mapsto \left(z, \frac{t}{z}\right).$$

We also have $X_0 = \mathbb{C} \coprod_0 \mathbb{C}$. Then flowing along a lift of $\frac{\partial}{\partial\theta}$ downstairs,

$$T_\theta : X_1 \to X_{e^{i\theta}}, \qquad T_\theta(u,v) = \left(e^{\frac{i\theta}{2}}u, e^{\frac{i\theta}{2}}v\right)$$

is the parallel transport map. In particular,

$$\begin{gathered} T_{2\pi} : X_1 \to X_1 \\ T_{2\pi}(u,v) = (-u,-v) \end{gathered}$$

is the antipodal map. So the *local* monodromy is the identity, since on an odd-dimensional sphere the antipodal map is a rotation,

$$T : H^*(X_1^l) \xrightarrow{\text{id}} H^*(X_1^l).$$

What about the global monodromy? See Figure 27.

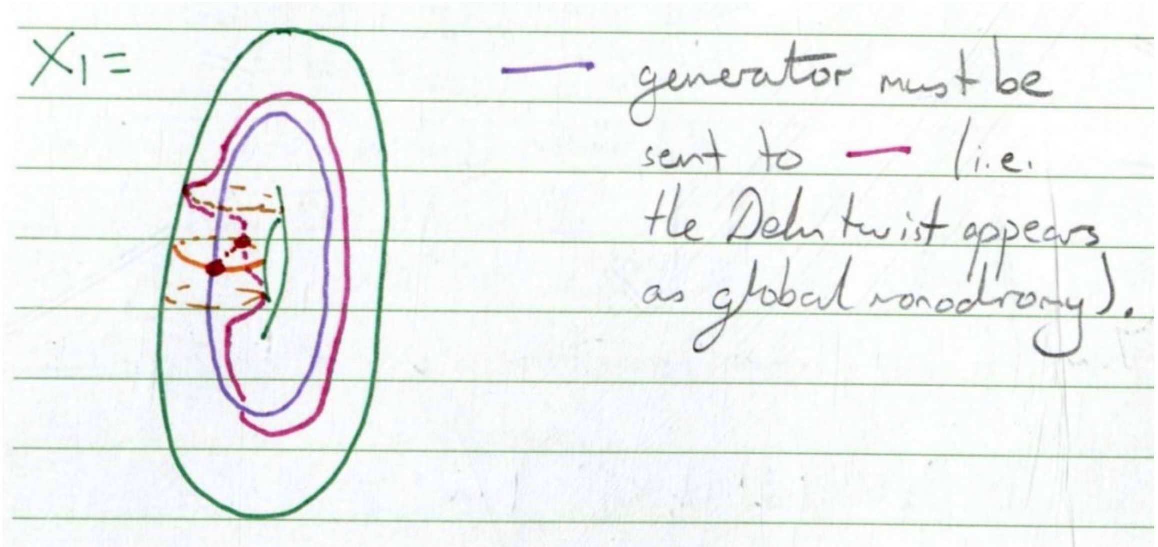


FIGURE 27. Global monodromy for X_1.

8. A preview of the Riemann-Hilbert correspondence.

8.1. C^∞ **Riemann-Hilbert.** Let X be a C^∞-manifold of dimension n. Consider the sheaf of rings $\mathscr{C}_X^\infty$ on X. Then there is a correspondence

$$\left\{ \begin{array}{c} \mathscr{C}_X^\infty\text{-vector bundles} \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Locally free sheaves} \\ \text{of } \mathscr{C}_X^\infty\text{-modules} \end{array} \right\}$$

$$(V \to X) \longmapsto \Gamma(-, V)$$

$$(V^{\mathscr{M}} \to X) \longleftarrow\!\!\shortmid\ \mathscr{M}$$

We define the bottom map as follows. Given a locally free $\mathscr{C}_X^\infty$-module $\mathscr{M}$, the trivializations

$$\phi_i^{\mathscr{M}} : \mathscr{M}|_{U_i} \xrightarrow{\sim} (\mathscr{C}_{U_i}^\infty)^{\oplus r}$$

give rise to transition functions

$$c_{ij}^{\mathscr{M}} = \phi_i \circ \phi_j^{-1}|_{U_i \cap U_j} \in C^\infty(U_i \cap U_j, GL_n).$$

These can then be glued together to give a vector bundle

$$V^{\mathscr{M}} = \frac{\coprod_i U_i \times \mathbb{C}^r}{\sim}$$

There is another correspondence

$$\left\{ \begin{array}{c} \text{Locally constant} \\ \text{sheaves of} \\ \mathbb{C}\text{-vector spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Vector bundles with} \\ \text{flat connection} \end{array} \right\}$$

$$\mathscr{F} \longmapsto (\mathscr{C}_X^\infty \otimes_{\mathbb{C}_X} \mathscr{F} = \mathscr{M}, d^{\mathscr{M}}(f \otimes s) = df \otimes s)$$

$$\ker(d^{\mathscr{M}}) \longleftarrow\!\!\shortmid\ (\mathscr{M}, d^{\mathscr{M}})$$

Remark
- The data of the $c_{ij}^{\mathscr{M}}$ determines a class in $H^1(X; \mathscr{C}^\infty(-, GL_n))$.
- In the above, s is a local section of $\mathscr{F}$ and f is a local section of $\mathscr{C}_X^\infty$.

- The data of a vector bundle with flat connection is equivalent to a class in $\check{H}^1(X, \underline{GL_n})$ (*locally constant* transition functions).

Given a flat vector bundle $(\mathscr{M}, d^{\mathscr{M}})$, define

$$\mathrm{dR}(\mathscr{M})^\bullet = \mathscr{M} \xrightarrow{d^{\mathscr{M}}} \mathscr{M} \otimes_{\mathscr{C}^\infty_X} \mathscr{A}^1_X \xrightarrow{d^{\mathscr{M}}} \mathscr{M} \otimes_{\mathscr{C}^\infty_X} \mathscr{A}^2_X \xrightarrow{d^{\mathscr{M}}} \cdots \xrightarrow{d^{\mathscr{M}}} \mathscr{M} \otimes_{\mathscr{C}^\infty_X} \mathscr{A}^n_X,$$

where $\mathscr{A}^i_X$ are the smooth i-forms on X, and we extend $d^{\mathscr{M}}$ via the Leibniz rule.

Proposition 8.1. $0 \to \ker(d^{\mathscr{M}}) \to dR(\mathscr{M})^\bullet$ *gives a quasi-isomorphism.*

Proof. Can check exactness locally, and then this is immediately implied by the Poincaré lemma. □

Corollary 8.2. $H^*(X; \ker(d^{\mathscr{M}})) = H^*(\Gamma(dR(\mathscr{M})^\bullet)) = H^*_{dR}(\mathscr{M}, d^{\mathscr{M}})$.

Proof. Use that the sheaves $\mathscr{M} \otimes_{\mathscr{C}^\infty_X} \mathscr{A}^m_X$ are fine (thus soft, thus acyclic). □

8.2. **Complex geometry.** Now suppose X is a complex manifold, $\dim_{\mathbb{C}} X = n$. Write

- $\mathscr{O}_X$ for the sheaf of holomorphic functions, and
- Ω^m_X for the sheaf of holomorphic m-forms.

We get a correspondence

$$\left\{ \begin{array}{c} \text{Locally constant sheaves} \\ \text{of } \mathbb{C}\text{-vector spaces} \\ \text{(locally free } \mathbb{C}_X\text{-modules)} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Holomorphic vector bundles} \\ \text{with flat connection} \\ d^{\mathscr{M}} : \mathscr{M} \to \mathscr{M} \otimes_{\mathscr{O}_X} \Omega^1_X \end{array} \right\}$$

$$\mathscr{F} \longmapsto (\mathscr{O}_C \otimes_{\mathbb{C}_X} \mathscr{F} = \mathscr{M}, d^{\mathscr{M}}(f \otimes s) = \bar{\partial} f \otimes s)$$

$$\ker(d^{\mathscr{M}}) \longleftarrow\!\shortmid (\mathscr{M}, d^{\mathscr{M}})$$

Remark $\mathscr{O}_X$-modules are not acyclic for $\Gamma(X; -)$.

Example 57. If $X \hookrightarrow \mathbb{CP}^N$ is projective, then X is an algebraic variety, and

$$\text{Coherent } \mathscr{O}_X\text{-modules} \xleftrightarrow{\sim} \text{Coherent } \mathscr{O}^{\text{alg}}_X\text{-modules}$$

Example 58. If X is Stein, $X \hookrightarrow \mathbb{C}^N$, then coherent $\mathscr{O}_X$-modules are acyclic.

Definition 25. An $\mathscr{O}_X$-module $\mathscr{M}$ is called *coherent* if

(1) it is finitely generated as an $\mathscr{O}_X$-module, (i.e. for each $x \in X$ there exists a neighbourhood $U \ni x$ and $\mathscr{O}_U^{\oplus r} \twoheadrightarrow \mathscr{M}|_U$); and,
(2) for each $U \subseteq X$ open and any
$$\phi : (\mathscr{O}_X|_U)^{\oplus r} \to \mathscr{M}|_U,$$
we have that $\ker(\phi)$ is finitely generated.

Theorem 8.3 (Oka)**.** $\mathscr{O}_X$ *is coherent as a module over* $\mathscr{O}_X$.

Exercise 8.1. Compare to the smooth case: $\mathscr{C}^\infty_X$ is **not** coherent as a $\mathscr{C}^\infty_X$-module.

Hint: Consider multiplication by the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Remark Coherent $\mathscr{O}_X$-modules are the smallest subcategory of sheaves which

(1) contains vectors bundles, and
(2) is closed under finite $\oplus$, kernels and cokernels.

Note that f^* does not preserve $\mathscr{O}$-modules.

Example 59. Flat connections on $X = \mathbb{C}^\times$ (i.e. locally constant sheaves). Every holomorphic vector bundle on $\mathbb{C}^\times$ is trivial, so

$$\left\{ \begin{array}{c} \text{holomorphic flat connections} \\ \text{on } \mathbb{C}^\times \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{square matrices of holomorphic 1-forms} \\ A(z)dz \end{array} \right\}$$

$$d^A(s) = ds - Asdz \longleftarrow\!\!\!\shortmid \; A(z)$$

where d^A is a connection on $\mathscr{O}_X^{\oplus r}$. We know that

$$\{\text{flat connections on } \mathbb{C}^\times\} \simeq \{\text{Reps of } \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}\} \simeq \{\text{vector space with an automorphism}\}.$$

So how do we determine the monodromy automorphism? Take A to be 1×1 in what follows – in general the ideas below require us to consider the *path-ordered exponential*.

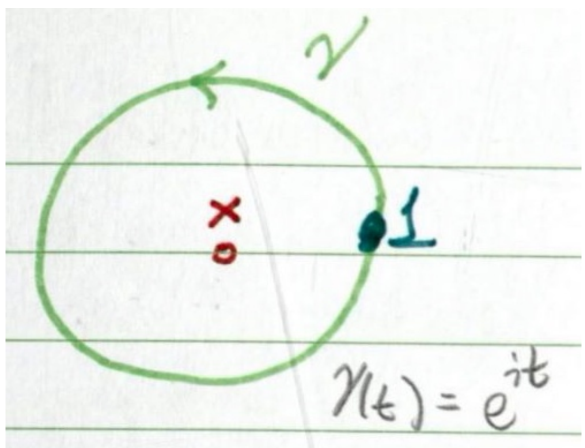


FIGURE 28. Choice of path in $\mathbb{C}^\times$ for parallel transport.

Given a connection matrix A, let $s_0 \in \mathbb{C}^r$ be a section at 1. We want to find $s(t)$ such that $d^A(s) = 0$, i.e.

$$s'(t) = A(\gamma(t))\gamma'(t)s(t), \quad \text{and} \quad s(0) = s_0.$$

To solve this, write

$$\frac{s'}{s} = A(\gamma(t))\gamma'(t),$$

and observe[1] that the solution is given by

$$s(t) = \exp\left(\int_0^t A(\gamma(t))\gamma'(t)dt\right) \cdot s_0.$$

Hence the monodromy automorphism is

$$s(2\pi) = \exp\left(\oint_\gamma Adz\right) \cdot s_0.$$

[1] One can motivate this solution by observing that the LHS is a log-derivative, so we expect naively to see

$$\log\left(\frac{s(t)}{s_0}\right) = \int_0^t A(\gamma(t))\gamma'(t)dt.$$

Theorem 8.4 (Cauchy)**.** *The monodromy matrix is*

$$M(z) = \exp\left(Res_0(A)\cdot 2\pi i\right).$$

I.e. we only need to look at connections of the form

$$B(z) = \frac{\mathrm{Res}_0(A(z))}{z}.$$

Exercise 8.2. It follows that every flat connection on $\mathbb{C}^\times$ is equivalent to one of the form $d^{\frac{B}{z}}$, B constant, i.e.

$$d^{\frac{B}{z}} = d + \frac{B}{z}.$$

So, find an invertible matrix $G(z)$ such that

$$\frac{B}{z} = \pm G^{-1}AG \pm G^{-1}dG,$$

where working out the correct signs is part of the exercise.

9. Constructible sheaves.

We now wish to understand a more general class of sheaves.

Definition 26. A *partition* $\mathcal{P}$ of a topological space X is a collection of disjoint, locally closed subsets of X, X_i, such that

$$\bigcup_i X_i = X.$$

Definition 27. A sheaf $\mathscr{F} \in \mathrm{Sh}(X;\mathbb{C})$ is called *constructible* (w.r.t. $\mathcal{P}$) if $\mathscr{F}|_{X_i}$ is a locally constant sheaf of finite rank.

Example 60. Let $f : X \to \Delta$ be a proper holomorphic map such that $f|_{f^{-1}(\Delta^*)}$ is a submersion. Then $\mathscr{H}^i = R^if_*(\mathbb{C}_X)$ is a constructible sheaf on $\Delta = \Delta^* \cup \{0\}$.

Definition 28. A complex $\mathscr{F} \in D^+\mathrm{Sh}(X;\mathbb{C})$ is called a *constructible complex* if $\mathscr{H}^i(\mathscr{F})$ is constructible for each i.

Example 61. Let $f : S^3 \to S^2$ be the Hopf fibration. $Rf_*(\mathbb{C}_{S^3})$ is a constructible complex of sheaves w.r.t. the trivial partition,

$$R^if_*(\mathbb{C}_{S^3}) = \begin{cases} \mathbb{C}, & i = 0,1, \\ 0, & \text{else.} \end{cases}$$

Recall that

$$Rf_*(\mathbb{C}_{S^3}) \ncong \mathbb{C}_{S^2} \oplus \mathbb{C}_{S^2}[-1].$$

So,

$$D^+_{c,\emptyset}(S^2) \ncong D^+(\mathrm{Sh}_{c,\emptyset}(S^2)),$$

where $D^+_{c,\emptyset}$ means the bounded derived category of constructible complexes with respect to the empty partition, and $\mathrm{Sh}_{c,\emptyset}$ means local systems.

Example 62. $X = \Delta = \Delta^* \cup \{0\}$. Let a partition be given by $\mathcal{P} = \{\Delta^*, \{0\}\}$, and choose nonzero $t \in \Delta$. What kind of data describes a constructible sheaf?

Let $\mathscr{F} \in \mathrm{Sh}_{c,\mathcal{P}}(\Delta)$, and let

$$V_0 = \mathscr{F}_0, \qquad V_t = \mathscr{F}_t.$$

We can consider small open sets to determine $\mathscr{F}_0$ and $\mathscr{F}_t$,

$$V_0 = \Gamma(U;\mathscr{F}) = \mathscr{F}(U)$$
$$V_t = \Gamma(V;\mathscr{F}) = \mathscr{F}(V)$$

where U and V are as shown in Figure 29.

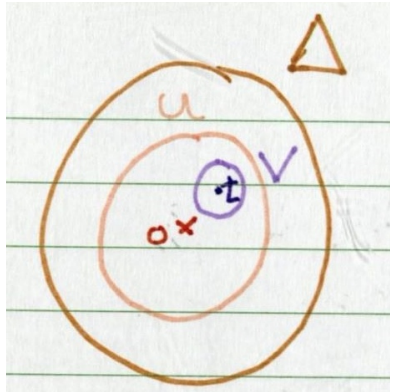


FIGURE 29. Obtaining a quiver from specialization maps.

So we get a quiver

$$\begin{tikzcd} V_0 \arrow[d, "\text{(restriction map) } \alpha"'] \\ V_t \arrow[loop below, "\beta \text{ (monodromy)}"] \end{tikzcd}$$

with the relation

$$\beta\alpha = \alpha \qquad \text{(sections in } V_t \text{ that extend to } V_0 \text{ must have trivial monodromy).}$$

Fact: This data (representation of a quiver with given relation) is *equivalent* to the data of a contructible sheaf.

10. Preliminaries on $\mathcal{D}$-modules.

Let X be a complex manifold. An $\mathscr{O}_X$-module is a sheaf of modules for the sheaf of rings $\mathscr{O}_X$.

Definition 29. A $\mathcal{D}$*-module* on X is an $\mathscr{O}_X$ module $\mathscr{M}$ together with **flat** connections

$$d^{\mathscr{M}} : \mathscr{M} \to \mathscr{M} \otimes_{\mathscr{O}_X} \Omega^1_X$$

$$d^{\mathscr{M}}(f \cdot m) = df \otimes m + f \otimes d^{\mathscr{M}}(m), \quad f \in \mathscr{O}_X, m \in \mathscr{M}.$$

Let $\mathscr{T}_X$ be the sheaf of holomorphic vector fields. For each vector field $\xi \in \mathscr{T}_X$,

$$d^{\mathscr{M}}_{\xi} : \mathscr{M} \to \mathscr{M},$$

and flatness means

$$d^{\mathscr{M}}_{\xi} d^{\mathscr{M}}_{\nu} - d^{\mathscr{M}}_{\nu} d^{\mathscr{M}}_{\xi} = d^{\mathscr{M}}_{[\xi,\nu]} \quad \forall \xi, \nu \in \mathscr{T}_X.$$

Example 63. If $x_1, \ldots, x_n$ are local coordinates on X, then

$$\partial_i := \frac{\partial}{\partial x_i} \in \mathscr{T}_X, \quad [\partial_i, \partial_j] = 0,$$

and we have commuting operators

$$d^{\mathscr{M}}_{\partial_i} : \mathscr{M} \to \mathscr{M}.$$

For now on we write $\xi(m)$ or ξm instead of $d^{\mathscr{M}}_{\xi}(m)$.

Let $\mathcal{D}_X$ denote the subsheaf of rings of $\mathscr{E}nd_{\mathbb{C}_X}(\mathscr{O}_X)$ generated by

- multiplication by $g \in \mathscr{O}_X$;
- $\mathscr{T}_X$ acting by derivations.

$\mathcal{D}_X$ is called the *ring of differential operators.*

In local coordinates, $P \in \mathcal{D}_X$ looks like

$$P = \sum_\alpha f_\alpha(x)\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$

Fact: A $\mathcal{D}$-module $\mathscr{M}$ is the same thing as a $\mathcal{D}_X$-module.

Definition 30. A *stratification* of X is a partition whose strata are manifolds (+ other conditions not listed here).

Theorem 10.1. *There is an equivalence*

$$\left\{ \begin{array}{c} \textit{Bounded constructible complexes for} \\ \textit{some analytic stratification of } X \end{array} \right\} \xleftrightarrow[\sim]{\textit{Riemann-Hilbert correspondence}} D^b \left\{ \begin{array}{c} \textit{regular holonomic} \\ \mathcal{D}_X\textit{-modules} \end{array} \right\}$$

$$dR(\mathscr{M}) \longleftarrow\!\!\!\shortmid \mathscr{M}$$

Sitting inside of these categories we have

$$\begin{tikzcd} D_c(X;\mathbb{C}) \arrow[r, leftrightarrow] & D^b(\mathcal{D}_X\text{-}mod^{rh}) \\ Perv(X;\mathbb{C}) \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \mathcal{D}_X\text{-}mod^{rh} \arrow[u, hook] \end{tikzcd}$$

where the bottom left category is the category of perverse sheaves.

We would like to understand this correspondence.

10.1. **Differential equations.** A (linear) differential operator on X is

$$Pu = 0 \quad \text{where} \quad P \in \mathcal{D}_X.$$

Given $P \in \mathcal{D}_X$, define

$$\mathscr{M}_P = \mathcal{D}_X / \mathcal{D}_X \cdot P, \quad \text{a left } \mathcal{D}_X\text{-module.}$$

Warning! $\mathcal{D}_X$ is non-commutative, so there is a distinction between right and left modules.

$\mathscr{M}_P$ represents the following functor,

$$\mathscr{S} \mapsto \{u \in \mathscr{S} \mid Pu = 0\};$$

$\mathscr{S}$ is thought of as a space of functions in which we might look for solutions. Or, more specifically, $\mathscr{S}$ is some space of functions on X which is a $\mathcal{D}_X$-module.

Example 64. $\mathscr{O}_X$, $\mathscr{S}_X$ the space of Schwarz functions, Sobolev spaces, etc.

Then

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathscr{M}_P, \mathscr{S}) = \{u \in \mathscr{S} \mid Pu = 0\}.$$

Example 65. $X = \mathbb{C}$, coordinate z.

(1) $P = \partial_z$ gives the left $\mathcal{D}$-module

$$\mathscr{M}_P = \mathcal{D}_X / \mathcal{D}_X \cdot \partial_z = \mathscr{O}_X.$$

(2) For $P = z$, what is the $\mathcal{D}$-module $\mathscr{M}_P = \mathcal{D}_X/\mathcal{D}_X \cdot z$? Note that the $\mathcal{D}$-modules $\mathscr{M}_P$ are cyclic; i.e. generated as a $\mathcal{D}_X$-module by one section u, the image of 1 in the quotient. So we can think of the module as
$$\mathscr{M}_P = \mathcal{D}_X \cdot u.$$
For instance, for the prior example, $\mathscr{M}_{\partial_z} = \mathcal{D}_X \cdot 1_X$, $\partial_z(1_X) = 0$. What about for this example? We have
$$\mathscr{M}_z = \mathcal{D}_X \cdot u, \quad z \cdot u = 0.$$
We call this
$$u = \delta(z), \quad \text{the } \delta\text{-}\textit{function}.$$
For us, $\delta(z)$ is just the name for the generator of $\mathscr{M}_z$, which satisfies $z \cdot \delta(z) = 0$. You can make sense of this with distributions if you want, but we won't have to in this course.

Then $\mathscr{M}_z$ will be generated by
$$\partial_z\delta(z) = \delta'(z), \ldots, \partial_z^n\delta(z) = \delta^{(n)}(z), \ldots$$
So
$$\mathscr{M}_z = \mathbb{C}\langle \delta, \delta', \ldots\rangle \cong \mathbb{C}[\partial_z],$$
i.e. the $\mathcal{D}$-module of *constant coefficient differential operators.*

(3) $\mathcal{D}_X \cdot z^\lambda$ with $\lambda \in \mathbb{C}$. We won't worry (yet) about troubles with solutions being multivalued. The object z^λ solves a differential equation,
$$\begin{aligned} \partial_z(z^\lambda) &= \lambda z^{\lambda-1} \\ z\partial_z(z^\lambda) &= \lambda z^\lambda \\ \Rightarrow (z\partial_z - \lambda)z^\lambda &= 0. \end{aligned}$$
If $\lambda \notin \mathbb{Z}$, then
$$\mathcal{D}_X \cdot z^\lambda = \mathcal{D}_X/\mathcal{D}_X(z\partial_z - \lambda).$$
On $\mathbb{C}^\times$ this gives a flat connection
$$d^{\mathscr{M}} = d + \frac{\lambda}{2}dz$$
with monodromy $e^{2\pi i\lambda}$. Why?
$$\partial_z(f(z)z^\lambda) = \frac{\partial f}{\partial z}z^\lambda + \frac{\lambda}{2}dz(z^\lambda).$$
One can compute that the stalk at 0 is
$$(\mathcal{D}_X z^\lambda)_0 = 0.$$

10.2. **The ring $\mathcal{D}_X$.** We now wish to study the ring structure on $\mathcal{D}_X$; recall from above that this is the sheaf $\mathcal{D}_X$ on a complex manifold X,
$$\mathcal{D}_X \subset \mathscr{E}nd_{\mathbb{C}_X}(\mathscr{O}_X).$$
which is generated by $\mathscr{O}_X \subset \mathscr{E}nd_{\mathbb{C}_X}(\mathscr{O}_X)$ acting by multiplication, and $\mathscr{T}_X = \mathrm{Der}(\mathscr{O}_X, \mathscr{O}_X) \subset \mathscr{E}nd_{\mathbb{C}_X}(\mathscr{O}_X)$.

Note: If $\theta \in \mathscr{T}_X$, $f \in \mathscr{O}_X$, let $[\theta, f] = \theta f - f\theta$. Then
$$[\theta, f](g) = \theta(fg) - f\theta(g) = \theta(f)g + f\theta(g) - f\theta(g) = \theta(f)g.$$
So
$$[\theta, f] = \theta(f) \in \mathscr{O}_X \subset \mathcal{D}_X.$$

Definition 31. We say $P \in \mathcal{D}_X$ has *order* $\leq m$ if P can be written as a sum of
$$\theta_1 \cdots \theta_l, \quad \theta_i \in \mathscr{T}_X, \quad l \leq m.$$
Define
$$\mathcal{D}_X(m) = \{\text{differential operators of order } \leq m\}.$$

Proposition 10.2. $[\mathcal{D}_X(m), \mathcal{D}_X(0)] \subset \mathcal{D}_X(m-1)$. *Observe that* $\mathcal{D}_X(0) = \mathscr{O}_X$.

Definition 32 (Alternative definition due to Grothendieck)**.** Given a commutative $\mathbb{C}$-algebra A, define a $\mathbb{C}$-algebra $\mathcal{D}_A \subseteq \mathrm{End}_{\mathbb{C}}(A)$ as follows:

- $\mathcal{D}_A(0) := A \subseteq \mathrm{End}_{\mathbb{C}}(A)$
- $\mathcal{D}_A(m) := \{P \in \mathrm{End}_{\mathbb{C}}(A) \,|\, [P, f] \in \mathcal{D}_A(m-1) \,\forall f \in A = D_A(0)\}$

Remark This definition makes sense for a sheaf of commutative $\mathbb{C}_X$-algebras $\mathscr{A}$.

Example 66. If $\mathscr{A} = \mathscr{O}_X$ then $\mathcal{D}_{\mathscr{O}_X} = \mathcal{D}_X$. This is not hard, but not trivial. E.g. for $m = 1$ we suppose $P : \mathscr{O}_X \to \mathscr{O}_X$ has the property $[P, f] \in \mathscr{O}_X$. Then we claim that $P - P(1)$ is a derivation.

Exercise 10.1. Prove the claim of the example.

Example 67. If A is the coordinate ring of a smooth affine algebraic variety $\mathrm{Spec}(A)$, we call $\mathcal{D}_A$ the ring of algebraic differential operators.

More generally:

- If X^{alg} is a smooth variety over $\mathbb{C}$, then $\mathcal{D}_{X^{\mathrm{alg}}}$ is a sheaf of rings in the Zariski topology.
- If X^{alg} is smooth, then $\mathcal{D}_{X^{\mathrm{alg}}}$ is generated by $\mathscr{O}_{X^{\mathrm{alg}}}$ and $\mathscr{T}_{X^{\mathrm{alg}}}$.
- **But** if X is singular this is no longer true in general.

Example 68. Let $A = \mathbb{C}[x_1, \dots, x_n]$. Then

$$\mathcal{D}_{\mathbb{C}[x_1,\dots,x_n]} =: W_n, \quad \text{the } \textit{Weyl algebra.}$$

This is a ring generated by symbols

$$x_1, \dots, x_n, \partial_1, \dots, \partial_n,$$

subject to the relations

$$[x_i, x_j] = 0, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij}.$$

Example 69. Let

$$A = \mathscr{O}_n := (\mathscr{O}_{\mathbb{C}^n})_0 =: \mathbb{C}\{\{x_1, \dots, x_n\}\},$$

(think of as power series with positive radius of convergence). Define $\mathcal{D}_n := \mathcal{D}_{\mathscr{O}_n}$, and note that at a point $x \in X$ a complex manifold,

$$(\mathcal{D}_X)_x \cong \mathcal{D}_n \quad \text{after picking coordinates around } x.$$

Remark There is a PBW type theorem for $\mathcal{D}_n$,

$$\mathcal{D}_n = \bigoplus_\alpha \mathscr{O}_n \cdot \underbrace{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}_{\partial^\alpha,\, \alpha = (\alpha_1, \dots, \alpha_n)},$$

where $\partial_i = \frac{\partial}{\partial x_i}$. I.e. any $P \in \mathcal{D}_n$ has a *unique* expression of the form

$$P = \sum f_\alpha(x) \partial^\alpha.$$

The *order* of P as a differential operator is the weight

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Proposition 10.3. *(1) $\mathcal{D}_n$ is a filtered ring,*

$$\mathcal{D}_n(0) \subseteq \mathcal{D}_n(1) \subseteq \mathcal{D}_n(2) \subseteq \cdots \quad , \textit{and} \quad \mathcal{D}_n(m)\mathcal{D}_n(k) \subseteq \mathcal{D}_n(m+k).$$

Note that this is **not** *graded, as the bracket can reduce the degree, e.g. $[\partial_x, x] = 1$.*

(2) The associated graded ring *of $\mathcal{D}_n$ is*

$$Gr(\mathcal{D}_n) := \bigoplus_{m=0}^{\infty} \mathcal{D}_n(m)/\mathcal{D}_n(m-1).$$

The symbol maps *are the quotients*

$$\sigma_m : \mathcal{D}_n(m) \to \mathcal{D}_n(m)/\mathcal{D}_n(m-1) =: Gr(\mathcal{D}_n)(m).$$

This is graded,

$$Gr(\mathcal{D}_n)(m) \cdot Gr(\mathcal{D}_n)(k) \subseteq Gr(\mathcal{D}_n)(m+k).$$

Note that $\sigma_1(x\partial_x) = \sigma_1(\partial_x x)$, *for instance. Then we have that*

$$Gr(\mathcal{D}_n) = \mathscr{O}_n[\xi_1, \ldots, \xi_n],$$

where $\xi_i = \sigma_1(\partial_i)$. *Observe that this is a commutative ring, which is holomorphic in* $\mathbb{C}^n$ *and polynomial in the dual space* $(\mathbb{C}^n)^*$.

(3) *Define* $\mathcal{D}_n^{op}$ *to have the same underlying vector space as* $\mathcal{D}_n$, *but with multiplication rule*

$$P \cdot^{op} Q := QP, \quad P, Q \in \mathcal{D}_n^{op}.$$

Then we have $\mathcal{D}_n \cong \mathcal{D}_n^{op}$ *via the map*

$$f \in \mathscr{O}_n \xmapsto{\ a\ } f \in \mathscr{O}_n$$

$$\partial_i \xmapsto{\ a\ } -\partial_i$$

so that

$$\sum_\alpha f_\alpha(x)\partial^\alpha \mapsto \sum_\alpha (-1)^\alpha \partial^\alpha f_\alpha(x).$$

$\mathcal{D}_X$ is also a filtered sheaf of rings,

$$\begin{array}{ccccc} \mathrm{Gr}(\mathcal{D}_X) & = & \mathrm{Sym}_{\mathscr{O}_X}(\mathscr{T}_X) & \hookrightarrow & \pi_*\mathscr{O}_{T^*X} \\ \uparrow & & \uparrow & & \uparrow \\ \text{sheaf of rings} & & \begin{array}{c}\text{holomorphic functions on } T^*X \\ \text{which are polynomial in the fibres}\end{array} & & \text{holomorphic functions on } T^*X \end{array}$$

where $\pi : T^*X \to X$.

To look up (if interested): There is also a ring $\mathscr{E}_X$ of *micro differential operators.*

Proof. We prove the second claim of the proposition. Why is the associated graded commutative?

$$P \in \mathcal{D}_n(m),\ Q \in \mathcal{D}_n(k) \quad \text{implies} \quad [P, Q] \in \mathcal{D}_n(m+k-1).$$

Thus,

$$\sigma_m(P)\sigma_k(Q) - \sigma_k(Q)\sigma_m(P) = \sigma_{m+k}([P, Q]) = 0.$$

Then the PBW theorem implies that

$$\mathcal{D}_n(m)/\mathcal{D}_n(m-1) \cong \bigoplus_{|\alpha|=m} \mathscr{O}_n \cdot \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

$\square$

With respect to the third part of the proposition: first recall that left A^{op}-modules are the same thing as right A-modules. In general (e.g. X a complex manifold), we have that $\mathscr{O}_X$ is a left $\mathcal{D}_X$-module. We also have a holomorphic line bundle (locally trivial sheaf of $\mathscr{O}_X$-modules of rank 1), Ω_X^n.

Claim: Ω_X^n is naturally a right $\mathcal{D}_X$-module.

Motivation: Given $\eta \in \Omega_X^n$, $f \in \mathscr{O}_X$, and ignoring for the moment the fact that we are working with holomorphic top forms, consider the desired module structure

$$\int (\eta \cdot \theta) f = \int \eta(\theta(f)), \quad \theta \in \mathscr{T}_X.$$

I.e. we want to define $(\eta \cdot \theta)$ such that

$$(\eta \cdot \theta)(f) - \eta(\theta(f)) \text{ is exact.}$$

Definition 33. $\eta \cdot \theta = -\mathcal{L}_\theta(\eta)$, the Lie derivative on top forms.

Then

$$-\mathcal{L}_\theta(\eta) = -d\iota_\theta(\eta) - \iota_\theta(d\eta) = -d\iota_\theta(\eta)$$

since η is a top form.

Claim: This defines a right $\mathcal{D}$-module structure on Ω_X^n.

Proposition 10.4. *There is an equivalence of categories*

$$\begin{aligned} \mathcal{D}_X\text{-}mod &\longrightarrow \mathcal{D}_X^{op}\text{-}mod \\ \mathscr{M} &\longmapsto \Omega_X^n \otimes_{\mathscr{O}_X} \mathscr{M} \end{aligned}$$

11. Algebraic geometry.

Let X be a complex manifold, $\mathscr{O}_X$ the sheaf of holomorphic functions on X.

Definition 34. A closed subset $Z \subseteq X$ is called *analytic* if Z is locally of the form

$$V(f_1, \dots, f_r) = \{x \mid f_1(x) = \dots = f_r(x) = 0\},$$

where $f_1, \dots, f_r \in \mathscr{O}_X$.

Given an analytic subset $Z \subseteq X$ we define a sheaf of ideals

$$\mathscr{I}_Z = \{f \in \mathscr{O}_X \mid f|_Z = 0\} \subseteq \mathscr{O}_X.$$

Note that $\mathscr{I}_Z$ is coherent, since locally $\mathscr{I}_Z = (f_1, \dots, f_r)$ (by definition).

Note also that $\mathscr{I}_Z$ is radical, i.e. if $f^N \in \mathscr{I}_Z$ then $f \in \mathscr{I}_Z$. I.e.

$$\mathscr{I}_Z = \sqrt{\mathscr{I}_Z} = \{f \mid f^N \in \mathscr{I}_Z \text{ for some } N > 0\}.$$

Theorem 11.1 (Analytic Nullstellensatz)**.**

$$\left\{ \begin{array}{c} \textit{coherent sheaves of} \\ \textit{radical ideals} \\ \mathscr{I} = \sqrt{\mathscr{I}} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \textit{closed analytic subsets} \\ Z \subseteq X \end{array} \right\}$$

$$\begin{aligned} \mathscr{I} &\longmapsto V(\mathscr{I}) \\ \mathscr{I}_Z &\longleftarrow\!\shortmid\; Z \end{aligned}$$

In fact, for any coherent ideal sheaf $\mathscr{I}$,

$$\mathscr{I}_{V(\mathscr{I})} = \sqrt{\mathscr{I}}.$$

The ring of germs of holomorphic functions at $x \in X$ is $\mathscr{O}_{X,x}$ – it is a noetherian local ring, with maximal ideal the functions vanishing at x. Prime ideals in $\mathscr{O}_{X,x}$ correspond to germs of irreducible analytic subsets of X near x.

Given an $\mathscr{O}_X$-module $\mathscr{F}$, define

$$\operatorname{supp}^\circ(\mathscr{F}) := \{x \in X \mid \mathscr{F}_x \neq 0\}.$$

and let $\operatorname{supp}(\mathscr{F})$ be its closure. There is a sheaf of ideal $\operatorname{Ann}(\mathscr{F}) \subseteq \mathscr{O}_X$,

$$\operatorname{Ann}(\mathscr{F}) := \{f \in \mathscr{O}_X \mid f \cdot s = 0 \quad \forall s \in \mathscr{F}\}.$$

Proposition 11.2. *If $\mathscr{F}$ is of finite type (locally finitely generated over $\mathscr{O}_X$), then*

$$supp(\mathscr{F}) = V(Ann(\mathscr{F})).$$

In particular, it is an analytic subset.

Proof. Suppose $\mathscr{F}_x = 0$, i.e. $x \notin \operatorname{supp}(\mathscr{F})$. Choose local generators $s_1, \ldots, s_r$ near x. Note that

$$\mathscr{F}_x = 0 \quad \text{implies} \quad (s_1)_x = \cdots (s_r)_x = 0.$$

So there exists some $U \ni x$ such that $\mathscr{F}|_U = 0$. So,

$$\operatorname{Ann}(\mathscr{F})|_U = \mathscr{O}_U, \text{ and so } V(\operatorname{Ann}(\mathscr{F})|_U) = \emptyset.$$

In particular, $x \notin V(\operatorname{Ann}(\mathscr{F}))$. Conversely, if $\mathscr{F}_x \neq 0$, take $0 \neq s_x \in \mathscr{F}_x$, $f \in \operatorname{Ann}(\mathscr{F})$. Then $f_x \cdot s_x = 0$ implies $f(x) = 0$ (since if $f(x) \neq 0$ then f is invertible on a neighbourhood of x, and f_x would be invertible, implying that $s_x = 0$ – a contradiction). $\square$

Example 70. Let $X = \Delta = \{|x| < 1\}$. Let $\mathscr{F} = \mathscr{O}_X/(x^k)$. Then $\operatorname{Ann}(\mathscr{F}) = (x^k)$, $\sqrt{\operatorname{Ann}(\mathscr{F})} = (x)$, and $\operatorname{supp}(\mathscr{F}) = V(x) = \{0\} \subset \Delta$.

Example 71. As above, but let

$$\mathscr{F}' = \bigoplus_{i=1}^{k} \mathscr{O}_X/(x).$$

Both examples are skyscraper sheaves supported at zero with k-dimensional stalks, but the are **not** the same sheaf.

Given an analytic set $i : Z \subseteq X$ we get a sheaf of rings

$$i^{-1}(\mathscr{O}_X/\mathscr{I}_Z),$$

where i^{-1} denotes the functor we previously had called i^* (from here on out we adopt this change in notation). Z is called a *complex analytic variety*, and decomposes as

$$Z = Z^{\text{reg}} \cup Z^{\text{sing}},$$

where the nonsingular points Z^{reg} form a complex manifold. Z is said to be *irreducible* if whenever we express Z as a union of analytic subsets

$$Z = Z_1 \cup Z_2,$$

we either have $Z = Z_1$ or $Z = Z_2$.

Fact: $Z = \cup_i Z_i$ where the Z_i are irreducible and the union is locally finite.

11.1. **Cycle of a coherent sheaf.** Suppose $\mathscr{F} \in \operatorname{Coh}(X)$, the category of coherent sheaves, and Z is an irreducible component of $\operatorname{supp}(\mathscr{F})$. We define the *multiplicity of $\mathscr{F}$ along Z*, $m(\mathscr{F}; Z) \in \mathbb{Z}_{>0}$ as follows.

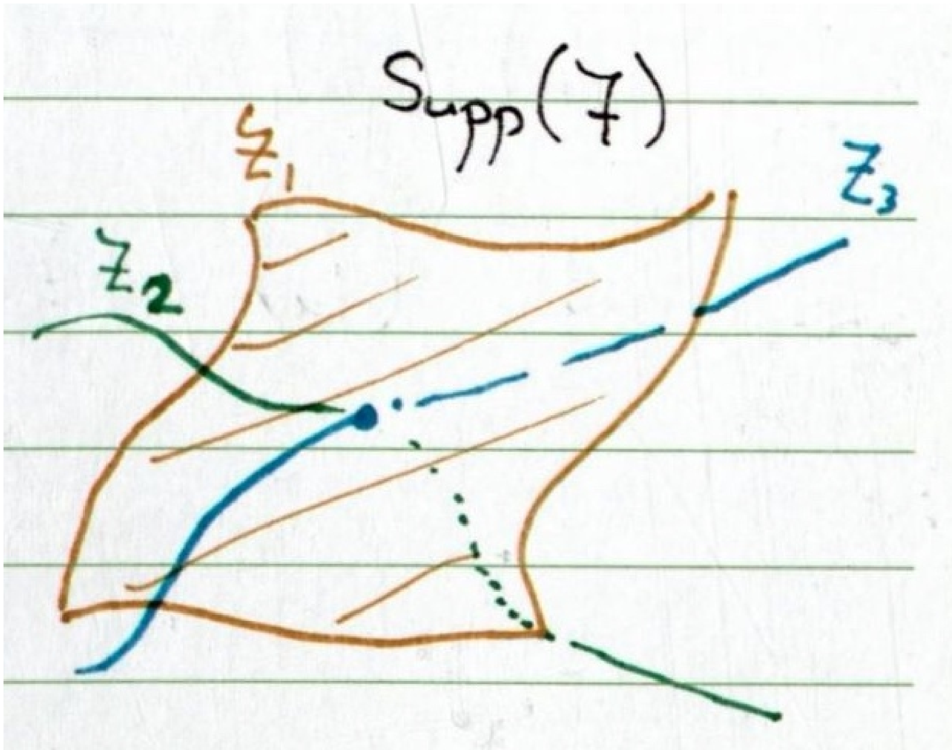


Figure 30. Decomposition of the support into irreducible components.

Pick $x \in Z^{\mathrm{reg}}$ (i.e. a smooth point). Then $\mathscr{F}_x$ is an $\mathscr{O}_{X,x}$-module, and $\mathfrak{p} = \mathscr{I}_{Z,x}$ is a prime ideal in $\mathscr{O}_{X,x}$ (prime because x is regular in Z). Localize at $\mathscr{I}_{Z,x}$. Then $\mathscr{F}_{x,\mathfrak{p}}$ is an $\mathscr{O}_{X,x,\mathfrak{p}}$-module (i.e. invert everything in $\mathscr{O}_{X,x} - \mathfrak{p}$). Recall that $\mathrm{length}_A(M)$ is the length of the longest chain of submodules

$$M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M.$$

Then we define

$$m(\mathscr{F}; Z) = \mathrm{length}_{\mathscr{O}_{X,x,\mathfrak{p}}}(\mathscr{F}_{x,\mathfrak{p}}).$$

(Claim that this is independent of the choice of x.)

More explicitly: $\mathscr{F}_x$ has a filtration

$$0 = (\mathscr{F}_x)_0 \subseteq (\mathscr{F}_x)_1 \subseteq \cdots \subseteq (\mathscr{F}_x)_k = \mathscr{F}_x,$$

such that

$$(\mathscr{F}_x)_i / (\mathscr{F}_x)_{i-1} \cong \mathscr{O}_X / \mathfrak{p}_i$$

for some prime ideals $\mathfrak{p}_i \subseteq \mathscr{O}_{X,x}$. Then

$$m(\mathscr{F}; Z) = \text{ number of times } \mathfrak{p} \text{ appears as a } \mathfrak{p}_i.$$

In the algebraic setting, we can take ξ_Z the generic point of Z. Then localize at ξ_Z, to obtain $\mathscr{F}_{\xi_Z}$ as an $\mathscr{O}_{X,\xi_Z}$-module. Then the multiplicity is the length of this module.

Definition 35. The *cycle* of $\mathscr{F}$ is

$$\mathrm{Cyc}(\mathscr{F}) = \sum_{Z \subset \mathrm{supp}(\mathscr{F})} m(\mathscr{F}; Z) \cdot [Z]$$

where the sum is locally finite, and is taken over the irreducible components of $\mathrm{supp}(\mathscr{F})$. We can also define

$$\mathrm{Cyc}_d(\mathscr{F}) = \sum_{\dim(Z) = d} m(\mathscr{F}; Z) \cdot [Z].$$

Finally, we define

$$m_d(\mathscr{F})_x := \sum_{Z \ni x,\, \dim(Z)=d} m(\mathscr{F}; Z).$$

Example 72. $X = \Delta$, $\mathscr{F} = \mathscr{O}_X/(x^k)$, $\mathrm{Ann}(\mathscr{F}) = (x^k)$. So $\mathrm{supp}(\mathscr{F}) = \{0\}$, but

$$\mathrm{Cyc}(\mathscr{F}) = k \cdot [0] = \underbrace{[0] + \cdots + [0]}_{k \text{ times}}.$$

I.e. $m(\mathscr{F}, \{0\}) = k$. A chain of length k is

$$0 \to \mathscr{O}_X/(x) \xrightarrow{\cdot x} \mathscr{O}_X/(x^2) \xrightarrow{\cdot x} \cdots \xrightarrow{\cdot x} \mathscr{O}_X/(x^k).$$

We have the same result for $\mathscr{F}' = \bigoplus_{i=1}^k \mathscr{O}_X/(x)$, i.e.

$$\mathrm{Cyc}(\mathscr{F}') = k \cdot [0].$$

So the cycle does **not** distinguish between these two different sheaves.

12. $\mathcal{D}$-modules: Singular support and filtrations.

Recall that $\mathcal{D}_X$ is a sheaf of rings on X, and we have a filtration

$$F_0\mathcal{D}_X \subseteq F_1\mathcal{D}_X \subseteq \cdots ;$$

previously we called $F_i\mathcal{D}_X = \mathcal{D}_X(i)$, the differential operators of order $\leq i$. Recall that

$$\mathrm{Gr}^F(\mathcal{D}_X) = \mathrm{Sym}^\bullet_{\mathscr{O}_X}(\mathscr{T}_X) \subseteq (\pi_X)_* \mathscr{O}_{T^*X}.$$

If $\mathscr{M}$ is a coherent $\mathcal{D}_X$-module, we want to define a subset, the *singular support*,

$$SS(\mathscr{M}) \subseteq T^*X,$$

a closed analytic conic subset (i.e. it is stable under the $\mathbb{C}^\times$ action on $T * X$ given by $t \cdot (x, \xi) = x, t\xi)$). We will also define the associated cycle of $SS(\mathscr{M})$ in T^*X, which we call $CC(\mathscr{M})$ (the *characteristic cycle*).

Assume we have a filtration $F_i\mathscr{M}$ $(F_k(\mathcal{D}_X) \cdot F_l(\mathscr{M}) \subseteq F_{k+l}(\mathscr{M}))$ such that

$$\mathrm{Gr}^F(\mathscr{M}) \text{ is coherent as a } \mathrm{Gr}^F(\mathcal{D}_X)\text{-module.}$$

Locally we can pick generators $u_1, \ldots, u_r$ of $\mathscr{M}$ and define

$$F_k\mathscr{M} := \sum_{i=1}^r F_k(\mathcal{D}_X) \cdot u_i.$$

$\mathrm{Gr}(\mathscr{M})$ is a coherent $\mathrm{Sym}_{\mathscr{O}}(\mathscr{T}_X)$-module, and we extend scalars

$$\mathrm{Gr}(\mathscr{M})^\sim := \mathscr{O}_{T^*X} \otimes_{\pi_X^{-1}\mathrm{Sym}(\mathscr{T}_X)} \pi_X^{-1}\mathrm{Gr}(\mathscr{M}) \in \mathrm{Coh}(T^*X).$$

Then

$$SS(\mathscr{M}) = \mathrm{supp}(\mathrm{Gr}(\mathscr{M})^\sim),$$
$$CC(\mathscr{M}) = \mathrm{Cyc}(\mathrm{Gr}(\mathscr{M})^\sim).$$

Recall: A (left) $\mathcal{D}_X$-module is equivalent to the data of an $\mathscr{O}_X$-module with a flat connection.

Let $\mathscr{M}$ be a $\mathcal{D}_X$-module. Say that $\mathscr{M}$ is *filtered* when

$$\cdots \subseteq F_i\mathscr{M} \subseteq F_{i+1}\mathscr{M} \subseteq \cdots$$

such that

(1) $F_i\mathscr{M} = 0$ for $i << 0$,
(2) $\bigcup_i F_i\mathscr{M} = \mathscr{M}$, and
(3) $F_i\mathcal{D}_X \cdot F_j\mathscr{M} \subseteq F_{i+j}\mathscr{M}$.

A filtration is called *good* if additionally

$$\mathrm{Gr}^F(\mathscr{M}) := \bigoplus_i (F_i\mathscr{M}/F_{i-1}\mathscr{M})$$

is a coherent $\mathrm{Gr}^F(\mathcal{D}_X)$-module.

Remark The good condition implies that $\mathscr{M}$ is coherent as a $\mathcal{D}_X$-module.

Remark If $\mathscr{M}$ is coherent (as a $\mathcal{D}_X$-module) then locally it is generated by some sections, $s_1, \dots, s_r$. I.e. there exists U such that

$$\mathscr{M}|_U = \mathscr{D}_U \cdot s_1 + \cdots + \mathcal{D}_U \cdot s_r.$$

Define

$$F_i\mathscr{M}|_U := F_i\mathcal{D}_U \cdot s_1 + \cdots + F_i\mathcal{D}_U \cdot s_r.$$

Claim: $(\mathscr{M}|_U, F)$ is a good filtration. I.e. good filtrations exist locally for a coherent $\mathcal{D}_X$-module.

Given $(\mathscr{M}, F)$ a good filtered $\mathcal{D}_X$-module,

$$\mathrm{Gr}^F(\mathscr{M}) \text{ is a coherent } \mathrm{Sym}^\bullet_{\mathscr{O}_X}(\mathscr{T}_X) = \mathrm{Gr}(\mathcal{D}_X)\text{-module.}$$

Aside: If V is a finite dimensional vector space over $\mathbb{C}$, the ring of polynomial functions on V is

$$\mathrm{Sym}^\bullet_{\mathbb{C}}(V^*) = \mathbb{C} \oplus V^* \oplus \mathrm{Sym}^2(V^*) \oplus \cdots$$

I.e. if $e_1, \dots, e_r$ is a basis for V with dual basis $\epsilon_1, \dots, \epsilon_r$,

$$\mathrm{Sym}^\bullet(V^*) = \mathbb{C}[\epsilon_1, \dots, \epsilon_r].$$

If M is a $\mathrm{Sym}^\bullet(V^*)$-module, then for $\mathrm{Ann}(M) \subseteq \mathrm{Sym}^\bullet(V^*)$

$$V(\mathrm{Ann}(M)) = \mathrm{supp}(M) \subseteq V.$$

V is also a $\mathbb{C}$-manifold, so we have $\mathscr{O}_V$. We can think of $\mathrm{Sym}^\bullet(V^*)$ as a constant sheaf of rings. Then we have

$$\mathscr{O}_V \otimes_{\mathrm{Sym}^\bullet(V^*)} M =: M^\sim,$$

which is a coherent sheaf. $\mathrm{supp}(M) = V(\mathrm{Ann}(M^\sim))$, i.e. we have the same notion of support.

If $M = \bigoplus_i M_i$ is a graded module over $\mathrm{Sym}^\bullet(V^*) = \bigoplus_i \mathrm{Sym}^i(V^*)$ (a graded ring), then $\mathrm{Ann}(M)$ is a homogeneous ideal. Then $\mathrm{supp}(M)$ is *conic*, i.e. it is preserved by the $\mathbb{C}^\times$ action on V. *End aside.*

Now, to make sense of $\mathrm{Sym}^\bullet_{\mathscr{O}_X}(\mathscr{T}_X)$, think of

$$V = T^*_xX, \quad V^* = T_xX \quad \text{for each point } x \in X.$$

So if $(\mathscr{M}, F)$ is a good coherent $\mathcal{D}_X$-module, we have that

$$\mathrm{Gr}^F(\mathscr{M}) \text{ is a coherent } \mathrm{Sym}^\bullet_{\mathscr{O}_X}(\mathscr{T}_X)\text{-module,}$$

and so we get

$$\mathrm{Gr}(\mathscr{M})^\sim = \mathscr{O}_{T^*X} \otimes_{\pi_X^{-1}\mathrm{Sym}(\mathscr{T}_X)} \pi_X^{-1}(\mathrm{Gr}^F(\mathscr{M})),$$

which is a coherent $\mathscr{O}_{T^*X}$-module, where $\pi_X : T^*X \to X$ is projection.

Definition 36. The *singular support* is

$$SS(\mathscr{M}, F) = \mathrm{supp}(\mathrm{Gr}^F(\mathscr{M})^\sim) = V(\mathrm{Ann}(\mathrm{Gr}^F(\mathscr{M})^\sim)) \subseteq T^*X.$$

$SS(\mathscr{M})$ is a conic subset of T^*X.

Proposition 12.1. *$SS(\mathscr{M}, F)$ is independent of the choice of good filtration F.*

We omit the proof.

Corollary 12.2. *If $\mathscr{M}$ is a coherent $\mathcal{D}_X$-module, $SS(\mathscr{M}) \subseteq T^*X$ is well-defined.*

Definition 37. The *characteristic ideal* of $\mathscr{M}$ is

$$J_{\mathscr{M}} := \sqrt{\operatorname{Ann}(\operatorname{Gr}(\mathscr{M})^{\sim})} \subseteq \operatorname{Sym}^{\bullet}(\mathscr{T}_X).$$

The *characteristic cycle* is

$$CC(\mathscr{M}) = \operatorname{Cyc}(\operatorname{Gr}(\mathscr{M})^{\sim}).$$

Example 73. Let $\mathscr{M} = \mathcal{D}_X$ and F be the order filtration.

$$\operatorname{Gr}^F(\mathcal{D}_X) = \operatorname{Sym}^{\bullet}(\mathscr{T}_X)$$

and

$$\operatorname{Gr}^F(\mathcal{D}_X)^{\sim} = \mathscr{O}_{T^*X},$$

so we have

$$SS(\mathcal{D}_X) = \operatorname{supp}(\mathscr{O}_{T^*X}) = T^*X.$$

Remark Think of $\mathcal{D}_X$ as a noncommutative deformation of $\operatorname{Sym}^{\bullet}(\mathscr{T}_X) \subseteq \mathscr{O}_{T^*X}$, i.e. that $\mathcal{D}_X$ is a *quantization* of T^*X.

Think of $\mathcal{D}_X$-modules as "living on T^*X". The first approximation to this idea is

$$\mathscr{M} \mapsto SS(\mathscr{M}) \subseteq T^*X.$$

Example 74. Given $P \in \mathcal{D}_X$, let $\mathscr{M}_P = \mathcal{D}_X/\mathcal{D}_X \cdot P$. (I.e. this roughly corresponds to solving the equation $Pu = 0$.) So P could be, for example, $x^2\partial_x^2 + \cdots$. Then the *principal symbol* of P is

$$\sigma(P) = \text{ image of } P \text{ in } \operatorname{Gr}_m(\mathcal{D}_X) \text{ where } m \text{ is maximal.}$$

Implicitly we are considering the ∂_* as coordinates on the cotangent bundle. For instance, $P = x^2\partial_x^2 + x\partial_x$ has

$$\sigma(P) = x^2\xi^2,$$

where (x, ξ) are coordinates on T^*X (here $\xi = \sigma(\partial_x)$).

Proposition 12.3. *$SS(\mathscr{M}) = V(\sigma(P)) \subseteq T^*X$.*

Proof. We have a SES

$$0 \to \mathcal{D}_X \xrightarrow{\cdot P} \mathcal{D}_X \xrightarrow{q} \mathscr{M}_P \to 0.$$

q induces a good filtration on $\mathscr{M}_P$ by taking the image of the filtration on $\mathcal{D}_X$. Then

$$\operatorname{Gr}(\mathscr{M}_P) = \operatorname{Sym}(\mathscr{T}_X)/(\sigma(P)).$$

□

Example 75. Let $X = \Delta = \{|x| < 1\} \subset \mathbb{C}$. Take $P = \partial_x$, so $\mathscr{M}_P = \mathscr{O}_X$. Then

$$SS(\mathscr{O}_X) = V(\xi) = X \subset T^*X,$$

where X includes into $T * X$ as the zero section.

Example 76. Now take $P = x$. Then

$$SS(\mathscr{M}_P) = V(x) = T_0^*X \subseteq T^*X.$$

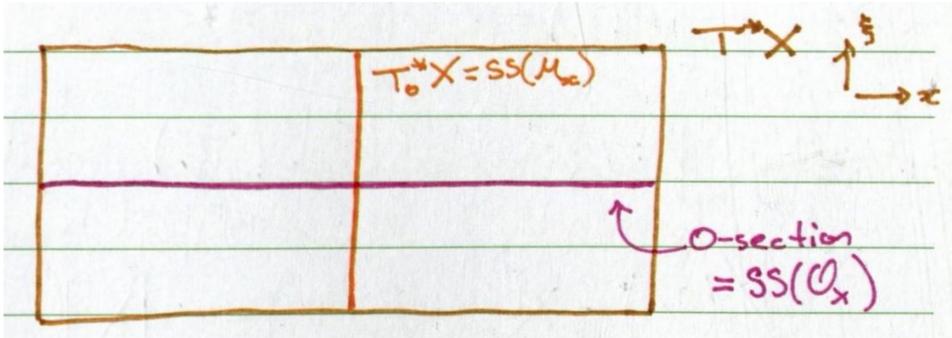


FIGURE 31. Singular supports of $\mathscr{O}_X$ and $\mathscr{M}_x$ in T^*X.

Example 77. Now take $P = x\partial_x$. Then $\sigma(P) = x\xi$, so $SS(\mathscr{M}_P) = V(x\xi) = X \cup T_0^*X$.

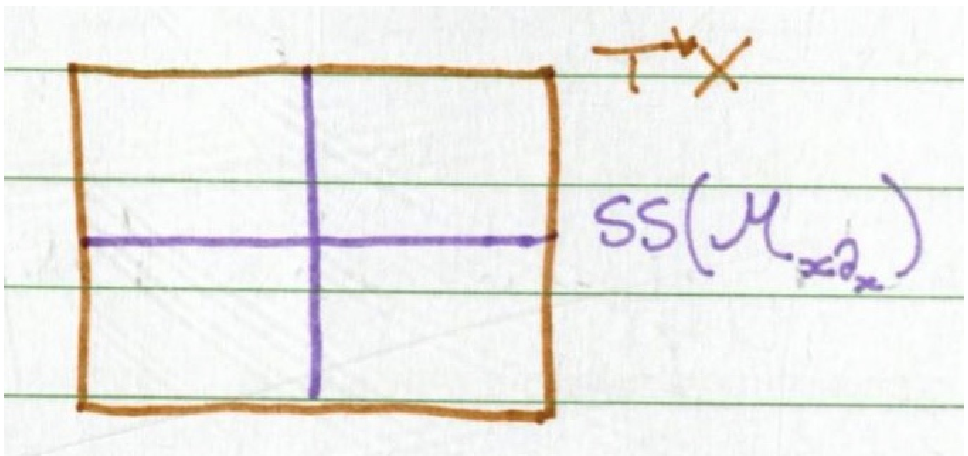


FIGURE 32. Singular support of the Euler operator $x\partial_x$.

In the above examples, the characteristic cycles are

$$CC(\mathscr{O}_X) = [X], \quad CC(\mathscr{M}_x) = [T_0^*X], \quad CC(\mathscr{M}_{x\partial_x}) = [X] + [T_0^*X].$$

Suppose $(\mathscr{V}, \Delta)$ is a flat connection, i.e. $\mathscr{V}$ is a locally free $\mathscr{O}_X$-module of finite rank.

Proposition 12.4. *$SS(\mathscr{V}) = X \subset T^*X$.*

Proof. The induced filtration is

$$0 = F_{-1}\mathscr{V} \subseteq F_0\mathscr{V} = \mathscr{V} \subseteq \mathscr{V} \subseteq \mathscr{V} \subseteq \cdots,$$

so $\mathrm{Gr}^F(\mathscr{V}) = \mathscr{V}$. Locally we can choose a horizontal frame and express $\mathscr{V} = \mathscr{O}_X^{\oplus r}$ as a $\mathrm{Sym}(\mathscr{T}_X)$-module, where $\mathscr{T}_X$ acts by 0. So, $\mathrm{Ann}(\mathscr{V}) = (\mathscr{T}_X)$, and

$$SS(\mathscr{V}) = X, \quad CC(\mathscr{V}) = r \cdot [X].$$

□

In fact, the following are equivalent for a coherent $\mathcal{D}_X$-module $\mathscr{M}$:

(1) $\mathscr{M}$ is a flat connection (i.e. locally free $\mathscr{O}_X$-module).
(2) $SS(\mathscr{M})$ is contained in the zero section of T^*X.

(3) $\mathscr{M}$ is coherent as an $\mathscr{O}_X$-module.

Lemma 12.5 (Bernstein's Lemma). *If $\mathscr{M}$ is a coherent $\mathcal{D}_X$-module,*

$$\dim(SS(\mathscr{M})) \geq \dim(X) = \frac{1}{2}\dim(T^*X).$$

We defer the proof to later.

Recall that if (W^{2n}, ω) is a symplectic vector space, $V \subseteq W$ a subspace, then V is called

- *isotropic* if $\omega|_V = 0$ (which implies that $\dim(V) \leq n$),
- *coisotropic* if $\omega|_{V^\perp} = 0$ (which implies that $\dim(V) \geq n$),
- *Lagrangian* if it is isotropic and coisotropic (in which case $\dim(V) = n$).

Theorem 12.6 (Gabber's Theorem). *$SS(\mathscr{M})$ is coisotropic in T^*X.*

Definition 38. A $\mathcal{D}_X$-module is called *holonomic* if $\dim(SS(\mathscr{M})) = \dim(X)$.

By Gabber's theorem, this is equivalent to $SS(\mathscr{M})$ being Lagrangian in T^*X.

13. Holonomic $\mathcal{D}$-modules.

We now consider a generalization of the theory of differential equations.

If $Pu = 0$, $P \in \mathcal{D}_{\mathbb{C}}$, we have that the space of solutions $u \in \mathscr{O}_{\mathbb{C}}$ is finite dimensional. If X is a complex manifold, recall that a (coherent) $\mathcal{D}_X$-module $\mathscr{M}$ is called *holonomic* if $SS(\mathscr{M}) \subseteq T^*X$ is Lagrangian (equivalently, $\dim SS(\mathscr{M}) = \dim(X)$).

Example 78. If $\dim X = 1$ and $\mathcal{D}_X = \mathcal{D}_X/\mathcal{D}_X \cdot P = \mathscr{M}_P$ for $P \in \mathcal{D}_X$, then

$$SS(\mathscr{M}_P) = V(\sigma(P)) \subseteq T^*X,$$

and $\dim SS(\mathscr{M}_P) = 1$ unless P is constant.

Theorem 13.1 (Kashiwara, PhD thesis). *Let $\mathscr{M}$ be a holonomic $\mathcal{D}_X$-module. Then we have two complexes of sheaves,*

$$\begin{aligned} Sol(\mathscr{M}) &:= R\mathscr{H}om_{\mathcal{D}_X}(\mathscr{M}, \mathscr{O}_X), && \text{“solutions of } \mathscr{M}\text{”},\\ DR(\mathscr{M}) &:= \mathscr{M} \xrightarrow{\nabla} \mathscr{M} \otimes_{\mathscr{O}_X} \Omega^1_X \to \cdots \to \mathscr{M} \otimes_{\mathscr{O}_X} \Omega^n_X, && \text{“de Rham complex”}. \end{aligned}$$

These are both constructible complexes with regards to some “nice” analytic stratification of S. In particular, the stalks of both complexes are finite dimensional.

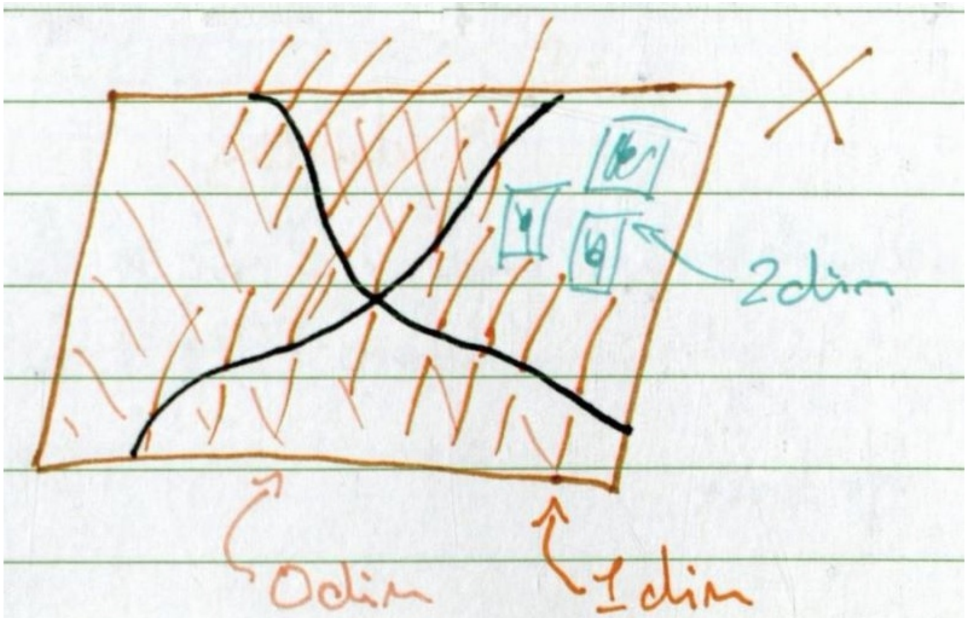


Figure 33. Picture of a constructible sheaf.

14. Functors for $\mathcal{D}$-modules.

Recall that for sheaves we had the "6-functor formalism": $*$- and !- adjoint pushforwards and pullbacks, $\mathscr{E}xt$ and $\otimes$. We would like similar adjoint functors for $\mathcal{D}$-modules – we will see, however, that these may not always exist, and even when they do exist they will only be adjoint for holonomic $\mathcal{D}$-modules.

Given a holomorphic map of complex manifolds, $f : X \to Y$, we will define functors between the bounded derived categories of $\mathcal{D}$-modules

$$\begin{tikzcd} D^b(\mathcal{D}_Y) \arrow[r, bend left, "f_*^{\mathrm{dR}}"] & D^b(\mathcal{D}_X) \arrow[l, bend left, "f^\dagger"] \end{tikzcd}.$$

Beware: In general these are **not** adjoint!

We recall and compare previous situations. Some notation has been changed from previous lectures to avoid overloading.

14.0.1. $\mathbb{C}_X$*-modules.*

$$\begin{tikzcd} \mathrm{Sh}(X;\mathbb{C}) \arrow[r, bend left, "f_\bullet"] & \mathrm{Sh}(Y;\mathbb{C}) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

f^{-1} is exact, and is left adjoint to $f_\bullet$ (which is therefore left exact). These give rise to derived functors f^{-1} and $Rf_\bullet$.

14.0.2. $\mathscr{O}$*-modules.*

$$\begin{tikzcd} \mathscr{O}_X\text{-mod} \arrow[r, bend left, "f_\bullet"] & \mathscr{O}_Y\text{-mod} \arrow[l, bend left, "f^*_{\mathscr{O}}"] \end{tikzcd}$$

where

$$f^*_{\mathscr{O}}(\mathscr{F}) := \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{F}).$$

$f^*_{\mathscr{O}}$ is left adjoint to $f_\bullet$, so is right exact. $\mathscr{O}_X$ is generally not $f^{-1}(\mathscr{O}_Y)$-flat, so $f^*_{\mathscr{O}}$ is not exact. Thus, we have a left derived functor $Lf^*_{\mathscr{O}}$. Similarly, $f_\bullet$ is left exact, so gives a right derived functor $Rf_\bullet$.

14.0.3. *Some examples.*

(1) If V is a vector bundle on Y,

$$\begin{tikzcd} f^{-1}(V) = V \times_Y X \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

and

$$\Gamma(-; f^{-1}(V)) = f^*_{\mathscr{O}}(\Gamma(-; V)).$$

(2) If $i : \{x\} \hookrightarrow X$, $\mathscr{F} \in \mathscr{O}_X$-mod, then $i^{-1}(\mathscr{F}) = \mathscr{F}_x$, the *stalk of* $\mathscr{F}$ *at* x. This is often an ∞-dimensional space of convergent power series. Conversely and comparatively,

$$i^*_{\mathscr{O}}(\mathscr{F}) = \left(\frac{\mathscr{O}_{X,x}}{\mathfrak{m}_x}\right) \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x = \mathscr{F}_x / \mathscr{F}_x \cdot \mathfrak{m}_x,$$

which we call the *fibre of* $\mathscr{F}$ *at* x. If $\mathscr{F}$ is a vector bundle, this is literally the fibre.

14.0.4. *$\mathcal{D}$-modules.* Define a functor

$$f^\circ : \mathcal{D}_Y\text{-mod} \to \mathcal{D}_X\text{-mod}$$

as follows. Let

$$f^\circ(\mathscr{M}) = f^*_{\mathscr{O}}(\mathscr{M})$$

as an $\mathscr{O}_X$ module. Then to define a $\mathcal{D}_X$-module structure, we need to say how $v \in \mathscr{T}_X$ acts on $m \in f^*_{\mathscr{O}}(\mathscr{M})$. Tangent vectors pushforward, $T_{X,x} \to T_{Y,f(x)}$, and these can be put together to give a map

$$\begin{aligned} \mathscr{T}_X &\to f^*_{\mathscr{O}}(\mathscr{T}_Y) \\ v &\mapsto \tilde{v} \end{aligned}$$

$f^*_{\mathscr{O}}(\mathscr{M}) = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{M})$. Given $g \otimes m$, $g \in \mathscr{O}_X$ and $m \in f^{-1}\mathscr{M}$,

$$v \cdot (g \otimes m) = v(g) \otimes m + g \otimes \tilde{v}(m).$$

Now define (since f° is right exact) a left derived functor

$$f^\dagger := Lf^\circ.$$

14.1. **Transfer bimodule.** Define

$$\mathcal{D}_{X\to Y} := f^\circ(\mathcal{D}_Y).$$

I.e.

$$\mathcal{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathcal{D}_Y),$$

so this carries a *left* $\mathcal{D}_X$-module action, and a *right* $f^{-1}(\mathcal{D}_Y)$-module action. Then we have functors (of *left* modules)

$$\begin{tikzcd} \mathcal{D}_Y\text{-mod} \arrow[r, "f^{-1}"'] \arrow[rr, bend left, "f^\circ"] & f^{-1}(\mathcal{D}_Y)\text{-mod} \arrow[r, "\mathcal{D}_{X\to Y}\otimes_{f^{-1}(\mathcal{D}_Y)}(-)"'] & \mathcal{D}_X\text{-mod}. \end{tikzcd}$$

We call $\mathcal{D}_{X\to Y}$ the *transfer bimodule.* Then

$$f^\dagger = \mathcal{D}_{X\to Y} \otimes^L_{f^{-1}(\mathcal{D}_Y)} (f^{-1}(-)),$$

the left derived tensor product. We also have a functor in the opposite direction, but for *right* modules:

$$\mathcal{D}_X^{\mathrm{op}}\text{-mod} \xrightarrow{(-)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}} f^{-1}(\mathcal{D}_Y)^{\mathrm{op}}\text{-mod} \xrightarrow{f_\bullet} \mathcal{D}_Y^{\mathrm{op}}\text{-mod}.$$

Recall that there is a nontrivial equivalence of categories between $\mathcal{D}_X$-mod and $\mathcal{D}_X^{\mathrm{op}}$-mod (and that this is a special property of $\mathcal{D}$-modules, **not** something that holds for arbitrary noncommutative rings). To get a functor on *left* $\mathcal{D}$-modules we take

$$\begin{tikzcd}
\mathcal{D}_X^{\mathrm{op}}\text{-mod} \arrow[r, "(-)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}"] & f^{-1}(\mathcal{D}_Y)^{\mathrm{op}}\text{-mod} \arrow[r, "f_\bullet"] & \mathcal{D}_Y^{\mathrm{op}}\text{-mod} \\
\mathcal{D}_X \otimes -mod \arrow[u, leftrightarrow, "\Omega_X^n\otimes_{\mathscr{O}_X}(-)", "\sim"'] \arrow[rr, "f_\circ"] & & \mathcal{D}_Y\text{-mod} \arrow[u, leftrightarrow, "\sim", "\Omega_Y^n\otimes_{\mathscr{O}_Y}(-)"']
\end{tikzcd}$$

Define

$$f_*^{\mathrm{dR}} : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$$
$$f_*^{\mathrm{dR}}(\mathscr{M}) := Rf_\bullet(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X}^{L} \mathscr{M})$$

What is $\mathcal{D}_{Y\leftarrow X}$? There is an equivalence of categories

$$\mathcal{D}_X - f^{-1}(\mathcal{D}_Y)\text{-bimodules} \simeq \mathcal{D}_X^{\mathrm{op}} - f^{-1}(\mathcal{D}_Y)^{\mathrm{op}}\text{-bimodules} = f^{-1}(\mathcal{D}_Y) - \mathcal{D}_X\text{-bimodules}.$$

Then $\mathcal{D}_{X\to Y} \leftrightarrow \mathcal{D}_{Y\leftarrow X}$ under this equivalence.

14.2. **Closed embeddings.** Let $i : Z \hookrightarrow X$ be a closed embedding (recall that this implies $i_\bullet$ is exact). Consider

$$i^\circ : \mathcal{D}_X\text{-mod} \to \mathcal{D}_Z\text{-mod}$$
$$i^\circ(\mathscr{M}) = \mathcal{D}_{Z\to X} \otimes_{i^{-1}(\mathcal{D}_X)} i^{-1}(\mathscr{M}),$$

and

$$i^! : \mathcal{D}_X\text{-mod} \to \mathcal{D}_Z\text{-mod}$$
$$i^!(\mathscr{M}) = i^{-1}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Z}, \mathscr{M})$$

Lemma 14.1 (Kashiwara's Lemma)**.** *There is a pair of adjoint functors* $(i_\circ, i^!)$

$$\begin{tikzcd}
\mathcal{D}_Z\text{-}mod \arrow[r, bend left, "i_\circ"] & \mathcal{D}_X\text{-}mod \arrow[l, bend left, "i^!"]
\end{tikzcd}$$

which induce an equivalence of categories between

$$\begin{tikzcd}
\mathcal{D}_Z\text{-}mod \arrow[r, bend left, "i_\circ"] & (\mathcal{D}_X\text{-}mod)_Z \arrow[l, bend left, "i^!"]
\end{tikzcd}$$

where the right hand side is the category of $\mathcal{D}_X$-modules supported on Z*.*

Remark This is **not** true for $\mathscr{O}$-modules.

Example 79. Consider the inclusion of a point $\{x\} \hookrightarrow X = \mathbb{C}$. Then $\mathscr{O}_{\{x\}}$-mod = Vect, while $(\mathscr{O}_X\text{-mod})_{\{x\}}$ contains nontrivial extensions, such as

$$0 \to \mathscr{O}_{X,x}/\mathfrak{m}_x \to \mathscr{O}_{X,x}/\mathfrak{m}_x^2 \to \mathscr{O}_{X,x}/\mathfrak{m}_x \to 0.$$

So $(\mathscr{O}_X\text{-mod})_{\{x\}}$ is not semisimple, thus not equivalent to Vect.

Lemma 14.2 (Kashiwara's Lemma part II)**.** $(i_\circ, i^!)$ *also gives an equivalence on the level of derived categories.*

14.3. **Where do the transfer bimodules come from?** $\mathcal{D}_X$-modules "live on T^*X", as a deformed version of quasicoherent sheaves. Given a map $f : X \to Y$, we don't get a map between cotangent bundles. Instead we get a *correspondence* or *span*

$$T^*X \xleftarrow{\rho_f} T^*Y \times_Y X \xrightarrow{\varpi_f} T^*Y.$$

In fact T^*X and T^*Y are symplectic, and thinking of the above as a map into the product $T^*X \times T^*Y$, the middle object is in fact Lagrangian. We then have

$$\begin{tikzcd} T^*X \arrow[d, rightsquigarrow] & T^*Y \times_Y X \arrow[l, "\rho_f"'] \arrow[r, "\varpi_f"] \arrow[d, rightsquigarrow, "\text{quantization}"] & T^*Y \arrow[d, rightsquigarrow] \\ \mathcal{D}_X & \mathcal{D}_{X\to Y} \text{ or } \mathcal{D}_{Y\to X} & \mathcal{D}_Y \end{tikzcd}$$

We can understand the reverse map, e.g. $\mathcal{D}_X \to T^*X$; $\mathcal{D}_X$ has a filtration as a $\mathcal{D}_X$-module, and the associated graded is functions on T^*X.

14.3.1. *Case 1: $f = i : X \hookrightarrow Y$ a closed embedding.*

Example 80. See Figure 34.

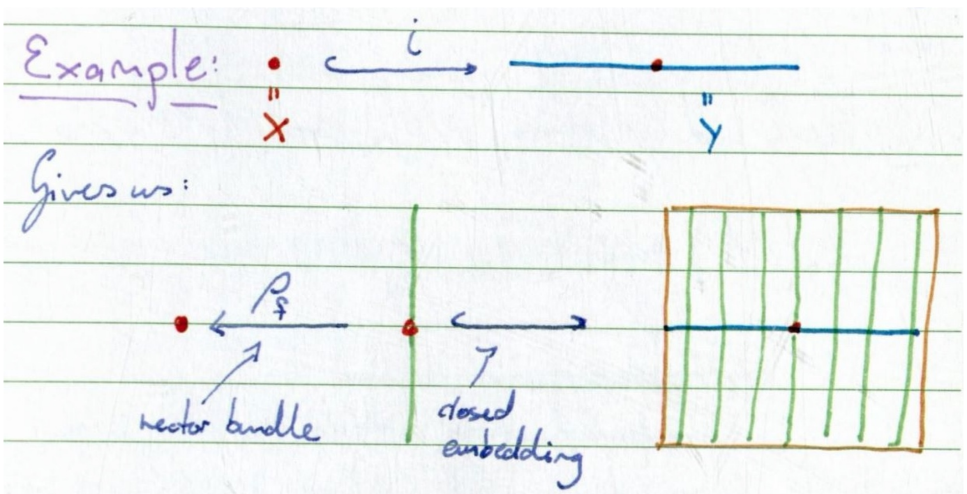


FIGURE 34. The span of a closed embedding.

In more generality, X is defined by a sheaf of ideals $\mathscr{I}_X \subseteq \mathscr{O}_Y$. Then

$$\mathscr{O}_X = i^{-1}\left(\mathscr{O}_Y/\mathscr{I}_X\right)$$
$$\mathcal{D}_{X\to Y} = \mathscr{O}_X \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}\mathcal{D}_Y = i^{-1}\left(\mathcal{D}_Y/\mathscr{I}_X \cdot \mathcal{D}_Y\right)$$

Exercise 14.1. Show that $\mathcal{D}_{Y\leftarrow X} = i^{-1}(\mathcal{D}_Y/\mathcal{D}_Y \cdot \mathscr{I}_X)$.

Definition 39. We define two underived functors:

$$i^\circ(\mathscr{N}) = \mathcal{D}_{X\to Y} \otimes_{i^{-1}\mathcal{D}_Y} i^{-1}\mathscr{N} = i^{-1}(\mathscr{N}/\mathscr{I}_X \cdot \mathscr{N}),$$
$$i^\natural(\mathscr{N}) = \mathscr{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, i^{-1}\mathscr{N}) \in \mathcal{D}_X\text{-mod}.$$

Remark Here and following, $\mathscr{N} \in \mathcal{D}_Y$-mod and $\mathscr{M} \in \mathcal{D}_X$-mod.

Observe that

$$i_\circ(\mathscr{M}) = i_\bullet(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathscr{M});$$

there is no need to derive, since pushforward along a closed embedding is exact.

Remark
- i° uses the forward facing bimodule $\mathcal{D}_{X\to Y}$.
- $i^\natural$ and $i_\circ$ use the backwards facing bimodule $\mathcal{D}_{Y\leftarrow X}$.

Proposition 14.3. *(1) $i_\circ$ is exact ($\mathcal{D}_{Y\leftarrow X}$ is flat over $\mathcal{D}_X$), i° is right exact, and $i^\natural$ is left exact.*
(2) $i^\dagger = Li^\circ = Ri^\natural[d]$ where $d = codim(X \hookrightarrow Y)$.
(3) $i^\natural$ is right adjoint to $i_\circ$, and moreover $Ri^\natural$ is right adjoint to $Li_\circ = i_*^{dR}$.
(4) $i_\circ$ induces an equivalence of categories

$$\begin{tikzcd} \mathcal{D}_X\text{-}mod \arrow[r, bend left, "i_\circ"] & (\mathcal{D}_Y\text{-}mod)_X \arrow[l, bend left, "i^\natural"] \end{tikzcd}$$

Again, there is a derived statement:

$$\begin{tikzcd} D^b(\mathcal{D}_X\text{-mod}) \arrow[r, bend left, "Li_\circ = i_*^{\mathrm{dR}}"] & D^b[(\mathcal{D}_Y\text{-mod})_X] \arrow[l, bend left, "Ri^\natural"] \end{tikzcd}$$

Example 81. Let's look at a special case:

$$X = \{z = 0\} \subseteq Y \quad \text{a hypersurface.}$$

We have local coordinates $y_1, \dots, y_n = z$ on Y (so the last coordinate cuts out the hypersurface X). In this case the transfer bimodule is

$$\mathcal{D}_{X\to Y} = i^{-1}(\mathcal{D}_Y / z \cdot \mathcal{D}_Y).$$

This is resolved by

$$i^{-1}(\mathcal{D}_Y / z \cdot \mathcal{D}_Y) \simeq (\underset{-1}{i^{-1}\mathcal{D}_Y} \xrightarrow{\cdot z} \underset{0}{i^{-1}\mathcal{D}_Y})$$

and similarly

$$\mathcal{D}_{Y\leftarrow X} \simeq (\underset{-1}{i^{-1}\mathcal{D}_Y} \xrightarrow{\cdot z} \underset{0}{i^{-1}\mathcal{D}_Y}).$$

So we have an explicit description

$$i^\natural(\mathscr{N}) = \mathscr{H}om_{i^{-1}\mathcal{D}_Y}(i^{-1}\mathcal{D}_Y / i^{-1}\mathcal{D}_Y \cdot z), \mathscr{N}) = \ker(z : \mathscr{N} \to \mathscr{N}).$$

Now, as a right $\mathcal{D}_X$-module,

$$\mathcal{D}_{Y\leftarrow X} = \mathbb{C}[\partial_z] \otimes_{\mathbb{C}} \mathcal{D}_X.$$

The difference between this and $\mathcal{D}_Y$ is that we don't have functions of the coordinate z (which vanishes on X) – however, we still have vector fields in the ∂_z-direction. Note that $\mathcal{D}_{Y\leftarrow X}$ is free as a right $\mathcal{D}_X$-module.

Thus, the functor $i_\circ$ is given by

$$i_\circ(\mathscr{M}) = i_\bullet(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathscr{M}) = \mathbb{C}[\partial_z] \otimes_{\mathbb{C}} \mathscr{M}.$$

Think: $i_\circ$ is "fattening $\mathscr{M}$ up" along the line $\langle \partial_z \rangle$.

What is the $\mathcal{D}_Y$-module structure on this?

$$i_\circ\mathscr{M} = \begin{tikzcd} \cdots \arrow[r, bend left, "\cdot z"] \arrow[r, phantom, "\oplus"] & \partial_z^2\mathscr{M} \arrow[l, bend left, "\partial_z"] \arrow[r, bend left, "\cdot z"] \arrow[r, phantom, "\oplus"] & \partial_z\mathscr{M} \arrow[l, bend left, "\partial_z"] \arrow[r, bend left, "\cdot z"] \arrow[r, phantom, "\oplus"] & \mathscr{M} \arrow[l, bend left, "\partial_z"] \arrow[r, "\cdot z"] & 0. \end{tikzcd}$$

Claim that since $z\mathscr{M} = 0$, the action of the other $\cdot z$ is determined by commutations with the ∂_z. Define

$$E := z\partial_z \quad \text{the } \textit{Euler operator}.$$

Claim that $\partial_z^i\mathscr{M}$ is the $(-i-1)^{\text{th}}$-eigenspace of E.

$$E = z\partial_z = \partial_z z - 1,$$

and $z\mathscr{M} = 0$. So E acts on $\mathscr{M}$ as multiplication by -1. Then we can check that

- multiplication by ∂_z lowers the eigenvalue by 1, and
- multiplication by z increases the eigenvalue by 1.

Claim: $i^\natural i_\circ \mathscr{M} \cong \mathscr{M}$.

Proof. The z and ∂_z operations are isomorphisms on $i_\circ\mathscr{M}$, *except* for in the last place where $z\mathscr{M} = 0$. So

$$i^\natural i_\circ \mathscr{M} = \ker(z : i_\circ\mathscr{M} \to i_\circ\mathscr{M}) = \mathscr{M}.$$

$\square$

Remark $i^\natural i_\circ \mathscr{M} \leftarrow \mathscr{M}$ will be the unit for an adjunction.

Now suppose $\mathscr{N} \in \mathcal{D}_Y$-mod. Look at

$$\mathscr{N}^i := \{i^{\text{th}} \text{ eigenspace of } E \text{ acting on } \mathscr{N}\}.$$

- We have no guarantee that i is integral.
- We have no idea whether this is bounded at either end:

$$\begin{tikzcd} \cdots \arrow[r, bend left, "\cdot z"] \arrow[r, phantom, "\oplus"] & \mathscr{N}^{i-2} \arrow[l, bend left, "\partial_z"] \arrow[r, bend left, "\cdot z"] \arrow[r, phantom, "\oplus"] & \mathscr{N}^{i-1} \arrow[l, bend left, "\partial_z"] \arrow[r, bend left, "\cdot z"] \arrow[r, phantom, "\oplus"] & \mathscr{N}^{i} \arrow[l, bend left, "\partial_z"] \arrow[r, bend left, "\cdot z"] \arrow[r, phantom, "\oplus"] & \mathscr{N}^{i+1} \arrow[l, bend left, "\partial_z"] \arrow[r, bend left, "\cdot z"] \arrow[r, phantom, "\oplus"] & \cdots \arrow[l, bend left, "\partial_z"] \end{tikzcd}$$

Note:

- This might not be all of $\mathscr{N}$ (e.g. $\mathscr{N}^{i+\frac{1}{2}}$ could exist).
- $\ker(z) \subseteq \mathscr{N}^{-1}$, so we see that the counit for the adjunction is

$$\cdots \oplus \partial_z \ker(z) \oplus \ker(z) = i_\circ i^\natural \mathscr{N} \to \mathscr{N}.$$

Now assume $\operatorname{supp}(\mathscr{N}) \subseteq X$. I.e.

$$z^N \cdot m = 0 \text{ for all } m \in \mathscr{N},\ N >> 0.$$

If $z \cdot m = 0$ then $m \in \mathscr{N}^{-1}$. By an induction argument, if $z^N \cdot m = 0$ then

$$m \in \mathscr{N}^{-N} \oplus \cdots \oplus \mathscr{N}^{-1}.$$

Exercise 14.2. In this case, $i_\circ i^\natural \mathscr{N} \xrightarrow{\sim} \mathscr{N}$.

Hence in this case we have shown an equivalence

$$\begin{tikzcd} \{\mathcal{D}\text{-modules on } X\} \arrow[r, leftrightarrow, "\sim"] \arrow[d, equal] & \{\mathcal{D}\text{-modules on } Y \text{ supported on } X\} \arrow[d, equal] \\ \mathcal{D}_X\text{-mod} \arrow[r, bend right, "i_\circ"'] & (\mathcal{D}_Y\text{-mod})_X \arrow[l, bend right, "i^\natural"'] \end{tikzcd}$$

Remark If $X \hookrightarrow Y$ is a closed embedding but *not* a hypersurface, it is cut out by a collection of functions (locally coordinate functions), $X = \{z_1 = \cdots = z_r = 0\}$. Let

$$K(z_i) = (i^{-1}(\mathscr{O}_Y \xrightarrow{\cdot z_i} i^{-1}(\mathscr{O}_Y)).$$

Then we define the *Koszul complex* of $X \subseteq Y$ to be

$$K(z_1) \otimes \cdots \otimes K(z_r).$$

14.4. **A concrete calculation.** Let's compute

$$p_*^{\mathrm{dR}} : D^b(\mathcal{D}_X) \to D^b(\mathbb{C}),$$

where $p : X \to \mathrm{pt}$. We have the transfer bimodules

$$\mathcal{D}_{X\to\mathrm{pt}} = \mathscr{O}_X \quad \text{left } \mathcal{D}\text{-module, right module for constant sheaf)}$$

and

$$\mathcal{D}_{\mathrm{pt}\leftarrow X} = \mathscr{O}_X \otimes \Omega_X^{\mathrm{top}} = \Omega_X^{\mathrm{top}} = \Omega_X^n,$$

where $n = \dim(X)$. So

$$p_*^{\mathrm{dR}}(\mathscr{M}) = \Omega_X^n \otimes_{\mathcal{D}_X}^L \mathscr{M}.$$

How do we compute $\otimes^L$? We either need to replace Ω_X^n of $\mathscr{M}$. To do this uniformly for all $\mathscr{M}$ we will replace Ω_X^n by a complex of locally free $\mathcal{D}_X^{\mathrm{op}}$-modules (right $\mathcal{D}_X$-modules).

We do this by taking a left module resolution of $\mathscr{O}_X$, then we will use the dualizing sheaf to produce the desired resolution.

Lemma 14.4. *There are (locally free) resolutions*

$$\mathcal{D}_X \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{T}_X \to \cdots \to \mathcal{D}_X \otimes_{\mathscr{O}_X} \bigwedge^2 \mathscr{T}_X \longrightarrow \mathcal{D}_X \otimes_{\mathscr{O}_X} \mathscr{T}_X \longrightarrow \mathcal{D}_X \to \mathscr{O}_X = \mathcal{D}_X/\mathcal{D}_X \cdot \mathscr{T}_X$$

$$P \otimes v \longmapsto P \cdot v$$

$$P \otimes (v_1 \wedge v_2) \longmapsto Pv_1 \otimes v_2 - Pv_2 \otimes v_1$$
$$-P \otimes [v_1, v_2]$$

and

$$\underbrace{\mathcal{D}_X = \Omega_X^0 \otimes_{\mathscr{O}_X} \mathcal{D}_X \to \cdots \to \Omega_X^{n-1} \otimes_{\mathscr{O}_X} \mathcal{D}_X \to \Omega_X^n \otimes_{\mathscr{O}_X} \mathcal{D}_X DR(\mathcal{D}_X)}_{\text{the de Rham complex of } \mathcal{D}_X \text{ though of as a } \mathcal{D}_X\text{-module.}} \to \Omega_X^n$$

The proof of these claims reduces to the commutative case. I.e. equip everything with compatible good filtrations. Then take associated graded – you will see this is

$$i_*\mathscr{O}_X \in \mathscr{O}_{T^*X}\text{-mod},$$

where $i : X \hookrightarrow T^*X$ is the zero section.

Why is this enough? There is an algebraic lemma to prove: if you have a good filtered complex of $\mathcal{D}$-modules, it is exact iff its associated graded is exact. This is a common technique – reduce a non-commutative problem to a problem in the associated graded.

So, we now have a quasi-isomorphism (so equality in the derived category)

$$\Omega_X^n \otimes_{\mathcal{D}_X}^L \mathscr{M} \simeq (\Omega_X^\bullet \otimes \mathscr{O}_X \mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathscr{M} = \Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M} = \mathrm{DR}(\mathscr{M}).$$

Caution: We have changed out grading convention. Now, $\mathrm{DR}(\mathscr{M})$ is concentrated in degrees $-n, -n+1, \ldots, 0$.

Now, to compute $p_*^{\mathrm{dR}}(\mathscr{M})$ we take the pushforward, i.e.

$$p_*^{\mathrm{dR}}(\mathscr{M}) = R\Gamma(\mathrm{DR}(\mathscr{M})).$$

Think: Analogue/generalization of taking the de Rham cohomology.

Example 82. Let $\mathscr{M} = \mathscr{O}_X$. Then

$$\mathrm{DR}(\mathscr{M}) = \underset{-n}{\mathscr{O}_X} \to \underset{-n+1}{\Omega_X^1} \to \underset{-n+2}{\Omega_X^2} \to \cdots \to \underset{0}{\Omega_X^n}.$$

Without the grading shift, this is a resolution of the constant sheaf, i.e.

$$\mathrm{DR}(\mathscr{O}_X) \simeq \mathbb{C}_X[n].$$

Then

$$p_*^{\mathrm{dR}}(\mathscr{O}_X) = R\Gamma(X;\mathbb{C}_X[n]) = \underbrace{H^{*-n}(X;\mathbb{C})}_{\text{in degrees } -n \text{ to } n}$$

Remark This will make Poincaré duality look symmetric around 0.

Remark Could play the same game with $\mathscr{O}_X$ replaced by a local system.

More generally, suppose $f : X \to Y$ is smooth (i.e. is a submersion – this is smoothness in the algebraic geometric sense). We define $\Omega^1_{X/Y}$ as

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0.$$

Locally (by the implicit function theorem) $X \cong Y \times Z$, so $\Omega^1_{X/Y}$ has elements of the form $f(y,z)dz$. Then we can form

$$\Omega^k_{X/Y} = \textstyle\bigwedge^k_{\mathscr{O}_X} \Omega^1_{X/Y},$$

which carries a de Rham differential $d_{X/Y}$ (apparent from the local form of the elements).

Proposition 14.5.

$$f_*^{dR}(\mathscr{M}) = Rf_\bullet(DR_{X/Y}(\mathscr{M})).$$

Remark For any $f : X \to Y$ we can factor as

$$X \xhookrightarrow{\Gamma_f} X \times Y \xrightarrow{p_2} Y,$$

a closed embedding (via the graph of f) followed by a submersion.

Remark We say a submersion is smooth because the fibres have the structure of a smooth manifold – describing a map by the properties of its fibres is a general principle in algebraic geometry.

Let $f : X \to Y$ be smooth. Then we have

$$\begin{tikzcd} & T^*Y \times_Y X \arrow[dl, "\rho_f"'] \arrow[dr, "\varpi_f"] & \\ T * X & & T^*Y \end{tikzcd}$$

Proposition 14.6. *(1) $f^\circ : \mathcal{D}_Y$-mod $\to \mathcal{D}_X$-mod is exact (so $f^\dagger \simeq f^\circ$).*

(2) f° preserves coherent $\mathcal{D}$-modules.

(3) $SS(f^\circ\mathscr{M}) = \rho_f \varpi_f^{-1}(SS(\mathscr{M}))$.

(4) $f^\dagger[-d]$ is left adjoint to f_*^{dR}, where d is the relative dimension of f.

Example 83. Consider $X \to \mathrm{pt}$. The singular support must lie in the zero section, as shown in Figure 35.

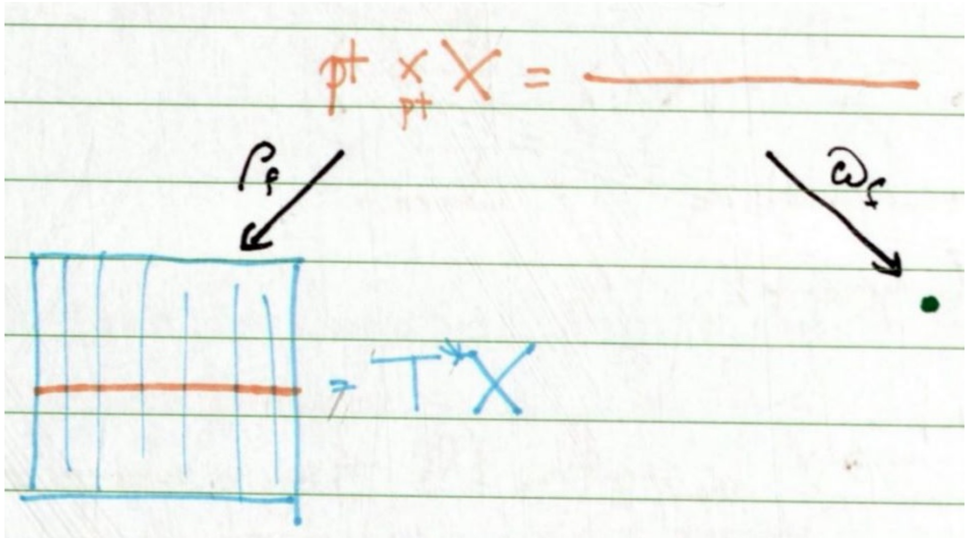


FIGURE 35. Correspondence for $X \to \mathrm{pt}$. $\mathrm{pt} \times_{\mathrm{pt}} X \cong X$, so by the proposition $SS(p^\circ \mathscr{M}) \subseteq X \subset T^*X$.

In this case, observe that ρ_f is a closed embedding and ϖ_f is smooth.

Proof of part 1. When f is smooth it is in particular flat, so $\mathscr{O}_X$ is flat as an $f^{-1}\mathscr{O}_Y$-module. Thus f° is exact. ☐

We defer the rest of the proof to later.

15. (Verdier) Duality for $\mathcal{D}$-modules.

Let $\mathscr{M} \in \mathcal{D}_X$-mod. We want to define a dual to $\mathscr{M}$, so our first obvious guess is

$$R\mathscr{H}om_{\mathcal{D}_X}(\mathscr{M}, \mathcal{D}_X).$$

This is a *right* $\mathcal{D}_X$-module; i.e. a $\mathcal{D}_X^{\mathrm{op}}$-module. To get a left $\mathcal{D}_X$-module, tensor with the inverse of the dualizing sheaf,

$$R\mathscr{H}om_{\mathcal{D}_X}(\mathscr{M}, \mathcal{D}_X) \otimes_{\mathscr{O}_X} (\Omega_X^n)^\vee \in \mathcal{D}_X\text{-mod}.$$

Define

$$\mathbb{D}_X(\mathscr{M}) := (R\mathscr{H}om_{\mathcal{D}_X}(\mathscr{M}, \mathcal{D}_X) \otimes_{\mathscr{O}_X} (\Omega_X^n)^\vee)\,[\dim X].$$

Example 84. Let $X = \mathbb{C}$, $\mathscr{M}_P = \mathcal{D}_X / \mathcal{D}_X \cdot P$, $P \in \mathcal{D}_X$. Then there is a quasi-isomorphism

$$\mathscr{M}_P \simeq \underset{-1}{\mathcal{D}_X} \xrightarrow{\cdot P} \underset{0}{\mathcal{D}_X}.$$

So

$$R\mathscr{H}om_{\mathcal{D}_X}(\mathscr{M}_P, \mathcal{D}_X) = \mathscr{H}om_{\mathcal{D}_X}(\mathcal{D}_X \xrightarrow{\cdot P} \mathcal{D}_X, \mathcal{D}_X)$$

$$\underset{0}{\mathcal{D}_X} \xrightarrow{P\cdot} \underset{1}{\mathcal{D}_X}$$

Definition 40. For $\mathscr{M} \in \mathcal{D}_X$-mod and $\mathscr{N} \in \mathcal{D}_X^{\mathrm{op}}$-mod, define

$$\mathscr{M}^r = \mathscr{M} \otimes_{\mathscr{O}_X} \Omega_X^n \in \mathcal{D}_X^{\mathrm{op}}\text{-mod}$$
$$\mathscr{N}^l = (\Omega_X^n)^\vee \otimes_{\mathscr{O}_X} \mathscr{N} \in \mathcal{D}_X\text{-mod}$$

Since $\Omega_X^n \cong \mathscr{O}_X$ for $X = \mathbb{C}$, we therefore have

$$\mathbb{D}_X(\mathscr{M}_P) = \mathcal{D}_X/\mathcal{D}_X \cdot P^*,$$

in degree 0 due to the degree shift in the definition. Here P^* is the adjoint of P – recall that there is a nontrivial isomorphism

$$\begin{tikzcd}
\mathcal{D}_X \arrow[r, "\sim"] & \mathcal{D}_X^{\mathrm{op}} \\
x \arrow[r, mapsto] & x \\
\partial_x \arrow[r, mapsto] & -\partial_x \\
x\partial_x \arrow[r, mapsto] & -\partial_x \cdot x
\end{tikzcd}$$

and P^* is the image of P under this map.

Proposition 15.1. *Let $D^b_{coh}(\mathcal{D}_X)$ be the derived category of complexes of $\mathcal{D}_X$-modules with coherent cohomology. Then*

- $\mathbb{D}_X : D^b_{coh}(\mathcal{D}_X) \xrightarrow{\sim} D^b_{coh}(\mathcal{D}_X)^{op}$.
- $\mathbb{D}_X \circ \mathbb{D}_X \simeq id_X$.
- *$\mathscr{M}$ is holonomic (in degree 0) if and only if $\mathbb{D}_X(\mathscr{M})$ is in degree 0.*

15.1. **Big Picture.** We started the semester with sheaves on a space X:

$$\begin{tikzcd}
 & H^*(X) \\
 & H^*_c(X) \\
X \text{ a topological space} \arrow[ur, rightsquigarrow] \arrow[uur, rightsquigarrow] \arrow[r, rightsquigarrow] \arrow[dr, rightsquigarrow] \arrow[dd, rightsquigarrow] & H_*(X) \\
 & H^{\mathrm{BM}}_*(X) \\
D^b(\mathbb{C}_X), \text{ the derived category of sheaves on } X. &
\end{tikzcd}$$

The 6-functor formalism gave us a nice framework by which to perform calculations in topology: recall that given $f : X \to Y$ we had functors

$$Rf_*, \quad Rf_!, \quad f^*, \quad f^!,$$

where

- Rf_* is *right adjoint* to f^*, and
- $Rf_!$ is *left adjoint* to $f^!$.

There is also a duality functor, $\mathbb{D}_X$. Its cohomology sheaves $\mathscr{H}^i\mathbb{D}_X(\mathscr{F})$ are given by the sheafification of

$$U \mapsto H^i_c(U; \mathscr{F})^\vee.$$

We constructed $\mathbb{D}_X(\mathbb{C}_X) = \omega_X$. Then

$$\mathbb{D}_X(\mathscr{F}) = R\mathscr{H}om_{\mathbb{C}_X}(\mathscr{F}, \omega_X).$$

It is useful to use this duality functor to talk about relations between our functors. To do this we have to restrict the category we look at:

$$D^b_c(\mathbb{C}_X) = \text{ bounded derived category of constructible complexes (with respect to \textit{some} stratification).}$$

Then $\mathbb{D}_X$ gives an equivalence

$$\mathbb{D}_X : D^b_c(\mathbb{C}_X) \xrightarrow{\sim} D^b_c(\mathbb{C}_X)^{\mathrm{op}}, \quad \mathbb{D}_X \circ \mathbb{D}_X \simeq \mathrm{id}.$$

This is a **quite nontrivial** self-duality.

Example 85. $\mathbb{D}_X(\mathbb{C}_X) = \omega_X$, and $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}(\mathbb{D}\mathscr{G}, \mathbb{D}\mathscr{F})$.

Fact/slogan: Commuting past the duality functor turns ! into $*$,

- $Rf_* \circ \mathbb{D}_X \simeq \mathbb{D}_Y \circ Rf_!$.
- $f^* \circ \mathbb{D}_Y \simeq \mathbb{D}_X \circ f^!$.

This will, for instance, swap various kinds of (co)homology around. So this is a "sheafy" version of Poincaré duality.

Definition 41. $\mathbb{D}_X$ is called the *Verdier duality functor*.

We would like a similarly nice (6 functor) formalism for $\mathcal{D}$-modules. As motivations, recall the picture of the Riemann-Hilbert correspondence (Figure 36).

Example 86. Given $f : \mathbb{C}^N \to \mathbb{C}$ a holomorphic function, consider $f^{-1}(0) = Z \subseteq \mathbb{C}^N$. Supposing some niceness properties (e.g. 0 an isolated singular value), we had some nice properties of the following functors and sheaves/ For instance, consider

$$Rf_*(\mathbb{C}_{\mathbb{C}^N}) \in D^b_c(\mathbb{C}_{\mathbb{C}}).$$

We considered the fibre Z as the central fibre X_0 in a family as per Figure 37.

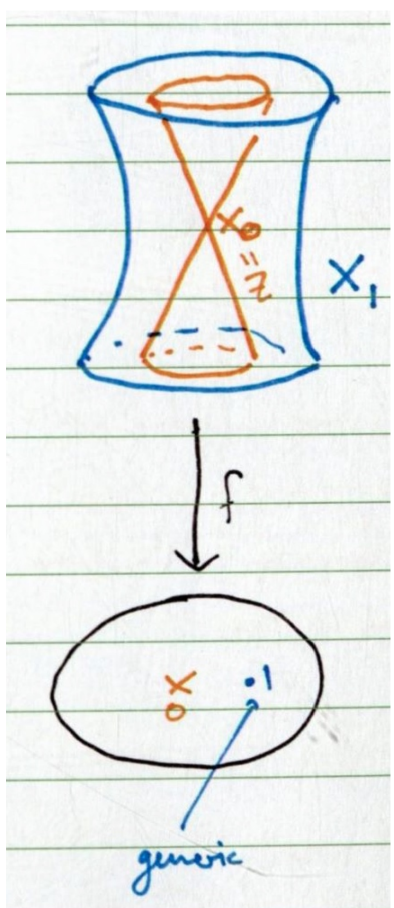


Figure 37. Studying a space Z as the central fibre in a family.

We then studied $Rf_*(\mathbb{C}_X)$, but also some more interesting sheaves of *vanishing* and *nearby* cycles, $\phi_f, \psi_f \in D^b_c(\mathbb{C}_{X_0})$.

To compute $f^!, f^*, Rf_*, Rf_!, \psi_f, \phi_f$, we will ask what they correspond to under the Riemann-Hilbert correspondence. We will also ask what the singular support $SS(\mathscr{M})$ of a $\mathcal{D}$-module corresponds to under Riemann-Hilbert.

Plan:

- Functoriality for coherent and holonomic $\mathcal{D}$-modules.
- Regular singularities, Riemann-Hilbert.
- Perverse sheaves, intersection cohomology.
- V-filtration, nearby and vanishing cycles, specialization to the normal cone.

15.1.1. *Aside: Algebraic versus analytic $\mathcal{D}$-modules.* We've been talking about X a complex manifold, with sheaves $\mathscr{O}_X$ of holomorphic functions and $\mathcal{D}_X$ of holomorphic differential operators.

We could instead have started with X a smooth algebraic variety (over $\mathbb{C}$). Then talking complex points gives a complex manifold, but this does not have a converse – e.g. the disk $\Delta = \{x \mid |x| < 1\}$ is not an algebraic variety.

X a complex manifold (complex topology)	X a smooth algebraic variety over $\mathbb{C}$ (Zariski topology)
$\mathscr{O}_X^{\mathrm{an}}$ holomorphic functions	$\mathscr{O}_X^{\mathrm{alg}}$ algebraic functions
$\mathcal{D}_X^{\mathrm{an}}$ holomorphic differential operators	$\mathcal{D}_X^{\mathrm{alg}}$ algebraic differential operators

Table 1. Algebraic versus analytic comparison.

Example 87. $X = \mathbb{C}^n = \mathbb{A}^n \supset U_f = \mathbb{A}^n - \{f = 0\}$. Then[2]

$$\mathscr{O}_X^{\mathrm{alg}} = \mathbb{C}[x_1, \dots, x_n]$$
$$\mathscr{O}_X^{\mathrm{alg}}(U_f) = \mathbb{C}[x_1, \dots, x_n][f^{-1}].$$

This is manifestly a much smaller space than $\mathscr{O}_X^{\mathrm{an}}$, the holomorphic functions on $\mathbb{C}^n$.

Now, $\mathcal{D}_X^{\mathrm{an}}$ contains elements of the form

$$\sum_i f_i(x)\partial^i$$

where the f_i are holomorphic functions – i.e. elements which are a power series in x and a polynomial in ∂. The description of $\mathcal{D}_X^{\mathrm{alg}}$ is much simpler. For example, we can present it in terms of generators and relations:

$$\mathcal{D}_X^{\mathrm{alg}} = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle / \sim, \quad [\partial_i, x_j] = \delta_{ij}, \quad [x_i, x_j] = 0 = [\partial_i, \partial_j].$$

15.2. **Base change for $\mathcal{D}$-module functors.** Given $f : X \to Y$, we produced

$$\begin{tikzcd} D^b(\mathcal{D}_X) \arrow[r, bend left, "f_*^{\mathrm{dR}}"] & D^b(\mathcal{D}_Y) \arrow[l, bend left, "f^!_{\mathrm{dR}}"] \end{tikzcd}$$

where $f^!_{\mathrm{dR}} := f^\dagger[\dim X - \dim Y]$. **Warning:** Recall that these are not necessarily adjoint.

Consider the base change

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

[2]Abusing notation by conflating a sheaf with its global sections – okay since we are working with affine varieties.

Theorem 15.2 (Base Change Theorem)**.** $f^!_{dR} g^{dR}_* \simeq \tilde{g}^{dR}_* \tilde{f}^!_{dR}$.

There are two special cases:

- f is *proper* (e.g. f is a closed embedding).
  - f^{dR}_* will preserve *coherent* complexes.
  - f^{dR}_* is *left adjoint* to $f^!_{\mathrm{dR}}$.
  - $f^{\mathrm{dR}}_* \circ \mathbb{D}_X \simeq \mathbb{D}_Y \circ f^{\mathrm{dR}}_*$.
- f is *smooth* (i.e. a submersion) of relative dimension d.
  - $f^!_{\mathrm{dR}}$ will preserve coherent complexes.
  - $f^!_{\mathrm{dR}}[-2d]$ is *left adjoint* to f^{dR}_*.
  - $f^!_{\mathrm{dR}} \circ \mathbb{D}_Y \simeq \mathbb{D}_X \circ f^!_{\mathrm{dR}}[-2d]$.
  - We have
  $$f^*_{\mathrm{dR}} = f^\dagger[-d] \quad \text{and} \quad f^!_{\mathrm{dR}} = f^\dagger[d].$$
  - $f^\dagger$ is *exact*.

16. Interpretation of singular support.

Question: What does $SS(\mathscr{M})$ mean?

Answer (to be explained): It measures directions in X for which solutions to $\mathscr{M}$ propagate.

Let $f : X \to Y$, X, Y complex manifolds (working in the analytic setting). We have

$$\begin{tikzcd}
 & T^*Y \times_Y X \arrow[dl, "\rho_f"'] \arrow[dr, "\varpi_f"] & \\
T^*X & & T^*Y \\
 & & SS(\mathscr{M}) \text{ lives here} \arrow[u, squiggly]
\end{tikzcd}$$

Fix $\mathscr{M} \in (\mathcal{D}_Y\text{-mod})_{\mathrm{coh}}$.

Definition 42. f is called *noncharacteristic for $\mathscr{M}$* if
$$\varpi_f^{-1}(SS(\mathscr{M})) \cap \mathring{T}^*_X Y = \emptyset.$$
Here $T^*_X Y = \ker(\rho_f)$ (a vector bundle on X, and
$$\mathring{T}^*_X Y = T^*_X Y - \{0\text{-section}\}.$$

If $X \hookrightarrow Y$ is a closed embedding, $T^*_X Y$ is called the *conormal bundle of* $X \hookrightarrow Y$. Another way of understanding this in the closed embedding case is as:
$$\varpi_f^{-1}(SS(\mathscr{M})) \cap T^*_X Y \subseteq 0\text{-section}.$$

Example 88. If f is smooth (submersion) then
$$T^*_X Y = \{0\} \times X.$$
Thus f is noncharacteristic for any $\mathscr{M}$.

Example 89. If $SS(\mathscr{M}) = T^*_Y Y = 0$-section, any f is noncharacteristic for $\mathscr{M}$.

Example 90. Consider

$$\begin{tikzcd}
X \arrow[r, hook, "f"] & Y \\
 & Z \arrow[u, hook, "g"']
\end{tikzcd}$$

where f and g are both closed embeddings. Let $\mathscr{M} = f_*^{\mathrm{dR}}(\mathscr{O}_X)$. When is Z a noncharacteristic submanifold for $\mathscr{M}$ (i.e. when is g noncharacteristic for $\mathscr{M}$)?

Recall from a previous lecture that

$$SS(\mathscr{M}) = T_X^*Y.$$

This is noncharacteristic if and only if

$$\varpi_g^{-1}(T_X^*Y) \cap (T_Z^*Y) = 0\text{-section},$$

if and only if

$$(T_X^*Y)_x \cap (T_Z^*Y)_x = 0 \quad \text{for all } x \in Z \cap X.$$

I.e. $SS(\mathscr{M})$ is noncharacteristic if and only if $X \pitchfork Z$, as in Figure 38.

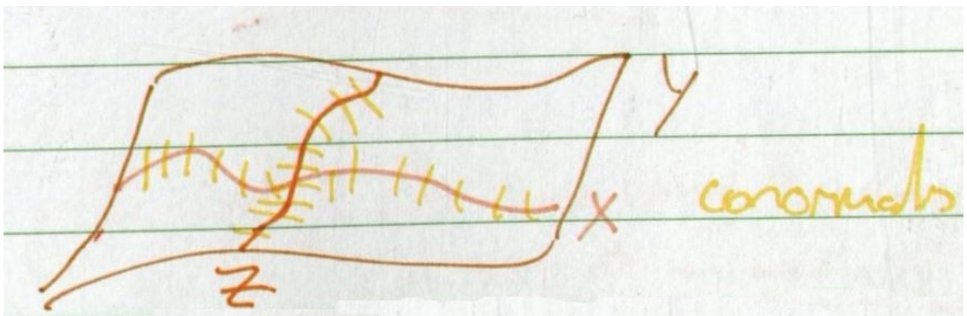


FIGURE 38. Transverse embedded submanifolds with their conormal bundles.

Theorem 16.1. *Suppose $f : X \to Y$ is noncharacteristic for $\mathscr{M}$.*

(1) $f^\dagger(\mathscr{M}) = \mathscr{H}^0 f^\dagger(\mathscr{M}) = f^\circ(\mathscr{M})$ (concentrated in degree zero – i.e. not being concentrated in degree zero is a non-transversality condition).
(2) $f^\dagger(\mathscr{M})$ is a coherent $\mathcal{D}_X$-module.
(3) $SS(f^\dagger\mathscr{M}) = \rho_f(\varpi_f^{-1}(SS(\mathscr{M})))$.
(4) $f^\dagger(\mathbb{D}_Y\mathscr{M}) =\cong \mathbb{D}_X(f^\dagger\mathscr{M})$.

Recall:

- $f^\dagger(\mathscr{M}) = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}$.
- $f^!_{\mathrm{dR}}(\mathscr{M}) = f^\dagger(\mathscr{M})[\dim(X) - \dim(Y)]$.

In general, $f^\dagger$ (or $f^!_{\mathrm{dR}}$) does not preserve coherence.

Example 91. $i : \{0\} \hookrightarrow \mathbb{A}^1$ induces $i^\dagger(\mathcal{D}_{\mathbb{A}^1})$, which is infinite dimensional.

Upshot: When is it reasonable to restrict coherent $\mathcal{D}$-modules as above? When the $\mathcal{D}$-module is *noncharacteristic.*

Remark The above theorem still makes sense in the algebraic setting – it might even be easier to prove there.

Theorem 16.2 (Cauchy-Kovalevskaya-Kashiwara)**.** *Let $f : X \to Y$ be noncharacteristic for $\mathscr{M}$, then*

$$Sol_X(f^\dagger\mathscr{M}) = f^{-1}Sol_Y(\mathscr{M}).$$

Recall that

$$\mathrm{Sol}_Y(\mathscr{M}) = R\mathscr{H}om_{\mathcal{D}_X}(\mathscr{M}, \mathscr{O}_Y).$$

Motivated by: $\mathscr{M} = \mathcal{D}_X / \mathcal{D}_X \cdot P$, where we interpret

$$\mathrm{Sol}_X(\mathscr{M}) = \{u \mid Pu = 0\}.$$

Consider the special case (the general proof always reduces to this special case):

$$X \subseteq Y \text{ a hypersurface,}$$
$$\mathscr{M} = \mathcal{D}_Y / \mathcal{D}_Y \cdot P, \quad P \in \mathcal{D}_Y.$$

Everything is local, so choose coordinates $y_1, \dots, y_n$ on X such that

$$X = \{y_1 = 0\} \subseteq Y.$$

What does it mean for X to be noncharacteristic for $\mathscr{M}$ (this should be some condition on the differential operator P)? Recall that

$$SS(\mathscr{M}) = V(\sigma_m(P)), \quad \text{where } P \text{ is of order } m.$$

Let $\xi = \sigma(\partial)$ (a function on the cotangent bundle), and write

$$P = \sum_{|\alpha| \leq m} a_\alpha(y) \partial^\alpha$$
$$\sigma_m(P) \sum_{|\alpha| = m} a_\alpha(y) \xi^\alpha.$$

X is noncharacteristic for $\mathscr{M}$ (or for P) iff for all $\xi \in \mathring{T}^*_X Y$ $\sigma_m(P)(\xi) \neq 0$, iff

$$\sigma_m(P) = (0, y_2, \dots, y_n, 1, 0, \dots, 0) \neq 0.$$

(Here the first coordinate is 0 since we are on X, and consequently the only conormal direction to X is in the ξ^1 position.) In coordinates we write P as

$$P = \sum a_{\alpha_1, \dots, \alpha_n}(y_1, \dots, y_n) \partial_{y_1}^{\alpha_1} \cdots \partial_{y_n}^{\alpha_n}.$$

So the above condition means

$$a_{m,0,\dots,0}(o, y_2, \dots, y_n) \neq 0.$$

Looking in a neighbourhood of X, then, we can invert this term so that, without loss of generality,

$$P = \partial_1^m + (\text{lower order terms in } \partial_1).$$

So: X being noncharacteristic for P of order m means that P is m^{th} order in the X direction, plus (potentially) other terms in other directions.

So, consider the claims of the theorem. We have

$$f^\dagger(\mathscr{M}) = \underset{-1}{\mathscr{M}} \xrightarrow{y_1 \cdot} \underset{0}{\mathscr{M}}.$$

This has potentially two cohomology groups. We want to show that $y_1 \cdot$ is injective. This would imply that

$$f^\dagger(\mathscr{M}) = f^\circ(\mathscr{M}) = \mathscr{M} / y_1 \mathscr{M} = \mathcal{D}_Y / (\mathcal{D}_Y \cdot P + y_1 \cdot \mathcal{D}_Y).$$

Claim:

$$\mathcal{D}_Y / (\mathcal{D}_Y \cdot P + y_1 \cdot \mathcal{D}_Y) \cong \mathcal{D}_X^{\oplus m}.$$

Example 92. Consider

$$f^\circ(\mathscr{M}) = \mathcal{D}_X[\partial_1] / \mathcal{D}_x[\partial_1] \cdot P.$$

Let $P = \partial_1^m$. Then we have that

$$f^\circ(\mathscr{M}) \cong \mathcal{D}_X^{\oplus m}.$$

In general,

$$1, \partial_1, \partial_1^2, \dots, \partial_1^{m-1} \quad \text{is a basis for } f^\circ(\mathscr{M}) \text{ as a } \mathcal{D}_X\text{-module.}$$

Let's pretend/assume that this corresponds to a proof of the first theorem we wrote down. What about the Cauchy-Kovalevskaya-Kashiwara theorem? The claim there was that

$$\begin{tikzcd}
f^{-1}\mathrm{Sol}_Y(\mathscr{M}) \arrow[rr, "\cong"] \arrow[d, equal] & & \mathrm{Sol}_X(f\dagger\mathscr{M}) \arrow[d, leftrightarrow, "\cong"] \\
f^{-1}R\mathscr{H}om_{\mathcal{D}_Y}(\mathcal{D}_Y/\mathcal{D}_Y\cdot P, \mathscr{O}_Y) \arrow[d, equal, "\text{(use the 2 term free resolution)}", "\mathcal{D}_Y \xrightarrow{\cdot P}\mathcal{D}_Y"'] & & R\mathscr{H}om_{\mathcal{D}_X}(\mathcal{D}_X^{\oplus m}, \mathscr{O}_X) \arrow[d, leftrightarrow, "\cong"] \\
f^{-1}(\underset{0}{\mathscr{O}_Y} \xrightarrow{P(-)} \underset{1}{\mathscr{O}_Y}) & & \mathscr{O}_X^{\oplus m}
\end{tikzcd}$$

The above map is then

$$\begin{gathered}
f^{-1}(\underset{0}{\mathscr{O}_Y} \xrightarrow{P(-)} \underset{1}{\mathscr{O}_Y}) \longrightarrow \mathscr{O}_X^{\oplus m} \\
u \in \mathscr{O}_X \longmapsto (u|_X, \partial_1 u|_X, \ldots, \partial_1^{m-1}u|_X)
\end{gathered}$$

This isomorphism is essentially an existence and uniqueness statement for an m^{th} order boundary value problem:

Theorem 16.3 (Cauchy-Kovalevskaya)**.** *The* Cauchy problem

$$\left\{\begin{array}{rcl} Pu & = & 0 \\ u|_X & = & v_1 \\ \partial_1 u|_X & = & v_2 \\ & \vdots & \\ \partial_1^{m-1}u|_X & = & v_n \end{array}\right\} \quad \textit{has a unique solution } u \in f^{-1}\mathscr{O}_Y.$$

(Here $P = \partial_1^m + \cdots$ is noncharacteristic.)

To answer the question, *"What does $SS(\mathscr{M})$ mean?"*, we will need a variant of the Cauchy-Kovalevskaya theorem. Let $\phi : X \to \mathbb{R}$ be a C^∞-function, X a complex manifold. Suppose

$$X_0 = \phi^{-1}(0) \text{ is a smooth real hypersurface.}$$

Let $\mathscr{M}$ be a coherent $\mathcal{D}_X$-module such that

$$SS(\mathscr{M}) \cap \mathring{T}^*_{X_0}(X_\mathbb{R}) = \emptyset.$$

Remark This uses the identification

$$\begin{gathered}
(T^*X)_\mathbb{R} \xrightarrow{\cong} T^*(X_\mathbb{R}) \\
(\partial\phi)_x \longleftrightarrow (d\phi)_x.
\end{gathered}$$

Then

$$(R\Gamma_{X_{\geq 0}}(\mathrm{Sol}_X(\mathscr{M})))_{X_0} \simeq 0.$$

We can picture this as in Figure 39.

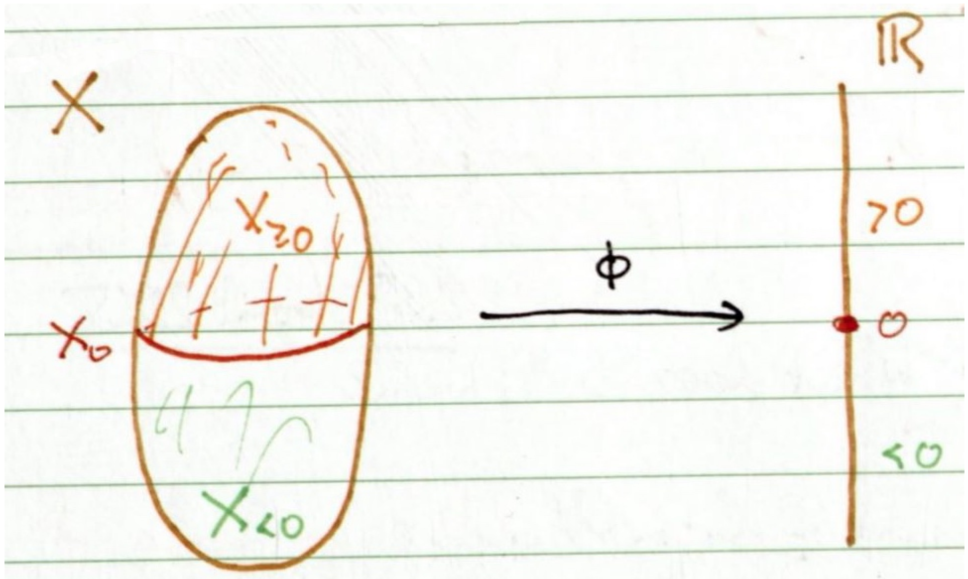


FIGURE 39. Solutions on $X_{<0}$ propagate across X_0 to solutions on $X_{>0}$.

Recall that if $k : W \hookrightarrow X$ is a locally closed embedding,

$$R\Gamma_W = Rk_* k^!, \quad (-)_W = Rk_! k^*.$$

We have a decomposition of our space into an open and a closed, so letting

$$\mathscr{F}^\bullet := \mathrm{Sol}_X(\mathscr{M}),$$

we have an exact triangle

$$R\Gamma_{X_{\geq 0}}(\mathscr{F}^\bullet) \to \mathscr{F}^\bullet \to R_{X_{<0}}(\mathscr{F}^\bullet) \xrightarrow{+1}$$

Looking stalkwise we see that applying $(-)_{X_{\leq 0}}$ and using the theorem gives

$$(R\Gamma_{X_{\leq 0}}(\mathscr{F}^\bullet))_{X_{\leq 0}} = 0,$$

so,

$$(\mathscr{F}^\bullet)_{\leq 0} \xrightarrow{\sim} R\Gamma_{X_{\leq 0}}(\mathscr{F}^\bullet).$$

We interpret this as saying that (holomorphic) solutions to $\mathscr{M}$ on $X_{<0}$ extend (to holomorphic solutions) over X_0.

Meaning: Solution to $\mathscr{M}$ on $X_{<0}$ can propagate over a "small' neighbourhood of X_0. The proof of this involves reducing to the case

$$\mathscr{M} = \mathcal{D}_X / \mathcal{D}_X \cdot P,$$

then expressing the result as a Cauchy problem.

16.1. **Microsupport.**

Definition 43. Let $\mathscr{F} \in D^b(\mathbb{C}_X)$, X a $C^\infty_{\mathbb{R}}$-manifold. Define the *microsupport* of $\mathscr{F}$, $\mu s(\mathscr{F})$ as follows. $\mu s(\mathscr{F}) \subseteq T^*X$, and

$T^*X \ni (x_0, \xi_0) \notin \mu s(\mathscr{F})$ if and only if There exists an open set $U \subseteq T^*X$ with $(x_0,\xi_0) \in U$ such that for all $x \in X$ and $\phi : X \to \mathbb{R}$ a smooth function with $\phi(x) = 0, (d\phi)_x \in U,$ we have that $R\Gamma_{\phi \geq 0}(\mathscr{F}) \simeq 0$.

Theorem 16.4. *If $\mathscr{M}$ is a coherent $\mathcal{D}_X$-module, then*

$$SS(\mathscr{M}) = \mu s(Sol_X(\mathscr{M})).$$

The generalized Cauchy-Kovalevskaya theorem provides $SS(\mathscr{M}) \supseteq \mu s(\mathrm{Sol}_X(\mathscr{M}))$. The other direction is harder, and we omit it.

Recall that

$$\mathrm{Sol}_X(\mathscr{M}) = R\mathscr{H}om_{\mathcal{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

Example 93. Let $\mathscr{M} = \mathscr{O}_X$ and recall that there exists a resolution of $\mathscr{O}_X$ as a $\mathcal{D}_X$-module, $\mathcal{D}_X \otimes_{\mathscr{O}_X} (\Omega_X^\bullet)$. Then

$$\begin{aligned} \mathrm{Sol}_X(\mathscr{O}_X) &\simeq \Omega_X^\bullet \simeq \mathbb{C}_X, \quad \text{(by the Poincaré lemma), so,} \\ SS(\mathscr{O}_X) &= X \subseteq T^*X. \end{aligned}$$

Example 94. Let $\mathscr{M} = \mathcal{D}_X$. Then $\mathrm{Sol}_X(\mathcal{D}_X) = \mathscr{O}_X$ – remember this is *not* as a $\mathcal{D}$-module, it is as the sheaf of solutions to a $\mathcal{D}$-module. Recall that

$$SS(\mathcal{D}_X) = T^*X.$$

Then $\mu s(\mathscr{O}_X) = T^*X$. I.e. there are no directions in which $\mathrm{Sol}_X(\mathcal{D}_X)$ looks like $\mathbb{C}_X$. So given holomorphic functions on some (open) domain with boundary, there will always be solutions that become singular on the boundary. Phrased another way:

$$\mathscr{O}_X(X_{\leq 0}) \to \mathscr{O}_X(X_{<0}) \text{ is never an isomorphism.}$$

Exercise 16.1. Determine what $\mu s(\mathscr{F})$ is measuring for points $(x_0, 0) \in T^*X$.

16.2. **Holonomic complexes.** Suppose $\mathscr{M} \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, complexes of $\mathcal{D}_X$-modules with holonomic cohomology sheaves. Recall $\mathscr{M}$ holonomic means that $SS(\mathscr{M}_\subseteq T^*X$ is Lagrangian. We can write

$$SS(\mathscr{M}) := \bigcup_i SS(\mathscr{H}^i(\mathscr{M})) = \Lambda = \bigcup_j \Lambda_j$$

where Λ is a conic Lagrangian, and the Λ_j are irreducible Lagrangians.

Example 95. See Figure 40.

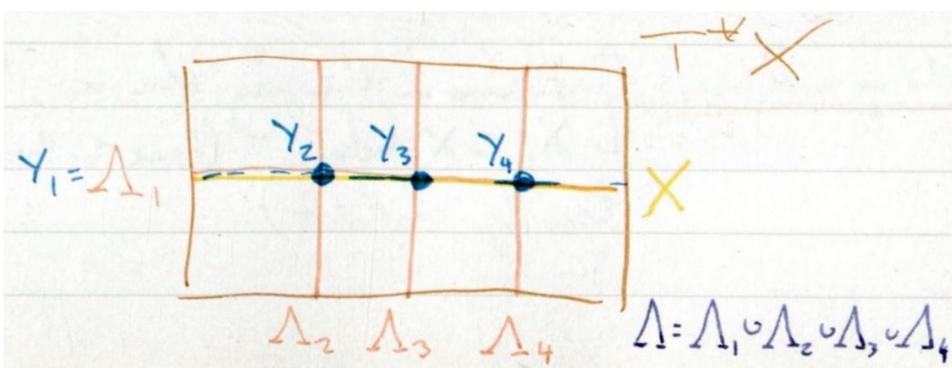


FIGURE 40. Decomposition of the singular support into irreducible Lagrangians.

Define

$$Y_i = \pi_X(\Lambda_i) = \Lambda_i \cap X,$$

some closed analytic irreducible subsets of X.

Lemma 16.5. $\Lambda_i = \overline{T^*_{Y_i^{reg}} X}$.

Idea: The Y_i are giving a stratification on which solutions to $\mathscr{M}$ are locally constant.

Theorem 16.6. *Given a conic Lagrangian* Λ*, there exists a* Whitney stratification *of* X*,*

$$X = \coprod_j X_j, \quad X_j \text{ smooth},$$

such that each Y_i *is a union of some* X_j*.*

Corollary 16.7. $\Lambda \subseteq \bigcup T^*_{X_i} X$.

Whitney conditions: These are conditions on the way the normal bundles to $X_j \subset X$ behave (roughly, they have to look locally constant).

Proposition 16.8. *If* $\mathscr{M}$ *is a holonomic* $\mathcal{D}_X$*-module,* $\mathscr{F} = Sol_X(\mathscr{M}) \in D^b(\mathbb{C}_X)$*, and* $\mu s(\mathscr{F}) = \Lambda$ *is a conic Lagrangian, then* $\mathscr{F}$ *is constructible with regards to* X_j*. I.e.* $\mathscr{H}^k(\mathscr{F})|_{X_j}$ *is locally constant with finite dimensional fibres.*

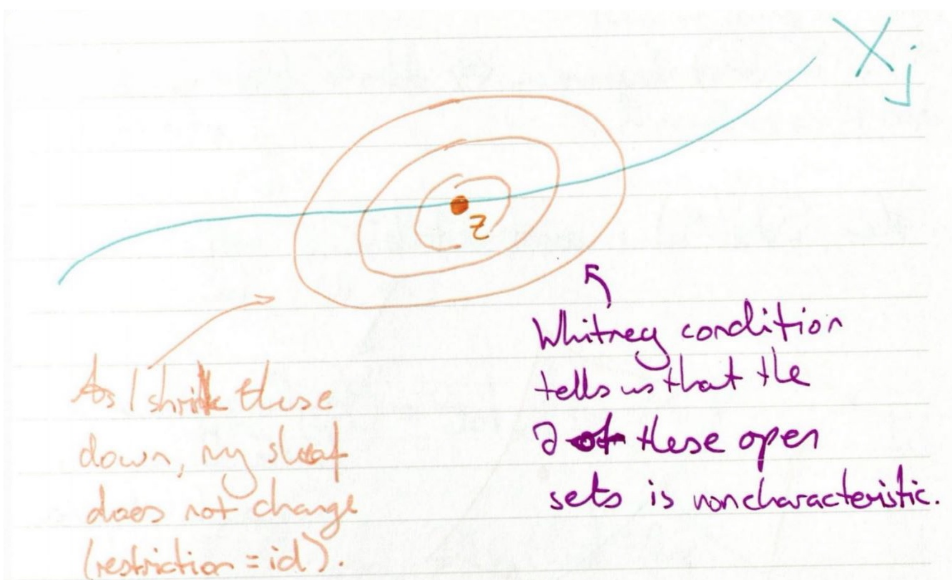


Figure 41. Rough picture of the Whitney condition.

17. Regular singularities.

We have now seen that if $\mathscr{M}$ is holonomic, then $\mathrm{Sol}_X(\mathscr{M})$ is constructible (locally constant and finite dimensional on each stratum, see Figure 42). We use this fact to motivate a new class of $\mathcal{D}$-modules – those with *regular singularities.*

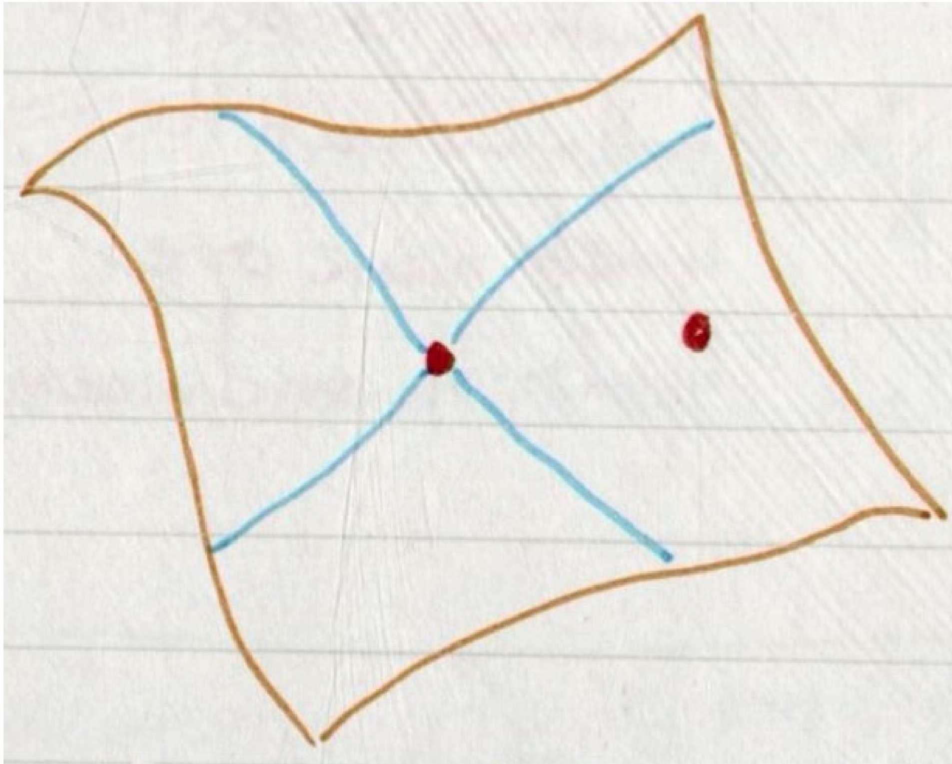


FIGURE 42. Stratification for a constructible sheaf.

Recall:

$$\mathrm{DR}_X(\mathscr{M}) = \underset{-n}{\mathscr{M}} \xrightarrow{d} \underset{-n+1}{\mathscr{M} \otimes_{\mathscr{O}_X} \Omega^1_X} \to \cdots \to \underset{0}{\mathscr{M} \otimes_{\mathscr{O}_X} \Omega^n_X}$$

We can identify this with

$$\mathrm{DR}_X(\mathscr{M}) = R\mathrm{Hom}_{\mathcal{D}_X}(\mathscr{O}_X, \mathscr{M})[n].$$

How?

- $\mathscr{O}_X$ has a resolution by free $\mathcal{D}_X$-modules

$$(\underset{-n}{\mathcal{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^n_{\mathscr{O}_X} \mathscr{T}_X} \to \cdots \to \underset{-2}{\mathcal{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^2_{\mathscr{O}_X} \mathscr{T}_X} \to \underset{-1}{\mathcal{D}_X \otimes_{\mathscr{O}_X} \mathscr{T}_X} \to \underset{0}{\mathcal{D}_X}) \xrightarrow{\sim} \underset{0}{\mathscr{O}_X}.$$

- So taking homs from this resolution by free $\mathcal{D}_X$-modules, we can move polyvector fields from the LHS to its dual (forms) on the RHS. Then to agree with the degrees of $\mathrm{DR}_X(\mathscr{M})$ we need to shift back down by n.

Now applying Verdier duality,

$$R\mathrm{Hom}_{\mathcal{D}_X}(\mathscr{O}_X, \mathscr{M})[n] = R\mathrm{Hom}(\mathbb{D}_X\mathscr{M}, \underbrace{\mathbb{D}_X\mathscr{O}_X}_{\mathscr{O}_X})[n] = \mathrm{Sol}_X(\mathbb{D}_X\mathscr{M}),$$

so,

$$\mathrm{DR}_X(\mathscr{M}) = \mathrm{Sol}_X(\mathbb{D}_X\mathscr{M}).$$

Example 96. If $\mathscr{M} = \mathscr{O}_X$, $\mathrm{DR}_X(\mathscr{O}_X) \simeq \mathbb{C}_X$. So we have a functor

$$\begin{tikzcd} D^b_{\mathrm{coh}}(\mathcal{D}_X) \arrow[r, "\mathrm{DR}_X"] & D^b(\mathbb{C}_X) \\ D^b_{\mathrm{hol}}(\mathcal{D}_X) \arrow[u, hook] \arrow[r, "\text{essentially surjective}"] & D^b_c(\mathbb{C}_X) \arrow[u, hook] \\ D^b_{\mathrm{rs}}(\mathcal{D}_X) \arrow[u, hook] & \end{tikzcd}$$

where the bottom most category is the bounded derived category of $\mathcal{D}_X$-modules with *regular singularities.*

Definition 44. A holonomic $\mathcal{D}_X$-module $\mathscr{M}$ has *regular singularities* at $x \in X$ if

$$\begin{tikzcd} \mathrm{Sol}_X(\mathscr{M})_x \arrow[r, "\sim"] & \widehat{\mathrm{Sol}_X}(\mathscr{M})_x \arrow[r, equal, "\mathrm{defn}"] & \mathrm{Hom}_{\mathcal{D}_{X,x}}(\mathscr{M}_x, \underbrace{\widehat{\mathscr{O}}_{X,x}}_{\mathbb{C}[[x_1,\dots,x_n]]}) \\ & \text{“\textit{Formal solutions of } } \mathscr{M}.\text{”} \arrow[u, squiggly] & \end{tikzcd}$$

I.e. “every formal solution has a positive radius of convergence”.

Warning: This is a definition for analytic and **not** algebraic $\mathcal{D}$-modules.

Then we define $\mathcal{D}^b_{\mathrm{rs}}(\mathcal{D}_X)$ to be complexes with cohomology objects regular singular $\mathcal{D}_X$-modules (i.e. regular singular at every point).

Theorem 17.1 (Riemann-Hilbert Correspondence)**.**

$$DR_X : D^b_{rs}(\mathcal{D}_X) \xrightarrow{\sim} D^b_c(\mathbb{C}_X)$$

is an equivalence of triangulated categories.

Remark We no longer have the flat connections $\leftrightarrow$ π_1-reps correspondence with complexes. Instead we need to look at the *exit path category.* See, e.g. [AFR].

Example 97. Let

$$\mathrm{Conn}(X) = \left\{ \begin{array}{c} \text{vector bundles on} \\ X \text{ with flat connections} \end{array} \right\} \subseteq D^b_{\mathrm{rs}}(\mathcal{D}_X).$$

Then

$$\begin{aligned} \mathrm{Conn}(X) &\xleftrightarrow{\sim} \{\text{locally constant sheaves}\} \\ (\mathscr{V}, \nabla) &\longmapsto \mathrm{DR}_X(\mathscr{V}, \nabla) \\ (\mathscr{F} \otimes_{\mathbb{C}_X} \mathscr{O}_X, d) &\longleftarrow\!\shortmid \mathscr{F}[n] \end{aligned}$$

There is a Poincaré lemma that says $\mathrm{DR}_X(\mathscr{V}, \nabla)$ has no higher cohomology – it is all concentrated in degree $-n$, thus

$$\mathrm{DR}_X(\mathscr{V}, \nabla) \simeq \ker(\nabla)[n].$$

Remark The Riemann-Hilbert correspondence was proven independently by Mebkhout (1979) and Kashiwara (1980).

17.1. **What does “regular singularities” mean?**

Example 98. Let

$$\mathscr{M} = \mathcal{D}_X / \mathcal{D}_X \cdot P \simeq (\mathcal{D}_X \xrightarrow{\cdot P} \mathcal{D}_X),$$

(we have an easy free resolution in this case). So

$$\mathrm{Sol}_X(\mathscr{M}) \simeq \underset{0}{\mathscr{O}_X} \xrightarrow{P(-)} \underset{1}{\mathscr{O}_X}.$$

Hence

$$\mathscr{H}^0\mathrm{Sol}_X(\mathscr{M}) = \ker(P)$$
$$\mathscr{H}^1\mathrm{Sol}_X(\mathscr{M}) = \mathrm{coker}(P)$$

We can localize everything to look at germs of solutions around $x \in X$,

$$\mathrm{Sol}_X(\mathscr{M})_x \simeq \underset{0}{\mathscr{O}_{X,x}} \xrightarrow{P(-)} \underset{1}{\mathscr{O}_{X,x}}$$
$$\mathscr{H}^0\mathrm{Sol}_X(\mathscr{M})_x = \ker P$$
$$\mathscr{H}^1\mathrm{Sol}_X(\mathscr{M})_x = \mathrm{coker} P$$

Formal solutions are:

$$\widehat{\mathrm{Sol}}_X(\mathscr{M})_x = \widehat{\mathrm{Sol}}_x(\mathscr{M}) = \widehat{\mathscr{O}}_{X,x} \xrightarrow{P} \widehat{\mathscr{O}}_{X,x}.$$

Assume X is 1d (so that $\mathscr{M}$ can be holonomic), and since we are working locally, without loss of generality let $X = \mathbb{C}$. Then

$$\mathscr{O}_{X,x} = \mathscr{O} = \mathbb{C}\{\{x\}\}$$
$$\widehat{\mathscr{O}}_{X,x} = \widehat{\mathscr{O}} = \mathbb{C}[[x]].$$

Example 99. Let $P = x\partial_x - \lambda$. Since $x\partial_x(x^n) = nx^n$, x^n is an eigenvalue for $x\partial_x$. So the solution to $Pu = 0$ is $u = x^\lambda$ – if λ is non-integral this requires a choice of branch of log (solution only makes sense on some small simply connected region). How can we think of this?

$$\begin{array}{ccc}
\vdots & & \vdots \\
x^4 & \xrightarrow{4-\lambda} & x^4 \\
x^3 & \xrightarrow{3-\lambda} & x^3 \\
x^2 & \xrightarrow{2-\lambda} & x^2 \\
x & \xrightarrow{1-\lambda} & x \\
1 & \xrightarrow{-\lambda} & 1 \\
 & \xrightarrow{P} &
\end{array}$$

How do the kernel and cokernel compare for power series versus formal power series?

Case 1: $\lambda \notin \mathbb{Z}_{\geq 0}$; then all of the maps above are isomorphisms, so

$$\mathrm{Sol}_X(\mathscr{M})_x \simeq \widehat{\mathrm{Sol}}_X(\mathscr{M})_x \simeq 0.$$

Case 2: $n = \lambda \in \mathbb{Z}_{\geq 0}$. Then x^n is in the kernel of P, and in fact

$$\ker(P) = \mathbb{C}x^n.$$

Also, x^n is **not** in the image of P, so

$$\mathrm{coker}(P) \cong \mathbb{C}x^n.$$

This holds for $\mathbb{C}\{\{x\}\}$ and $\mathbb{C}[[x]]$. Thus,

$$\mathcal{D}/\mathcal{D}\cdot(x\partial_x - \lambda) = \mathcal{D}\cdot x^\lambda$$

is *regular* for every λ.

Example 100. $P = x^2\partial_x - 1$. $e^{\frac{1}{x}}$ is a solution – this has an essential singularity in the analytic sense. Looking at this:

- **Observation 1:** $\ker(P) = \ker(\widehat{P}) = 0$. Why? $e^{\frac{1}{x}}$ is the unique solution (to this first order ODE), but it is not holomorphic at 0. So $\ker(P) = 0$.
- **Observation 2:** $\widehat{P}$ is an isomorphism.
- **Observation 3:** P is not surjective.

For $\widehat{P}$:

$$\begin{aligned}\widehat{P}(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) &= -a_0 + (a_1x^2 - a_1x) + (2a_2x^3 - a_2x^2) + (3a_3x^4 - a_3x^3) + \cdots \\ &= -a_0 - a_1x + (a_1 - a_2)x^2 + (2a_2 - a_3)x^3 + \cdots\end{aligned}$$

Thus $\ker \widehat{P} = 0$ (solve the above equation term-by-term from a_0).

Moreover, given a power series we can algebraically recursively solve for the values of $a_0, a_1, \ldots$. E.g. there exist $a_0, a_1, \ldots$ such that

$$\widehat{P}(a_0 + a_1x + a_2x^2 + \cdots) = x.$$

Here we need

$$\underset{a_0}{0} + \underset{(-1)}{a_1}\, x + \underset{a_2}{(-1)}x^2 - 2x^3 + \cdots \quad \text{(solving recursively).}$$

It is not hard to see that this will have 0 radius of convergence.

Conclusion: $\widehat{P}$ is an isomorphism, so $\widehat{\mathrm{Sol}}_x(\mathscr{M}) = 0$.

But $\mathrm{Sol}_x(\mathscr{M})$ turns out to have cokernel generated by x; so

$$\mathrm{Sol}_x(\mathscr{M}) \cong \mathbb{C}[-1].$$

Thus,

$$\mathscr{M} = \mathcal{D}/\mathcal{D}\cdot(x^2\partial_x - 1) \text{ is } \textit{irregular} \text{ at } 0.$$

Continue assuming X is 1d, and since we are working locally, continue assuming $X = \mathbb{C}$ and $x = 0$.

Question: When is $\mathcal{D}/\mathcal{D}\cdot P$ regular?

Definition 45. $\delta(P) = m - \mathrm{ord}_0(a_m)$, where

$$P = a_m(x)\partial + x^m + a_{m-1}(x)\partial_x^{m-1} + \cdots + a_0(x),$$

and ord_0 is the order of vanishing at 0.

We can think of $\mathcal{D}$ as $\mathcal{D} = \mathscr{O}[\partial]$.

Definition 46.

$$\widehat{\delta}(P) = \max_{0\le k\le m}\{k - \mathrm{ord}_0(a_k)\}.$$

By convention, $\mathrm{ord}_0(0) = +\infty$.

Example 101. If $P = x^N\partial^m$, then $\delta(P) = \widehat{\delta}(P) = m - N$.

Definition 47. The *index of P* is

$$\begin{aligned}\chi(\mathcal{D}/\mathcal{D}\cdot P) &:= \chi(\mathrm{Sol}_X(\mathcal{D}/\mathcal{D}\cdot P)_0) \quad \text{(which is a two term complex),} \\ &= \dim(\ker(P)) - \dim(\mathrm{coker}(P)).\end{aligned}$$

Similarly we define

$$\widehat{\chi}(\mathcal{D}/\mathcal{D}\cdot P) = \chi(\widehat{\mathrm{Sol}}_X(\mathcal{D}/\mathcal{D}\cdot P).$$

Exercise 17.1. Show that $\mathcal{D}/\mathcal{D}\cdot P$ is regular if and only if $\chi(\mathcal{D}/\mathcal{D}\cdot P) = \widehat{\chi}(\mathcal{D}/\mathcal{D}\cdot P)$.

Theorem 17.2 (Index Theorem)**.**

$$\delta(P) = \chi(\mathcal{D}/\mathcal{D}\cdot P)$$
$$\widehat{\delta}(P) = \widehat{\chi}(\mathcal{D}/\mathcal{D}\cdot P)$$

Corollary 17.3. *$\mathcal{D}/\mathcal{D}\cdot P$ is regular if and only if $\delta(P)=\widehat{\delta}(P)$.*

17.2. **Regular singularities in the local setting.** Recall the notation:

$$\mathscr{O} := \mathbb{C}\{\{x\}\} \cong (\mathscr{O}_{\mathbb{C}})_0 \supset \mathfrak{m} = x\mathscr{O}, \quad \text{and} \quad \widehat{\mathscr{O}} := \mathbb{C}[[x]].$$

Remark $\mathbb{C}[x] \subsetneq \mathbb{C}\{\{x\}\} \subsetneq \mathbb{C}[[x]]$.

$$\mathcal{D} := \mathscr{O}[\partial] \cong (\mathcal{D}_{\mathbb{C}})_0, \quad [\partial, f] = \partial(f).$$

Define the *field of fractions of $\mathscr{O}$* to be

$$\mathscr{K} := \mathscr{O}[x^{-1}],$$

(i.e. local meromorphic functions).

Example 102. For $\mathcal{D}/\mathcal{D}\cdot P$ a $\mathcal{D}$-module,

$$\begin{aligned}\mathrm{Sol}(\mathcal{D}/\mathcal{D}\cdot P) &= \mathscr{O} \xrightarrow{P} \mathscr{O},\\ \widehat{\mathrm{Sol}}(\mathcal{D}/\mathcal{D}\cdot P) &= \widehat{\mathscr{O}} \xrightarrow{\widehat{P}} \widehat{\mathscr{O}}.\end{aligned}$$

Notation convention: From now on,

$$\mathrm{Sol}(P) := \mathrm{Sol}(\mathcal{D}/\mathcal{D}\cdot P).$$

This is a two term complex, so we only have

$$\begin{aligned}\mathscr{H}^0(\mathrm{Sol}(P)) &= \ker(P)\\ \mathscr{H}^1(\mathrm{Sol}(P)) &= \mathrm{coker}(P)\end{aligned}$$

Remark $\ker(P) \subseteq \ker(\widehat{P})$ and $\mathrm{coker}(P) \twoheadrightarrow \mathrm{coker}(\widehat{P})$.

Recall we defined $\mathcal{D}/\mathcal{D}\cdot P$ to be *regular* if

$$\mathrm{Sol}(P) \xrightarrow{\sim} \widehat{\mathrm{Sol}}(P) \quad \text{(quasi-isomorphism)}.$$

This turns out to be equivalent to

$$\chi(P) = \widehat{\chi}(P).$$

Remark Our previous remark tells us that we always have $\chi(P) \leq \widehat{\chi}(P)$.

Let

$$P = \sum_{k=0}^{m} a_k(x)\partial^k, \quad a_m(x) \not\equiv 0.$$

Then we defined

$$\begin{aligned}\delta(P) &= m - \mathrm{ord}(a_m(x))\\ \widehat{\delta}(P) &= \max_{0\leq k\leq m}(k - \mathrm{ord}(a_k(x)))\end{aligned}$$

Theorem 17.4. *We have that*

$$\delta(P) = \chi(P) \quad \text{and} \quad \widehat{\delta}(P) = \widehat{\chi}(P).$$

Proof. The proof for δ uses analysis. The idea is that $P - a_m(x)\partial^m$ is a compact operator between certain Banach spaces (recall that P is Fredholm). Then there is an index theorem that says $\chi(P) = \chi(a_m(x)\partial^m)$. See [B] for details.

The proof that $\widehat{\delta}(P) = \widehat{\chi}(P)$ is a purely algebraic computation in manipulating power series. □

Claim: $\delta(P) = \widehat{\delta}(P)$ if and only if

$$P = \sum_{i=0}^{m} b_i(x)\theta^i, \quad b_i(x) \in \mathscr{K}, \quad \theta = x\partial,$$

and

$$\frac{b_i(x)}{b_m(x)} \text{ is holomorphic.}$$

Why? Start with $a_m(x) = x^{m-\delta(P)}\tilde{a}_m(x)$, $\tilde{a}_m(x)$ nonvanishing. Observe that

$$x^m\partial^m = \theta(\theta-1)\cdots(\theta-(m-1)).$$

Example 103. $P_1 = x\partial - \lambda = \theta - \lambda$ is regular.

Example 104. $P_2 = x^2\partial - \lambda$,

$$\delta(P_2) = -1$$
$$\widehat{\delta}(P_2) = 0$$

is not regular. What fails in this example?

$$P_2 = x(x\partial) - \lambda = x\theta - \lambda,$$

and $\frac{\lambda}{x}$ is not holomorphic.

Remark When we rewrite P in this form, $\mathrm{ord}(b_i(x)) = \mathrm{ord}(a_k) - k$.

Definition 48. A P of the type

$$P = \sum_{i=0}^{m} b_i(x)\theta^i, \quad b_i(x) \in \mathscr{K}, \quad \theta = x\partial,$$

with

$$\frac{b_i(x)}{b_m(x)} \text{ is holomorphic,}$$

is called *Fuchsian*.

Remark Locally, Fuchsian differential operators are regular. Globally, however, they may not be regular – this is in fact a failure of Hilbert's 23rd problem!

17.3. **Another approach (still in the local setting).**

(1) $Pu = 0$ an ODE.
(2) $\frac{d}{dx}\vec{u}(x) = \frac{\Gamma(x)}{x}\vec{u}(x)$, where

$$\Gamma(x) = \begin{pmatrix} 0 & 1 & & \cdots & 0 & 0 \\ 0 & 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ & & & & 0 & 1 \\ \frac{-b_0}{b_m} & \frac{-b_1}{b_m} & \cdots & \cdots & \cdots & \frac{-b_{m-1}}{b_m} \end{pmatrix}$$

is a matrix in rational canonical form.

Then solutions u of the first equation are in one-to-one correspondence with solutions of the second equation

$$\vec{u}(x) = \begin{pmatrix} u(x) \\ \theta u(x) \\ \vdots \\ \theta^{m-1}u(x) \end{pmatrix}.$$

The condition now becomes

$$P \text{ is Fuchsian } \textit{if and only if } \Gamma(x) \text{ is holomorphic.}$$

18. Meromorphic connections.

Definition 49. A *meromorphic connection* is a finite dimensional $\mathscr{K}$-vector space $\mathscr{M}$, together with

$$\nabla : \mathscr{M} \to \mathscr{M}$$

satisfying the Leibniz rule

$$\nabla(fm) = \frac{df}{dx}m + f\nabla m, \quad f \in \mathscr{K}, m \in \mathscr{M}.$$

Let $\tilde{\mathscr{K}}$ denote the space of (possibly multivalued) holomorphic functions on $\Delta_\epsilon^* = \{0 < |x| < \epsilon\}$ for arbitrarily small ϵ. "Possibly multivalued" should be interpreted as *functions on the universal cover.*

This generalizes the space of meromorphic functions in two ways:

- We are allowing multivalued functions.
- We are allowing essential singularities at 0.

We will use this as a solution space, after a remark on meromorphic connections.

Remark If $\mathscr{M}$ is a meromorphic connection, pick a $\mathscr{K}$-basis $e_1, \dots, e_n$ of $\mathscr{M} \cong \mathscr{K}^n$. Then we can write

$$\nabla = d - A, \quad d \text{ the de Rham differential, } A \text{ a matrix.}$$

So horizontal sections $\vec{u}$ of $\mathscr{M}$ are the same as solutions to the equation

$$\frac{d}{dx}\vec{u}(x) = A(x) \cdot \vec{u}(x).$$

Fact: If $\mathscr{M} \cong \mathscr{K}^m$ is a meromorphic connection, then is has a complex m-dimensional space of solutions in $\tilde{\mathscr{K}}$.

If $\vec{u}_1(x), \dots, \vec{u}_m(x) \in \tilde{\mathscr{K}}$ is a basis of horizontal sections, then we say that

$$S(x) := \left(\begin{array}{c|c|c|c} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{array} \right)$$

is a *fundamental solution.*

Definition 50. We say that $f \in \tilde{\mathscr{K}}$ has *moderate growth* if for all open $(a,b) \subseteq \mathbb{R}$ and $\epsilon > 0$ such that f is defined on $S^\epsilon_{(a,b)}$ as shown in Figure 43, there exists $N >> 0$ such that

$$|f(x)| \leq C|x|^{-N} \quad \text{for all } x \in S^\epsilon_{(a,b)}.$$

Note: $\mathscr{M}$ is a $\mathcal{D}$-module via $\nabla \leftrightarrow \partial$.

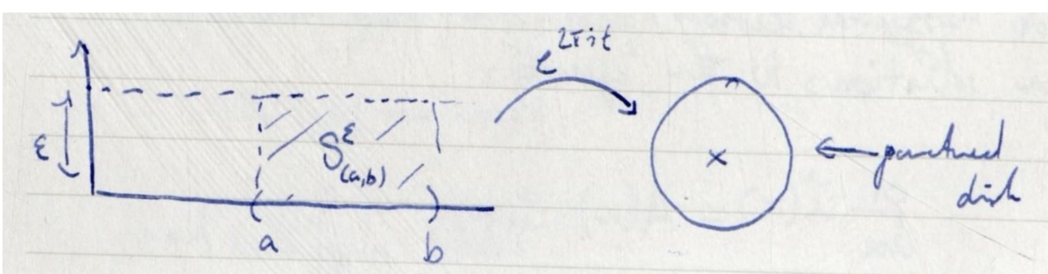


Figure 43. Universal cover of punctured disk, with strip $S^\epsilon_{(a,b)}$ shown.

Fact: If f is single valued, then moderate growth is equivalent to meromorphicity. (I.e. this is the way we say a multivalued function does not have essential singularities.)

Theorem 18.1. *For $\mathscr{M}$ a fixed meromorphic connection, the following are equivalent:*

(1) $\mathscr{M}$ is equivalent to a system of the form,

$$\frac{d}{dx}\vec{v}(x) = \frac{\Gamma(x)}{x}\vec{v}(x),$$

where $\Gamma(x) \in M_n(\mathscr{O})$ ($n \times n$ matrices).

(2) Same as above, but Γ is a constant *matrix.*

(3) All solutions to $\mathscr{M}$ in $\tilde{\mathscr{K}}^m$ have moderate growth.

Also, all of these are equivalent to $\mathscr{M}$ being regular*.*

Sketch that (3) implies (2). Let $S(x)$ be a fundamental solution matrix. Take

$$\lim_{t\to 1} S(e^{2\pi i t}x) = G \cdot S(x);$$

the LHS is another fundamental solution, so can be expressed in the form of the RHS. Call G the *monodromy matrix.* Let Γ be some matrix such that

$$e^{2\pi i \Gamma} = G,$$

where we can use the Jordan form for $G \in GL_n\mathbb{C}$ to make finding such a Γ easier. Note that $e^{\Gamma \log(x)}$ has the same monodromy as $S(x)$. So define

$$T(x) := S(x) \cdot e^{-\Gamma \log(x)}.$$

Then $T(x)$ is single valued with moderate growth (our original assumption), so $T(x)$ is meromorphic. Then $\vec{u}(x)$ is a solution of $\mathscr{M}$ if and only if $T(x)^{-1}\vec{u}(x)$ is a solution of

$$\frac{d}{dx}(-) = \frac{\Gamma(x)}{x}(-)$$

where Γ is a constant matrix. □

Remark For those who are interested, the key ingredient in (1) ⇒ (2) is *Grönwall's inequality.* We will not be discussing this.

Example 105. (1) $\frac{d}{dx}u(x) = \frac{\lambda}{x}u(x)$ has solutions $u(x) = x^\lambda$. This is a multivalued function: to make sense of it, consider choosing a branch of log, of considering it as a function on the universal cover of Δ^*. This is regular.

(2) $\frac{d}{dx}u(x) = -\frac{1}{x^2}u(x)$ has as solution the single-valued function $u(x) = e^{\frac{1}{x}}$. This is non-regular – $e^{\frac{1}{x}}$ can grow faster than any meromorphic function.

Corollary 18.2. *If $Pu = 0$ is an ODE (so $P \in \mathcal{D}$ – still considering the local situation) then P is Fuchsian if and only if all solutions to P in $\tilde{\mathscr{K}}$ have moderate growth.*

Definition 51 (/Proposition). A meromorphic connection is *regular* if there exists a finitely generated $\mathscr{O}$-submodule $L \subseteq \mathscr{M}$ such that

$$\Theta L \subseteq L, \quad \Theta = x\partial,$$

and L generated $\mathscr{M}$ over $\mathscr{K}$. We call L a *lattice.*

Remark This gives a coordinate free expression of regularity.

Given $\mathscr{M}$ as in the theorem, take a basis $e_1, \ldots, e_m$ of $\mathscr{M}$ such that the connection matrix looks like

$$\frac{\Gamma(x)}{x}, \quad \Gamma(x) \in M_n(\mathscr{O}).$$

Then take $L = \mathscr{O}e_1 + \cdots + \mathscr{O}e_m$.

Lemma 18.3. *Given L as in the definition/proposition, we have that*

$$L \cong \mathscr{O}^m \quad (L \text{ is free}).$$

Thus we can think of L as a vector bundle on the unpunctured *disk with connection*

$$\nabla : L \to L \otimes \Omega^1(d\log(x))$$

induced from the original connection on $\mathscr{M}$.

I.e. L is a vector bundle with connection that has logarithmic poles. Here

$$d\log(x) = \frac{dx}{x}.$$

Example 106. From the first example above,

$$\mathscr{M} = \mathscr{K} \quad \text{and} \quad \nabla = d - \frac{\lambda}{x}dx.$$

We already (implicitly) chose a trivialization $e \in \mathscr{M}$ to write this, and then $L = \mathscr{O}e \subseteq \mathscr{M}$.

19. Global theory of regular singularities.

There are two directions in which we could generalize this topic:

- *Irregular connections.* (Unfortunately we won't have time for this.)
- *Global theory.* (This will be our focus.)

Hilbert's 21^{st} problem: *Let $\{a_1, \dots a_k\} \subseteq \mathbb{A}^1 = \mathbb{C}$. Given $G_1, \dots, G_k \in GL_n\mathbb{C}$, does there exist a Fuchsian differential equation with singularities at $\{a_i\}$ and monodromy G_i at a_i?*

Remark In the local case, given $G \in GL_n\mathbb{C}$ we can always find $\Gamma \in M_n(\mathbb{C}) = \mathfrak{gl}_n$ such that

$$e^{2\pi i \Gamma} = G \rightsquigarrow \tfrac{d}{dx}\vec{u}(x) = \tfrac{\Gamma}{x}\vec{u}(x).$$

Note that:

(1) $\pi_1(\underbrace{\mathbb{A}^1 - \{a_1, \dots, a_k\}}_{=:U}) \cong F_k$, the free group on k letters.

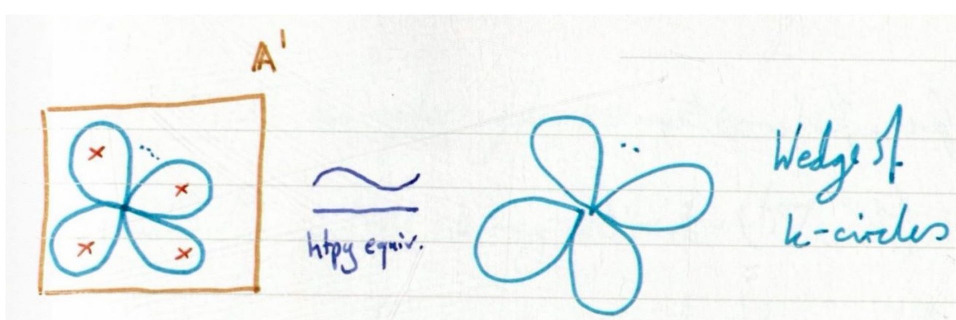


Figure 44. Determining $\pi_1(U)$.

So,

$$\{\text{Local systems on } U\} \simeq \text{Rep}(\pi_1(U)) \cong GL_n(\mathbb{C})^k / GL_n(\mathbb{C}),$$

i.e. a choice of k matrices up to simultaneous conjugation (change of basis).

(2) $\mathbb{A}^1 = \mathbb{C} \subseteq \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \cong S^2$. So we could also demand in Hilbert's 21^{st} problem that ∞ is not a singular point. For a long time, people thought that this problem would have a positive answer – in fact it does **not**, but something "very close" does.

19.1. **Deligne's "solution" to Hilbert's problem.**
First idea:

$$\mathrm{Rep}(\pi_1(U)) \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{Locally constant} \\ \text{sheaves on } U \end{array} \right\} \xleftrightarrow{\sim} \{\text{Flat connections on } U\}$$

$$\mathscr{L} \longleftrightarrow (\mathscr{O}_X \otimes_{\mathbb{C}_X} \mathscr{L}, d)$$

Given a flat connections $(\mathscr{V}^*, \nabla^*)$ on U, Deligne defined an extension $\mathscr{M}$ a meromorphic flat connection on $(\mathbb{P}^1, \{a_1, \dots, a_k\})$. I.e. $\mathscr{M}$ is a locally free $\mathscr{O}_{\mathbb{P}^1}\left[\frac{1}{x-a_1}, \cdots, \frac{1}{x-a_k}\right]$-module such that $\mathscr{M}|_U \cong \mathscr{V}^*$.

Note: For $j : U \hookrightarrow \mathbb{P}^1$ you can always take

$$j_*(\mathscr{V}^*) \in \mathscr{O}_{\mathbb{P}^1}\text{-modules.}$$

But in the world of analytic geometry, this sheaf allows for *essential singularities*, so is not a meromorphic connection. (This cannot happen in the algebraic world.) The $\mathscr{M}$ we construct will be a subsheaf

$$\mathscr{M} \subseteq j_*(\mathscr{V}^*).$$

Turns out: $\mathscr{M}$ has *regular* singularities at $a_1, \dots, a_k$; i.e. there is $L \subseteq \mathscr{M}$ a vector bundle on$\mathbb{P}^1$ such that ∇ has log poles with respect to L.

Note: L may not be trivial! So this doesn't necessarily correspond to what one might think of as a system of differential equations on the plane.

I.e. This does **not** imply that Hilbert's 21^{st} problem is true as stated above. But you could consider this the "correct" version (or "corrected" version) of the problem. Why? Because a differential equation is equivalent to

$$(\mathscr{M}, L, \nabla) \text{ together with a trivialization } L \cong \mathscr{O}^m.$$

But if L is not trivial, then the trivialization does not exist.

Remark $\mathscr{M}$ is a $\mathcal{D}$-module: take its de Rham complex,

$$\mathrm{DR}_{\mathbb{P}^1}(\mathscr{M}) = Rj_*(\underbrace{\mathrm{DR}_U(\mathscr{V}^*)}_{\text{local system on } U \text{ in degree } -1}).$$

19.2. **Higher dimensions.** If X is a complex manifold, $D \subseteq X$ a hypersurface, then Deligne proved

$$\begin{tikzcd} \mathrm{Conn}^{\mathrm{reg}}(X, D) \arrow[rr, leftrightarrow, "\sim"] & & \mathrm{Conn}(U) \\ \begin{array}{c}\text{flat connections on } X, \text{ meromorphic} \\ \text{along } D, \text{ with regular singularities}\end{array} \arrow[u, rightsquigarrow] & & \text{all flat connections on } U = X - D \arrow[u, rightsquigarrow] \end{tikzcd}$$

We call this *Deligne's Riemann-Hlbert correspondence.*

20. Algebraic story.

As far as Sam knows, there is no self-contained algebraic story – it has to pass through the analytic story.

Suppose X is projective, $X \subseteq \mathbb{P}^N$. Then X is an algebraic variety (Chow's theorem). GAGA then implies that

$$\mathrm{Conn}^{\mathrm{reg}}(X^{\mathrm{alg}}, D^{\mathrm{alg}}) \simeq \mathrm{Conn}^{\mathrm{reg}}(X^{\mathrm{an}}, D^{\mathrm{an}}) \simeq \mathrm{Conn}(U^{\mathrm{an}}).$$

Of course we can also restrict

$$\mathrm{Conn}^{\mathrm{reg}}(X^{\mathrm{alg}}, D^{\mathrm{alg}}) \xrightarrow{\sim\ (\text{defn})} \mathrm{Conn}^{\mathrm{reg}}(U^{\mathrm{alg}}) \subsetneq \mathrm{Conn}(U^{\mathrm{alg}}).$$

Example 107. On $\mathbb{A}^1 = \mathbb{C} = U$ define

$$\mathscr{V}^* = (\mathscr{O}_U, d - \lambda) \overset{\text{solutions}}{\longleftrightarrow} e^{\lambda x}.$$

$e^{\lambda x}$ is analytic and **not** algebraic (for $\lambda \neq 0$). But it *is* the solution to an algebraic equation. So $(\mathscr{O}_U, d-\lambda)$ is an *algebraic* $\mathcal{D}$*-module* on $\mathbb{A}^1$,

$$\mathcal{D}_U/\mathcal{D}_U(\partial - \lambda),$$

and these are all inequivalent for different λ. Analytically, however, $(\mathscr{V}^*)^{\text{an}} \simeq (\mathscr{O}_X^{\text{an}}, d)$ (i.e. we have equivalence for all λ).

Fact: This *algebraic* $\mathcal{D}$-module is not regular unless $\lambda = 0$.

Why? Need to embed $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. Then $e^{\lambda x} \mapsto e^{\lambda \frac{1}{w}}$ near infinity, which has an essential singularity.

Note that $\mathscr{M}^{\text{alg}} = j_*^{\text{alg}}(\mathscr{V}^*)$ is a meromorphic connection, but is not regular unless $\lambda = 0$.

21. Last remarks on Riemann-Hilbert.

We work in the analytic setting. Let X be a complex manifold, $D \subseteq X$ a complex hypersurface, $U = X - D$. Consider Figure 45:

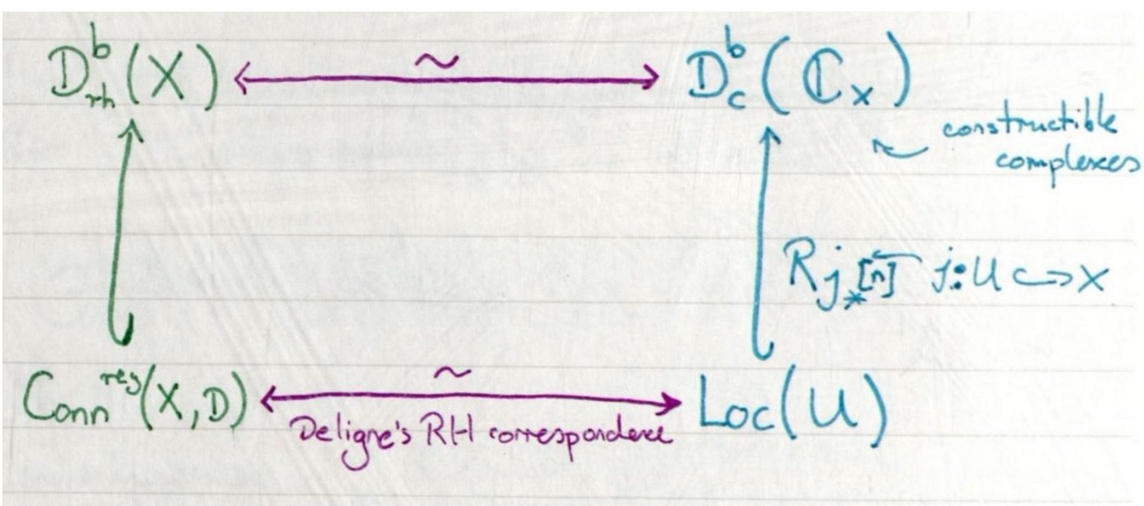


Figure 45. The Riemann-Hilbert correspondence.

Claim that this diagram in Figure 45 commutes.

Example 108. Some possible choices of (X, D):

(1) Punctured Riemann surface as in Figure 46.

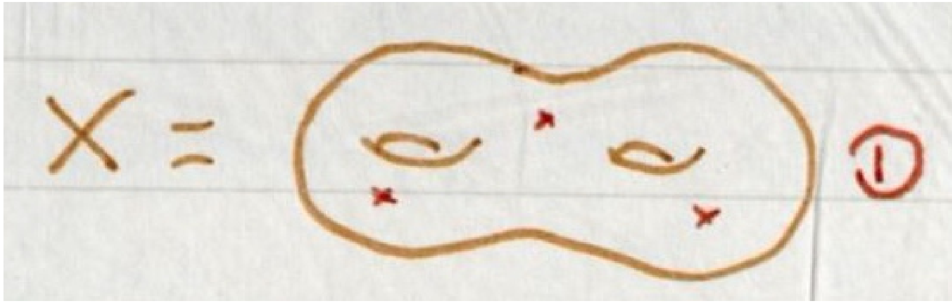


Figure 46. Riemann surface X with punctures D.

(2) $Z \subseteq X$ a closed subvariety, $E \subseteq Z$ a hypersurface such that $Z - E =: V$ is smooth – see Figure 47.

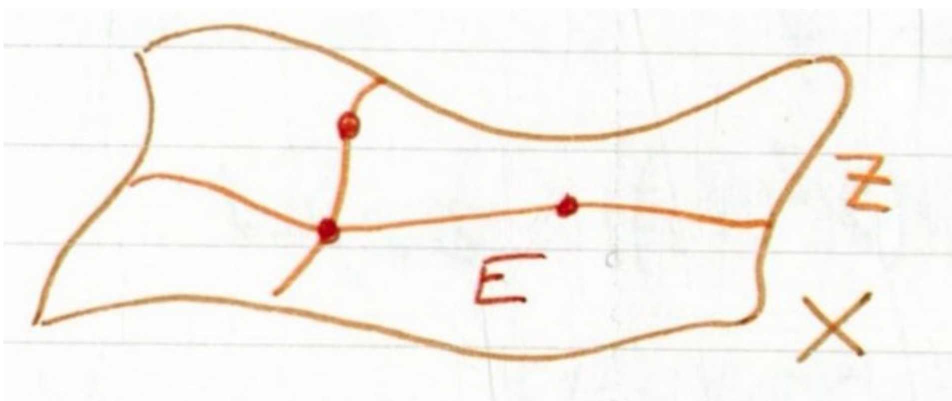


Figure 47. Situation described in example 2 above.

Let $k : V \hookrightarrow Z$, $i : Z \hookrightarrow X$. Given $\mathscr{L}$ a local system on V, we produce

$$i_* Rk_*(\mathscr{L}) \in D^b_c(X).$$

In fact, these generate the category $D^b_c(X)$ as we range over all possibilities of (Z, E).

We have an even more refined picture of the Riemann-Hilbert correspondence in Figure 48:

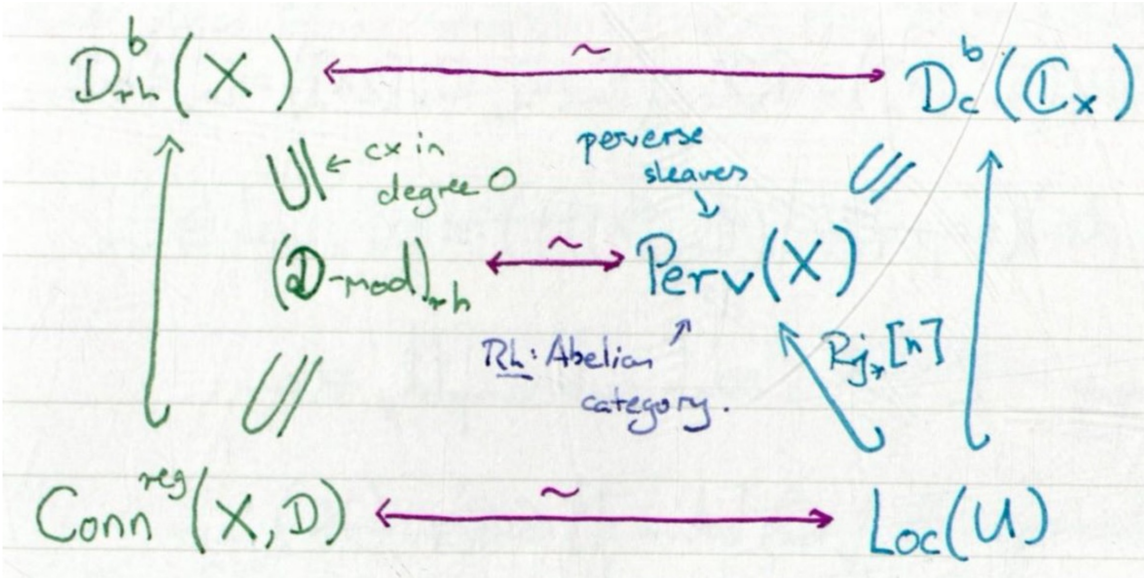


Figure 48. Refined Riemann-Hilbert correspondence.

We can define $\mathrm{Perv}(X)$ to be the image of $(\mathcal{D}\text{-mod})_{\mathrm{rh}}$ under Riemann-Hilbert, so that the equivalence in Figure 48 is tautological. But we can give an intrinsic characterisation as well.

Definition 52. An object $\mathscr{F} \in D^b_c(X)$ is called a *perverse sheaf* if

- $\dim(\mathrm{supp}(\mathscr{H}^j(\mathscr{F})) \leq -j$, and
- $\dim(\mathrm{supp}(\mathscr{H}^j(\mathbb{D}_X\mathscr{F})) \leq -j$.

Remark For X a $\mathbb{C}$-manifold with $\dim_{\mathbb{C}} X = n$,

$$\mathbb{D}_X(\mathscr{F}) = R\mathscr{H}om_{\mathbb{C}_X}(\mathscr{F}, \underbrace{\omega_X}_{\mathbb{C}_X[2n]}).$$

Example 109. $\mathbb{C}_X[n] \in \mathrm{Perv}(X)$. (Recall: $\mathbb{C}_X[n] \cong \mathrm{DR}_X(\mathscr{O}_X)$.)

- $\mathbb{D}_X(\mathbb{C}_X[n]) = R\mathscr{H}om(\mathbb{C}_X[n], \mathbb{C}_X[2n]) = \mathbb{C}_X[n]$.
- $\dim(\mathrm{supp}(\mathscr{H}^{-n}(\mathbb{C}_X[n])) = n$.

Example 110. Let $i : Z \subseteq X$ be a closed submanifold. Then

$$i_*\mathbb{C}_Z[\dim Z] \in \mathrm{Perv}(X).$$

Remark All our shifts make it so that Poincaré duality is a "symmetric flip" across degree 0, rather than a "shifted flip".

Fact: For irreducible $\mathscr{L} \in \mathrm{Loc}(U)[n]$, there is a unique irreducible subobject of

$$Rj_*(\mathscr{L}) \in \mathrm{Perv}(X).$$

We call it $IC(X, \mathscr{L}) \in \mathrm{Perv}(X)$, the *minimal extension/intermediate extension/Goresky-MacPherson extension. IC* stands for *intersection cohomology.* This has the property that

$$IC(X, \mathscr{L})|_U \cong \mathscr{L}.$$

What does this correspond to in the $\mathcal{D}$-module world?

$$\mathscr{L} \in \mathrm{Loc}(U) \leftrightarrow \mathscr{M} \in \mathrm{Conn}^{\mathrm{reg}}(X, D),$$

and there is a unique choice of lattice $L \subseteq \mathscr{M}$ such that the eigenvalues of $\Theta = f\partial_f$ have real part in $[0,1)$ (a fundamental domain for the exponential function). Then

$$IC(X, \mathscr{L}) \leftrightarrow \mathcal{D}_X \cdot L \subseteq \mathscr{M},$$

the $\mathcal{D}$-module generated by L.

Remark $\mathrm{Perv}(X)$ is an artinian abelian category. So *irreducible* means no nonzero subobjects, and such an object exists by the artinian property.

22. Wider context in maths.

Want to understand the topology of complex varieties.

Example 111. $X \xrightarrow{f} C$ a flat proper family of varieties over a curve as in Figure 49:

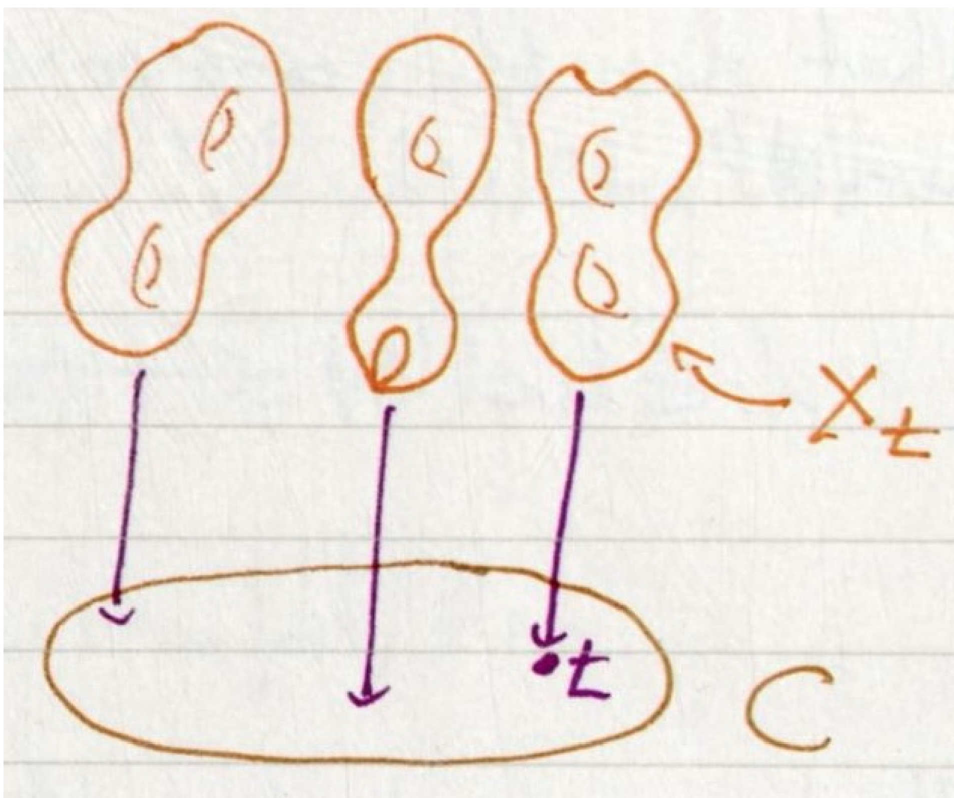


Figure 49. A flat proper family of varieties X over a curve C.

Under the Riemann-Hilbert correspondence we have

$$f_*^{\mathrm{dR}}(\mathscr{O}_X) \leftrightsquigarrow Rf_*(\mathbb{C}_X) \in D^b_c(C).$$

Generically on C, this is a vector bundle with a flat connection (the *Gauss-Manin connection*), and we have

$$R^i f_*(\mathbb{C}_X)_t = H^i(X_t; \mathbb{C}).$$

Example 112. Take the family

$$X = \{y^2 = x(x-1)(x-t)\} \xrightarrow{f} (\mathbb{P}^1, \{0, 1, \infty\})$$

whose fibre over t are the solutions to the given equation. Then

$$R^1 f_*(\mathbb{C}_X) \leftrightsquigarrow \text{Picard-Fuchs equation on } \mathbb{P}^1.$$

It is a classical result that this equation has regular singularities at $\{0, 1, \infty\}$. Solutions of this differential equation are *hypergeometric functions.*

If X is a compact Kähler manifold, then

$$H^i(X; \mathbb{C}) = \bigoplus_{p+q=i} \underbrace{H^q(X; \Omega_X^p)}_{=:H^{p,q}(X)}.$$

This is called a *Hodge structure.* What happens to this structure as we vary X in a family?

Remark The Hodge structure also tells us how $H^i(X;\mathbb{Z}) \subseteq H^i(X;\mathbb{C})$ intersects with the decomposition.

Theorem 22.1 (A Torelli Theorem)**.** *If X is an elliptic curve, then X is determined by $H^1(X)$ with Hodge structure:*

$$\begin{array}{ccccc} H^1(X;\mathbb{Z}) & \longrightarrow & H^{0,1}(X) & \xrightarrow{\text{quotient}} & X \\ \shortparallel & & \shortparallel & & \\ \mathbb{Z}^2 & & \mathbb{C} & & \end{array}$$

Hodge filtration: (Warning: Check that the p's and q's are correct below!)

$$F^p H^i(X, \mathbb{C}) = \bigoplus_{\substack{p' \geq p \\ p'+q=i}} H^{p',q}(X).$$

As the fibres X_t vary with t, the $F^p H^i(X_t)$ form a holomorphic subbundle of the corresponding flat connection on, e.g., $\mathbb{P}^1 - \{0, 1, \infty\}$ (i.e. on the smooth locus).

This leads to the notion of a *variation of Hodge structure*,

$$\begin{array}{ccccc} (\mathscr{V} & \nabla & F^\bullet & V_{\mathbb{Z}} & \alpha : \ker(\nabla) \xrightarrow{\sim} (V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C})), \\ \uparrow & & \updownarrow & \uparrow & \\ \text{vector bundle w/ flat conn.} & & \text{filtration of } \mathscr{V} & \text{locally const. sheaf of free abelian groups} & \end{array}$$

together with a condition called *Griffiths transversality.* ∇ does not preserve the filtration (since e.g. this would contradict the above Torelli theorem). Instead, we have the *Griffiths transversality theorem*:

$$\nabla F^i \subseteq F^{i-1}.$$

Then we have an analogy

$$\begin{array}{ccc} \mathcal{D}\text{-module} & : & \text{vector bundle with flat connection} \\ & \updownarrow & \\ \text{Hodge module} & : & \text{variation of Hodge structure} \end{array}$$

22.1. **Further topics.**

- Mixed Hodge modules (nearby/vanishing cycles)
- Moduli of flat connections/Moduli of Higgs bundles (Geometric Langlands)
- Beilinson-Bernstein (e.g. representations of $\mathfrak{sl}_2$ with trivial central character correspond to $\mathcal{D}$-modules on $\mathbb{P}^1$)

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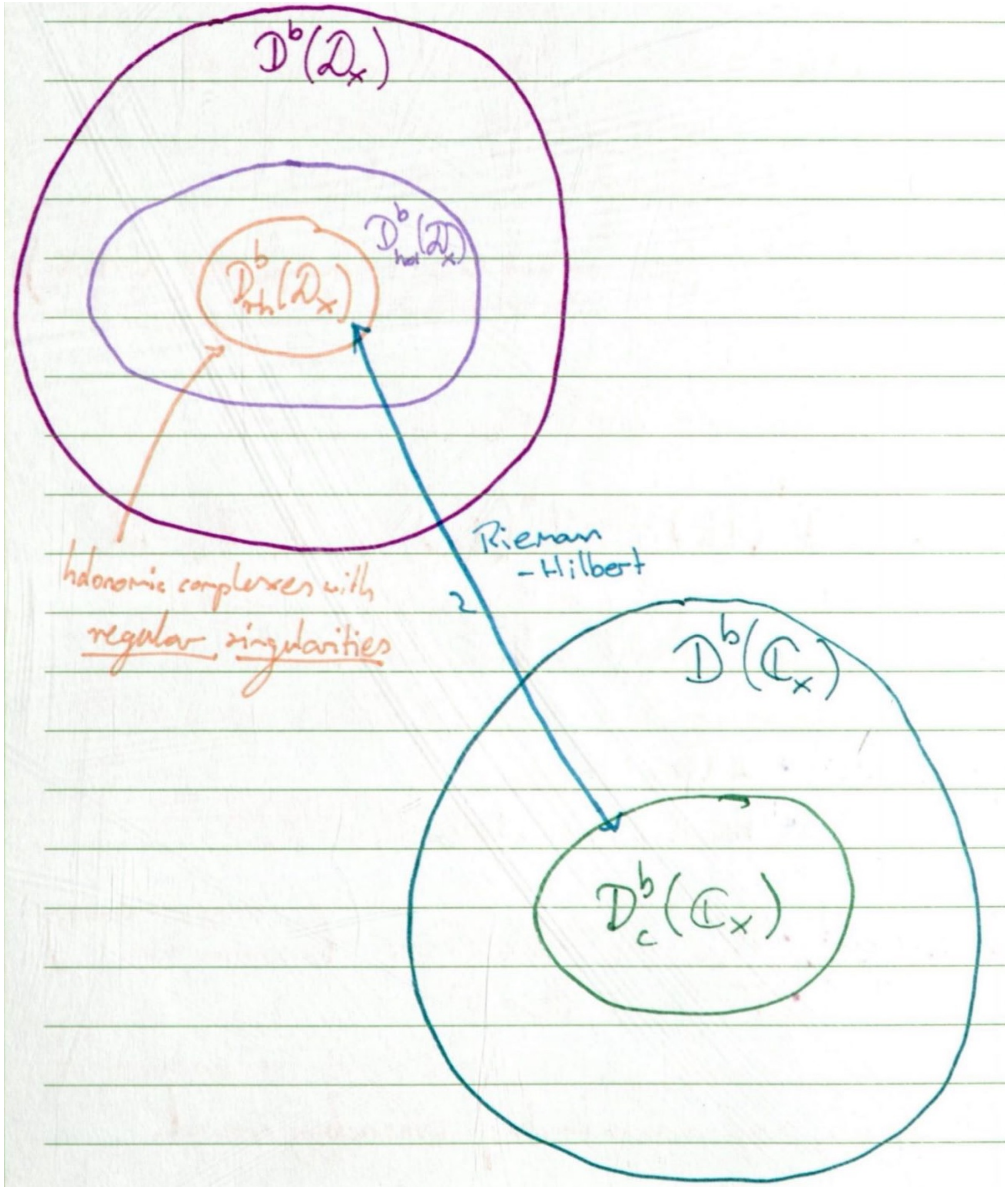

FIGURE 36. The Riemann-Hilbert correspondence.