QFT for Mathematicians (Workshop)

Richard Derryberry

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Notes from the QFT for Mathematicians workshop at PI.

References are pretty much entirely missing. They need to be added in.

Ben Webster's intro:

How will this work/what's the idea? Want to start at the beginning and get a sort of global view of the mathematical underpinnings of QFT.

People are encouraged to ask questions.

Contents

1 Factorization Algebras and the General Structure of QFT

1.1 Lecture 1 (Philsang Yoo)

1.1.1 The Plan (this lecture series)

In this lecture series:

- What sort of thing is a QFT?
- What shorts of things can one do with a QFT?

Some dichotomies to consider – we will be considering the italicised versions:

- non-perturbative vs *perturbative*
- unitary vs *complexified*
- Lorentzian vs *Euclidean*
- states vs *observables*

In particular: from the beginning we are excluding the non-perturbative approach. The goal is to be entirely rigorous, and we don't have a rigorous non-perturbative approach yet.

Remark 1.1.

- Even at the level of physics there is not a good framework for non-perturbative theories.
- Mathematically, complexified and Euclidean theories are important.
- The observables capture much of the important information in the theory.

Note that often in physics people often care about perturbative/unitary/Lorentzian (states vs observables is a little less clear).

1.1.2 The Plan (today)

Today we will focus on

1. Classical BV
2. Quantum BV

and we'll focus on the 0-dimensional case.

1.1.3 Classical field theory

For (d-dimensional) classical field theory, we have as input:

a. M^d a spacetime manifold

b. $\mathcal{F} = \mathcal{F}(M)$ a space of fields (usually sections of a bundles over M)

c. $S : \mathcal{F} \to \mathbb{C}$ an action functional

The output of this is

$$\mathrm{crit}(S) = \{\alpha \in \mathcal{F} \mid dS(\alpha = 0)\} \tag{1.1}$$

where dS are the "equations of motion".

Example 1.1 (Free scalar field theory)**.**

$$M = (M^d, g)$$
$$\mathcal{F}(M) = C^\infty(M)$$

(Note: not worrying about the precise space of fields for the moment – at first approximation use smooth functions.)

$$S : \begin{array}{ccc} \mathcal{F}(M) & \longrightarrow & \mathbb{C} \\ \phi & \longmapsto & \int_M \phi(D + m^2)\phi \end{array}$$

where D is the Laplacian. Then

$$\mathrm{crit}(S) = \{\phi \in C^\infty(M) \mid (D + m^2)\phi = 0\}.$$

(Note: not necessarily entirely rigorous – e.g. wouldn't be quite correct for non-closed manifolds – but we'll worry about that later, perhaps.)

Example 1.2 (Chern-Simons theory)**.**

$$M = M^3$$
$$\mathcal{F}(M) = \Omega^2(M; \mathfrak{g})$$

$$S : \begin{array}{ccc} \mathcal{F}(M) & \longrightarrow & \mathbb{C} \\ A & \longmapsto & \frac{1}{2}\int_M \langle A, dA\rangle + \frac{1}{6}\int_M \langle A, [A, A]\rangle \end{array}$$

Then

$$\mathrm{crit}(S) = \{A \in \Omega^1(M; \mathfrak{g}) \mid F(A) := dA + \frac{1}{2}[A, A] = 0\}$$

i.e. flat connections. (Note: A might not necessarily be a global 1-form, but the critical point equations will still single out flat connections.)

1.1.4 Classical field theory – BV formalism

From now on, fix

$$M = \mathrm{pt}$$
$$\mathcal{F}(M) = \mathcal{F}(\mathrm{pt}) = X \text{ a manifold (finite dimensional)}$$

and

$$S : \mathcal{F}(M) = X \to \mathbb{C}.$$

We want to revisit $\mathrm{crit}(S)$ **following the BV formalism.**

Note

$$\mathrm{crit}(S) = \{dS = 0\} = \mathrm{Graph}(dS) \times_{T^*X} X.$$

We want to consider $d\,\mathrm{Crit}(S)$ the *derived critical locus*. On the level of functions,

$$\mathcal{O}(\mathrm{Crit}(S)) = \mathcal{O}(\mathrm{Graph}(dS)) \otimes_{\mathcal{O}(T^*X)} \mathcal{O}(X);$$

but because we want to consider the derived critical locus, we will derive this tensor product:

$$\mathcal{O}(d\,\mathrm{Crit}(S)) = \mathcal{O}(\mathrm{Graph}(dS)) \otimes^{\mathbb{L}}_{\mathcal{O}(T^*X)} \mathcal{O}(X). \tag{1.2}$$

Problem 1. *Show* $\mathcal{O}(d\,\mathrm{Crit}(S)) \simeq (\mathcal{O}(T^*[-1]X), \iota_{dS})$.

Special case: $S = 0$: $\mathcal{O}(X) \otimes^{\mathbb{L}}_{\mathcal{O}(T^*X)} \mathcal{O}(X)$. Resolve $\mathcal{O}(X)$ with the Koszul resolution.

$$\mathcal{O}(X) \simeq \mathrm{Sym}_{\mathcal{O}_X}(T_X[1] \xrightarrow{\mathrm{id}} T_X),$$

and so the derived tensor product is

$$\mathrm{Sym}_{\mathcal{O}(X)} T_X[1] = \mathcal{O}(T^*[-1]X),$$

functions on the (-1)-shifted cotangent bundle.

So:

$$\mathcal{O}(T^*[-1]) = \mathrm{Sym}_{\mathcal{O}(X)}(T_X[1]),$$
$$\mathrm{PV}(X) = \bigoplus_{k=0}^{n} \Gamma(X, \wedge^k T_X)[k] = \bigoplus \mathrm{PV}^k[k]$$

where PV is *polyvector fields*.

These are equipped with the *Schouten-Nijenhuis bracket* $\{-,-\}$, defined by

- $\xi_1, \xi_2 \in \mathrm{PV}^1$: $\{\xi_1, \xi_2\} = [\xi_1, \xi_2]$
- $f, g \in \mathrm{PV}^0$, $\xi \in \mathrm{PV}^1$: $\{\xi, f\} = \xi(f)$, $\{f, g\} = 0$
- Extend by Leibniz rule.

This gives a Poisson bracket of cohomological degree 1.

Problem 2. *A (-1)-shifted symplectic structure yields $\{-,-\}$ a Poisson bracket of degree 1.*

Problem 3. *Check that*[1] $\iota_{dS} = \{S, -\}$.

So the input data yields,

$$(\mathcal{O}(T^*[-1]X), \iota_{dS} = \{S,-\}, \{-,-\}). \tag{1.3}$$

Definition 1.1. A $\mathbb{P}_0$-*algebra* is $(A, d, \{-,-\})$ where (A, d) is a cdga[2] and $\{-,-\}$ is a Poisson bracket of degree 1.

Poisson bracket: (A, d) cdga, $\{-,-\}$ Poisson satisfies

$$d\{a, b\} = \{da, b\} + (-1)^{|a|}\{a, db\}.$$

Problem 4. $(\mathcal{O}(T^*[-1]X), \iota_{dS} = \{S,-\}, \{-,-\})$ *is a* $\mathbb{P}_0$*-algebra.*

Remark 1.2. $T^*[-1]X$ is a derived "space" – work homotopically, so e.g. rather than working with usual functions one should appropriately (homotopically) resolve functions as a cdga (not an arbitrary cdga, but can worry about this later maybe).

[1] ι is contraction operator.

[2] commutative differential graded algebra

1.1.5 Gauge Theory

We will work with the "stack"

$$X = V/G$$

We are working with *perturbation theory* – i.e. we want to do perturbation theory about a specific point in V. So we fix $0 \in V$. Now by functions on V we mean things which can be seen *near* this point:

$$\mathcal{O}(V) = \widehat{\mathrm{Sym}}(V^*)$$

Moreover this formal information is captured by the Lie algebra of the group, so we can replace G by $\mathfrak{g}$. Now by functions on X we mean invariant functions on V:

$$\mathcal{O}(X) = \mathcal{O}(V)^{\mathfrak{g}} = \mathrm{Hom}_{U\mathfrak{g}}(\mathbb{C}, \mathcal{O}(V))$$

But of course we want everything to be derived! So really,

$$\mathcal{O}(X) = \mathbb{R}\,\mathrm{Hom}_{U\mathfrak{g}}(\mathbb{C}, \mathcal{O}(V)) = C^\bullet(\mathfrak{g}, \mathcal{O}(V)), \tag{1.4}$$

which we call *Chevalley-Eilenberg cochains.*

If $\mathfrak{g}$ acts on a module M, then **as a graded vector space**

$$C^\bullet(\mathfrak{g}, M) = \widehat{\mathrm{Sym}}(\mathfrak{g}^*[-1]) \otimes M,$$

and the differential comes by taking the maps

$$\mathfrak{g} \otimes M \to M, \qquad , \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

turning them into maps

$$\begin{aligned} M &\to \mathfrak{g}^* \otimes M, \\ \mathfrak{g}^* &\to \mathfrak{g}^* \otimes \mathfrak{g}^*, \end{aligned}$$

and then extending by the Leibniz rule.

So, as a graded vector space,

$$\begin{aligned} C^\bullet(\mathfrak{g}, \mathcal{O}(V)) &= \widehat{\mathrm{Sym}}(\mathfrak{g}^*[-1]) \otimes \widehat{\mathrm{Sym}}(V^*) \\ &= \widehat{\mathrm{Sym}}(\mathfrak{g}^*[-1] \oplus V^*) \\ &= \widehat{\mathrm{Sym}}((\mathfrak{g}[1] \oplus V)^*) \\ &= \mathcal{O}(\mathfrak{g}[1] \oplus V). \end{aligned}$$

The differential d_{CE} becomes a vector field ξ_{CE} of cohomological degree 1, satisfying

$$[\xi_{CE}, \xi_{CE}] = 0.$$

So: if we start with $X = V/G$ where G is "gauge symmetry" and work perturbatively at the point $0 \in V$, we map replace V/G by

$$\mathfrak{g}[1] \oplus V$$

equipped with a vector field ξ of cohomological degree 1.

1.1.6 Classical Master Equation

Now apply 1.1.5 to 1.1.4. Take

$$T^*[-1]X \qquad \text{with} \qquad X = V/G,$$

so obtain

$$\begin{aligned}\mathcal{E} &= T^*[-1](\mathfrak{g}[1] \oplus V)\\ &= \underbrace{\mathfrak{g}[1]}_{\text{ghost field}} \oplus \underbrace{V}_{\text{field}} \oplus \underbrace{V^*[-1]}_{\text{anti-field}} \oplus \underbrace{\mathfrak{g}^*[-2]}_{\text{anti-ghost}}\end{aligned}$$

So

$$\mathcal{O}(\mathcal{E}) = \mathcal{O}(T^*[-1]X)$$

has two sources of differential:

$$\begin{aligned}&\iota_{dS} = \{S, -\}\\ &\xi_{CE} \to \xi_{CE} = \{S_{\text{gauge}}, -\} \qquad\qquad \text{symplectic vector field}\end{aligned}$$

Note that:

$$\begin{aligned}\{S, S\} &= 0,\\ \{S_{gauge}, S\} &= 0,\\ \{S_{gauge}, S_{gauge}\} &= 0\end{aligned}$$

(The second equation comes from invariance of the classical action.)

Now define:

$$S_{BV} = S + S_{gauge} \tag{1.5}$$

Then $\{S_{BV}, S_{BV}\} = 0$.

So now we have a $\mathbb{P}_0$-algebra

$$(\mathcal{O}(T^*[-1]X), \{S_{BV}, -\}, \{-, -\})$$

For future reference, write

$$S_{BV} = \underbrace{S_{free}(\alpha)}_{\deg 2} + \underbrace{I(\alpha)}_{\deg \geq 3} \tag{1.6}$$

If

$$S_{free}(\alpha) = \frac{1}{2}\omega(\alpha, Q\alpha),$$

for a linear $Q : \mathcal{E} \to \mathcal{E}$, then

$$\{S_{BV}, -\} = Q + \{I, -\}.$$

More general perspective ahead!

Suppose we have a (-1)-shifted (derived) symplectic space $(\mathcal{O}(\mathcal{E}), Q, \{-, -\})$, and suppose that the the underlying derived space is a linear space $(\mathcal{E}, Q)$.

Definition 1.2. $I \in \mathcal{O}(\mathcal{E})$ is said to satisfy the *classical master equation (CME)* if

$$QI + \frac{1}{2}\{I, I\} = 0.$$

Claim: If I is a solution to the CME then

$$(\mathcal{O}(\mathcal{E}), Q + \{I, -\}, \{-, -\})$$

is a $\mathbb{P}_0$-algebra.

Upshot: So we have another way to construct a $\mathbb{P}_0$-algebra – we could have just started with the (-1)-shifted derived symplectic space.[3]

1.1.7 Chern-Simons theory revisited

Recall the setup:

- $M = M^3$
- $\mathcal{F}(M) = \Omega^1(M, \mathfrak{g})$
- $S : \mathcal{F}(M) \to \mathbb{C}$ given by

$$A \mapsto \frac{1}{2}\int_M \langle A, dA\rangle + \frac{1}{6}\int_M \langle A, [A, A]\rangle$$

Some observations – let's simplify matters first by working in the abelian case $\mathfrak{g} = \mathbb{C}$:

(1) The theory has gauge symmetry (1.1.5). $X = V/G = V \oplus \mathfrak{g}[1]$, and the gauge symmetry part is

$$\Omega^0(M)[1] \xrightarrow{d=\{S_{gauge}, -\}} \Omega^1(M)$$

(2) Running the BV formalism on $T^*[-1]X$ we have

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d=\{S,-\}} \Omega^2 \xrightarrow{d} \Omega^3$$

For arbitrary $\mathfrak{g}$ we get $(\Omega^\bullet(M; \mathfrak{g})[1], d)$.

1.1.8 Classical field theory to perturbative QFT

Idea:

$$\begin{tikzcd}
\text{Classical field theory} \arrow[r, dashed] \arrow[d, dotted] & \text{perturbative QFT} \arrow[d, dotted] & \\
d\,\mathrm{Crit} \arrow[r, dashed, "\text{picking a solution}"] & \text{formal moduli} \arrow[r, equal, "\text{Koszul duality}"] & \text{DG Lie algebra (or } L_\infty)
\end{tikzcd}$$

Example 1.3. For Chern-Simons the bottom row of this is

$$\mathrm{Loc}_G(M) \overset{P}{\dashrightarrow} \mathrm{Loc}_G(M)_P\hat{} \underset{C^\bullet(\mathfrak{g})}{\overset{[-1]}{\rightleftarrows}} (\Omega^\bullet(M) \otimes \mathfrak{g}_P, d_P)$$

So in $\Sigma[-1] = \Omega^\bullet \otimes \mathfrak{g}$, with fields $X, A, A^\vee, X^\vee$,

$$S_{BV}(\alpha) = \frac{1}{2}\int \langle \alpha, d\alpha\rangle + \frac{1}{6}\int \langle \alpha, [\alpha, \alpha]\rangle$$

[3] Check: was this the upshot?

for $\alpha \in \Omega^\bullet \otimes \mathfrak{g}$. The S_{gauge} contribution to this are the terms

$$\int \langle [X, A], A^\vee \rangle$$
$$\int \langle [X, X], X^\vee \rangle$$

– it is somewhat of a "miracle" of Chern-Simons theory that the resulting S_{BV} winds up looking precisely the same as the original action (just with terms now arising from forms located at every stage of the complex, not just the connections in Ω^1).

1.2 Lecture 2 (Kevin Costello)

Idea for today: Kevin wants to tell us how to go from

$$\text{Integration by parts (in } \infty \text{ dimensions)} \longrightarrow \text{Factorization Algebras}$$

1.2.1 Homological Integration

Take as input:

- M manifold
- $S \in C^\infty(M)$
- $d\mu \in \Omega^n(M)$ a measure

Then can consider the operation[4]

$$\int e^{S/\hbar} d\mu(-) : C^\infty(M) \to \mathbb{R}((\hbar)).$$

Can consider this as an *algebraic* operation by observing that

$$\int e^{S/\hbar} d\mu(\mathrm{Div}(V)) = 0$$

In local coordinates x_i,

$$d\mu = dx_1 \wedge \cdots \wedge dx_n,$$

and

$$V = \sum f_i \partial_{x_i},$$

then

$$\mathrm{Div}_{e^{S/\hbar} d\mu} V = \sum \frac{\partial f}{\partial x_i} + \underbrace{\frac{1}{\hbar} \sum f_i \frac{\partial S}{\partial x_i}}_{\text{dominates at } \hbar \text{ small}}$$

Lemma 1.1. *If S has 1 isolated critical point, then*

$$C^\infty(M)/\mathit{Im}(\mathrm{Div}) = \mathbb{R}((\hbar)).$$

[4]Think the target should be $\mathbb{R}((\hbar))$ – $\mathbb{R}$ was written during the lecture.

1.2.2 Divergence Complex

Can extend the observation $\int e^{S/\hbar}d\mu(\mathrm{Div}(V)) = 0$ by defining the *Divergence Complex* as the top line of the following isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & C^\infty(M, \wedge^2 TM) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Vect}(M) \arrow[r, "\mathrm{Div}"] \arrow[d, "\simeq"] & C^\infty(M) \arrow[d, "\simeq"', "e^{S/\hbar}d\mu"] \\
\cdots \arrow[r] & \Omega^{n-2}(M) \arrow[r] & \Omega^{n-1}(M) \arrow[r, "d"] & \Omega^n(M)
\end{tikzcd}$$

Why do this? In infinite dimensions the top line exists, while there is no corresponding notion of "top form".

Remark 1.3. The notation Div will always mean $\mathrm{Div}_{e^{S/\hbar}d\mu}$.

In coordinates x_i this complex is

$$\mathbb{R}[x_i, \epsilon^i], \quad |\epsilon^i| = -1$$

with differential

$$\sum \underbrace{\frac{\partial}{\partial x_i}\frac{\partial}{\partial \epsilon^i}}_{\text{"BV Laplacian"}} + \frac{1}{\hbar}\frac{\partial S}{\partial x_i}\frac{\partial}{\partial \epsilon^i}$$

We've set this up in finite dimensions – now let's blindly apply it in infinite dimensions and see what we get!

1.2.3 How does this work in ∞ dimensions?

Let X be a Riemannian manifold, and if $\varphi \in C^\infty(X)$, define

$$S(\varphi) = \int_X \varphi \Delta \varphi.$$

Remark 1.4. $C^\infty(X)$ is the infinite dimensional replacement for M.

We want to consider "$\int_{\varphi \in C^\infty(X)} e^{S(\varphi)/\hbar} d\mu$" – square quotes because of the non-existent infinite dimensional Lebesgue measure $d\mu$.[5]

We have to replace $\mathbb{R}[x_i]$. Need to consider polynomial functions of φ. If $U \subseteq X$ and $f \in C_c^\infty(U)$, define a distribution by

$$\mathcal{O}_f(\varphi) = \int_U f(x)\varphi(x) d\mathrm{vol}.$$

Polynomial functions of φ are expressions like

$$\mathcal{O}f_1\mathcal{O}_{f_2}\cdots\mathcal{O}_{f_n}, \quad f_i \in C_c^\infty(U).$$

This gives us our replacement for polynomial functions – now we also need to introduce an replacement for vector fields.

Analog of $\mathbb{R}[x_i, \epsilon^i]$. These will be polyvector fields, i.e. polynomial functions on $\mathbb{R}^n \oplus \mathbb{R}^n[-1]$.

Introduce new objects $\psi \in C^\infty(X)[-1]$. If $g \in C_c^\infty(U)$,

$$\mathcal{O}_g^\star(\psi) = \int_U g\psi.$$

[5]Or not necessarily existent – for 1d X there are ways to construct this.

Then the analog of $\mathbb{R}[x_i, \epsilon^i]$ is given by expressions like

$$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^\star_{g_1} \cdots \mathcal{O}^\star_{g_m}, \quad f_i, g_j \in C_c^\infty(U).$$

The above expression is in degree $-m$.

Example 1.4. Vector fields on $C^\infty(X)$ look like

$$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^\star_g.$$

This expression gives a derivation of the algebra of polynomial functions

$$\mathcal{O}_{h_1} \cdots \mathcal{O}_{h_m} \mapsto \sum \left(\int_U g \cdot h_i \right) \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_m} \mathcal{O}_{h_1} \cdots \widehat{\mathcal{O}_{h_i}} \cdots \mathcal{O}_{h_m}$$

Remark 1.5. Kevin has been hiding a little that we aren't looking at *all* vector fields, due to the compact support condition. Since $C^\infty(U)$ is a vector space, $TC^\infty(U) = C^\infty(U)$ at every point, and so polynomial vector fields should look like

$$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^\star_g$$

with $g \in C^\infty(U)$. We want $g \in C_c^\infty(U) \subset C^\infty(U)$. Why?

Fields(U) is a sheaf on X. **But**, S is not really a function! An expression like $\int \varphi \Delta \varphi$ does not converge.

Fields(U) has a foliation by 1st order variations with compact support, and

$$dS \in T^*_C \operatorname{Fields}(U)$$

is a closed 1-form along the leaves. (Here $T_C \operatorname{Fields}(U)$ is the subbundle of the tangent bundle defining the foliation.)

If $V \in \operatorname{Vect}(\operatorname{Fields}(U))$, divergence involves a contraction

$$V \vee dS,$$

but this only makes sense for V *pointing along the leaves*, i.e. $V \in \Gamma(\operatorname{Fields}(U), T_C \operatorname{Fields}(U))$.

How do we define divergence? In finite dimensions, if

$$S = \sum A^{ij} x_i x_j,$$

then

$$\operatorname{Div}\left(\sum f_i \partial_{x_i}\right) = \sum \frac{\partial f_i}{\partial x_i} + \frac{1}{\hbar} \sum A^{ij} f_i x_j.$$

In ∞ dimensions, if

$$S(\varphi) = \int \varphi \Delta \varphi,$$

then[6]

$$\operatorname{Div} : \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^\star_g \mapsto \sum \mathcal{O}_{f_1} \cdots \widehat{\mathcal{O}_{f_i}} \cdots \mathcal{O}_{f_n} \int g f_i + \frac{1}{\hbar} \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_{\Delta g}.$$

Remark 1.6. Once again, this will only make sense if we apply the compact support condition to g.

More generally we can define a divergence complex, where elements are of the form

$$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^\star_{g_1} \cdots \mathcal{O}^\star_{g_m}$$

and the differential sends

$$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_m} \mathcal{O}^\star_{g_1} \cdots \mathcal{O}^\star_{g_m} \mapsto \sum_{i,j} \pm \mathcal{O}_{f_1} \cdots \widehat{\mathcal{O}_{f_i}} \cdots \mathcal{O}_{f_n} \mathcal{O}^\star_{g_1} \cdots \widehat{\mathcal{O}^\star_{g_j}} \cdots \mathcal{O}^\star_{g_m} \int f_i g_j + \frac{1}{\hbar} \sum \pm \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_{\Delta g_j} \mathcal{O}^\star_{g_1} \cdots \widehat{\mathcal{O}^\star_{g_j}} \cdots \mathcal{O}^\star_{g_m}$$

[6] $x_{i_1} \cdots x_{i_n} \partial_{x_j}$ is analogous to $\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^\star_g$.

Definition 1.3. If $U \subseteq X$ we let $\mathrm{Obs}^q(U)$ be this cochain complex.

Definition 1.4. A *prefactorisation algebra* on X is an assignment of a cochain complex $\mathcal{F}(U)$ to each open $U \subseteq X$, and if

$$U_1, \dots, U_n \subseteq V$$

and the U_i are disjoint, then we have a cochain map

$$\bigotimes \mathcal{F}(U_i) \to \mathcal{F}(V).$$

These cochain maps satisfy an associativity constraint: if

$$W_1, \dots, W_n \subseteq V$$
$$U_{ij} \subset W_j, \quad i = 1, \dots, m_j,$$

with the U_{ij} disjoint and the W_k disjoint, then

$$\begin{tikzcd} \bigotimes \mathcal{F}(U_{ij}) \arrow[r] \arrow[d] & \mathcal{F}(V) \\ \bigotimes \mathcal{F}(W_k) \arrow[ur] & \end{tikzcd}$$

commutes.

This definition captures some familiar structures.

Example 1.5. If $\mathcal{F}$ is "topological" on $\mathbb{R}^n$ then $\mathcal{F}$ gives an E_n-algebra. In particular, on $\mathbb{R}$ it yields an associative[7] algebra.

Example 1.6. If $\mathcal{F}$ is "holomorphic" on $\mathbb{C}$, then $\mathcal{F}$ yields a vertex algebra.

Remark 1.7. Precisely what conditions are required for "topological" and "holomorphic" will be left a little vague for now.

Example 1.7 (On $\mathbb{R}$)**.** If $I \subseteq J$ are intervals, suppose that $\mathcal{F}(I) \to \mathcal{F}(J)$ is always a quasi-isomorphism. In particular

$$H^*(\mathcal{F}(a,b)) = H^*(\mathcal{F}(-\infty,\infty)) =: A$$

is an associative algebra. The associative product $A \otimes A \to A$ is induced by the inclusion of two disjoint intervals in a larger interval. (Note that this is not commutative – we cannot continuously exchange the two disjoint intervals while keeping them disjoint.)

Lemma 1.2. $U \mapsto \mathrm{Obs}^q(U)$ *is a prefactorisation algebra.*

Key point: $\mathrm{Obs}^q(U)$ is a commutative algebra, but the differential is *not* a derivation (i.e. does not respect the product).

Example 1.8. $\mathcal{O}_f \mathcal{O}_g^\star \mapsto \frac{1}{\hbar}\mathcal{O}_f \mathcal{O}_{\Delta g} + \left(\int fg\right) \cdot 1$ (acts as a second order differential operator).

If $U_1, U_2 \subset V$ are disjoint and $\alpha \in \mathrm{Obs}^2(U_1)$, $\beta \in \mathrm{Obs}^q(U_2)$, then using $\mathrm{Obs}^q(U_i) \subset \mathrm{Obs}^q(V)$ these satisfy

$$d(\alpha\beta) = (d\alpha)\beta \pm \alpha d\beta;$$

so the differential respects the product on elements with disjoint support.

Example 1.9. Set $\alpha = \mathcal{O}_f$ and $\beta = \mathcal{O}_g^\star$, $f \in C_c^\infty(U_1)$ and $g \in C_c^\infty(U_2)$, then

$$d(\mathcal{O}_f \mathcal{O}_g^\star) = \frac{1}{\hbar}\mathcal{O}_f \mathcal{O}_{\Delta g} + \underbrace{\left(\int fg\right) \cdot 1}_{=0 \text{ if disjoint support}}$$

Remark 1.8. In fact we even get a factorisation algebra – the extra local-to-global axiom is a little cumbersome however, and not particularly useful for us at the moment.

[7] $A_\infty = E_1$

1.2.4 Canonical commutation relations

Consider the free scalar field on $\mathbb{R}$:

- $X = \mathbb{R}$
- $\varphi \in C^\infty(\mathbb{R})$
- $S(\varphi) = \int_{\mathbb{R}} \varphi \Delta \varphi$

Claim: $H^*(\mathrm{Obs}^q(a,b))$ is $\mathbb{R}[p,q]$ with $[p,q] = \hbar$.

What are linear observables? $\mathcal{O}_f$ for $f \in C_c^\infty((a,b))$. Then in cohomology

$$[\mathcal{O}_{f_1}] = [\mathcal{O}_{f_2}]$$

if there exists g such that

$$\frac{\partial^2}{\partial x^2} g = f_1 - f_2,$$

as $d\mathcal{O}_g^\star = \mathcal{O}_{\Delta g}$. Then

$$C_c^\infty((a,b))/\partial_x^2 C_c^\infty((a,b))$$

is 2-dimensional, spanned by

- f a bump function normalised to $\int f = 1$, and
- $\frac{\partial f}{\partial x}$

Set $q = \mathcal{O}_f$, with $\int f = 1$, and $p = \mathcal{O}_{f'}$,

$$p(\varphi) = \int f' \varphi = -\int f \varphi'.$$

To make life easier introduce δ-functions. Represent

$$q = \mathcal{O}_{\delta_{x=0}}$$
$$p = \mathcal{O}_{\delta'_{x=0}}$$

How do we compute $[p,q]$?

$$\mathcal{O}_{\delta_{x=0}} \mathcal{O}_{\delta'_{x=\epsilon}} - \mathcal{O}_{\delta_{x=0}} \mathcal{O}_{\delta'_{x=-\epsilon}}$$

Notice that

$$d\left(\hbar \mathcal{O}^\star_{\delta_{[-\epsilon,\epsilon]}}\right) = \mathcal{O}_{\partial_x^2 \delta_{[-\epsilon,\epsilon]}} = \mathcal{O}_{\delta'_{x=\epsilon}} - \mathcal{O}_{\delta'_{x=-\epsilon}}$$

is a homotopy between measuring momentum at $x = \pm\epsilon$. Then

$$d\left(\hbar \mathcal{O}_{\delta_{x=0}} \mathcal{O}^\star_{\delta_{[-\epsilon,\epsilon]}}\right) = \mathcal{O}_{\delta_{x=0}} (\mathcal{O}_{\delta'_{x=\epsilon}} - \mathcal{O}_{\delta'_{x=-\epsilon}}) + \hbar \int \delta_{x=0} \delta_{[-\epsilon,\epsilon]}.$$

In cohomology this becomes

$$q_0 p_\epsilon - q_0 p_{-\epsilon} = -\hbar.$$

Upshot: Often in physics the canonical commutation relations are imposed by fiat. This calculation shows that they can in fact actually be derived from prior principles.

Example 1.10. Set $X = Y \times \mathbb{R}$, $\int \varphi \Delta \varphi$, and $f \in C^\infty(Y)$ and eigenvector for Δ_Y. Set

$$q = \mathcal{O}_{f \delta_{t=0}}$$
$$p = \mathcal{O}_{f \delta'_{t=0}}$$

Then we can compute

$$q(0)(p(\epsilon) - p(-\epsilon)) = \hbar + \text{terms which } \to 0 \text{ as } \epsilon \to 0.$$

1.2.5 Chern-Simons

Philsang introduced this last lecture: fields of CS for $U(1)$ are

$$(\Omega^*(M)[1], d).$$

The observables are

$$\mathrm{Obs}^q(U) = \mathrm{Sym}^*\left(\Omega^*_c(U)[2]\right)$$

which we interpret as polynomials on fields and antifields.

This example is a little different to the previous one – the $\mathcal{O}_g^\star$ functions are already built in since our space of generators is a complex. If $f_i \in \Omega^*_c(U)[2]$,[8]

$$d(\mathcal{O}_{f_1}\cdots\mathcal{O}_{f_n}) = \sum \pm\frac{1}{\hbar}\mathcal{O}_{f_1}\cdots\mathcal{O}_{df_i}\cdots\mathcal{O}_{f_n} + \sum \pm\mathcal{O}_{f_1}\cdots\widehat{\mathcal{O}_{f_i}}\cdots\widehat{\mathcal{O}_{f_j}}\cdots\mathcal{O}_{f_n}\int f_i f_j.$$

Example 1.11 (Linking number)**.** $H^0(\mathrm{Obs}^q(\mathbb{R}^3)) = \mathbb{R}((\hbar))$, because there is a spectral sequence

$$\mathrm{Sym}^*\left(H^*_c(\mathbb{R}^3[2])\right) \Rightarrow H^*(\mathrm{Obs}^q(\mathbb{R}^3)).$$

Then given two knots $K_1, K_2 \subset \mathbb{R}^3$,

$$[\mathcal{O}_{\delta_{K_1}}\cdots\mathcal{O}_{\delta_{K_2}}] \in H^0(\mathrm{Obs}^q(\mathbb{R}^3))$$

is equal to $\hbar \times$ (linking number).

Proof: $K_1 = \partial S$, S a surface. $\delta_{K_1} = d\delta_S$. Then

$$d\left(\hbar\mathcal{O}_{\delta_S}\cdot\mathcal{O}_{K_2}\right) = \mathcal{O}_{\delta_{K_1}}\cdot\mathcal{O}_{\delta_{K_2}} + \hbar\underbrace{\int \delta_S\delta_K}_{=\text{ linking number}}.$$

1.3 Lecture 3 (Philsang Yoo)

1.3.1 The story so far...

There are three dichotomies:

- 0-dimensional vs d-dimensional
- free vs interacting
- classical vs quantum

The blue items were the topic of the first lecture; the orange items were the topic of the second lecture.

The following table reviews what we have covered thus far (and also sets some of the notation for the lecture):

[8] This is precisely the space of generators for the symmetric algebra.

	0-diml	d-diml
free theory	$(\mathcal{O}(\mathcal{E}), Q, \{-,-\})$	$(\mathcal{O}_{\text{loc}}(\mathcal{E}), Q, \{-,-\})$
$\mathcal{E}$	$T^*[-1] = V \oplus V^*[-1]$	$C^\infty(M) \oplus \Omega^n(M)[-1]$
free action	$S_{free}(x) = \sum x^i A_{ij} x^j$	$S_{free}(\phi) = \int_M \phi D \phi \, d\text{vol}$
$\mathcal{O}(\mathcal{E})$	$\text{PV}(V) = \mathbb{C}[x^i, \xi_i]$, $\lvert\xi_i\rvert = -1$	$\text{Sym}(C_c^\infty(M) \oplus C_c^\infty(M)[1])$
linear observable - degree 0	x^i	$\mathcal{O}_f(\phi) = \int f\phi \, d\text{vol}$, $f \in C_c^\infty(M)$
linear observable - degree 1	ξ_i	$\mathcal{O}_g^\star(\psi) = \int g\psi$, $g \in C_c^\infty(M)$
classical Q	$\iota_{dS_{free}} = \frac{\partial S_{free}}{\partial x^i} \frac{\partial}{\partial \xi_i} = \sum A_{ij} x^j \frac{\partial}{\partial \xi_i}$	D
on vector fields	$p_n(x)\xi_k \mapsto p_n(x)\left(\sum A_{kj} x^j\right)$	$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_g^\star \mapsto \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_{Dg}$
Δ	$x^i \xi_j \mapsto \delta_j^i$	$\mathcal{O}_f \mathcal{O}_g^\star \mapsto \int fg$
quantum $Q + \hbar\Delta$	$\frac{\partial S_{free}}{\partial x^i} - \hbar \sum \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i}$	
factorization algebra	$(\mathcal{O}(T^*[-1]V), Q + \hbar\Delta)$	$(\text{Sym}(C_c^\infty(-) \oplus C_c^\infty(-)[1]), Q + \hbar\Delta)$

Here $(\mathcal{E}, Q)$ is a (-1)-shifted symplectic space. The subscript loc did not appear in previous lectures – but we will see that we need to include it later today.

Recall that we defined the divergence complex by considering the isomorphism from finite dimensions:

$$\begin{tikzcd} \Gamma(M; \wedge^n T_M) \arrow[r, "\Delta_\mu"] \arrow[d, "\simeq"', "\vee\mu"] & \cdots \arrow[r] & C^\infty(M, \wedge^2 TM) \arrow[r, "\Delta_\mu"] \arrow[d, "\simeq"', "\vee\mu"] & \text{Vect}(M) \arrow[r, "\Delta_\mu"] \arrow[d, "\simeq"] & C^\infty(M) \arrow[d, "\simeq"', "\vee\mu"] \\ \mathcal{O}(M) \arrow[r, "d"] & \cdots \arrow[r] & \Omega^{n-2}(M) \arrow[r, "d"] & \Omega^{n-1}(M) \arrow[r, "d"] & \Omega^n(M) \end{tikzcd}$$

If the measure is related to the usual Lebesgue measure by

$$\mu = e^{S/\hbar} d\mu_{Leb}$$

then

$$\Delta_\mu = \Delta_{Leb} + \frac{1}{\hbar} \iota_{dS}$$

so that

$$\hbar \Delta_\mu = \iota_{dS} + \hbar \Delta_{Leb}.$$

We are interested in the resulting complex:

$$(\mathcal{O}(T^*[-1]X)[[\hbar]], \iota_{dS} + \hbar\Delta).$$

BV/BD quantisation.

Definition 1.5. A *Beilinson-Drinfeld algebra (BD-algebra)* is a triple $(A, \cdot, \{-,-\})$ where $(A, \cdot)$ is a graded commutative algebra equipped with a differential d ($d^2 = 0$), flat over $\mathbb{R}[[\hbar]]$, and $\{-,-\}$ is a Poisson bracket of cohomological degree 1 such that

$$d(a \cdot b) = da \cdot b + (-1)^{|a|} a \cdot db + \hbar\{a, b\}.$$

Remark 1.9. Since d is not a derivation for the multiplication, the cohomology is no longer a commutative algebra – its multiplication

Recall that given a (-1)-shifted symplectic space with a vector field $(\mathcal{E}, Q)$ with functions $(\mathcal{O}(\mathcal{E}), Q, \{-,-\})$, we say that $I \in \mathcal{O}(\mathcal{E})$ satisfies the *classical master equation (CME)* if

$$QI + \frac{1}{2}\{I, I\} = 0.$$

We can also produce from $(\mathcal{E}, Q)$ the BD-algebra $(\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta, \{-,-\})$, and say that I satisfies the *quantum master equation* if

$$QI + \frac{1}{2}\{I, I\} + \hbar\Delta I = 0$$

or equivalently,

$$(Q + \hbar\Delta)e^{I/\hbar} = 0.$$

Definition 1.6. A *BV/BD quantisation* of a $\mathbb{P}_0$-algebra A is a BD-algebra $\tilde{A}$ such that $\tilde{A}|_{\hbar=0} = A$.

Moving on up

For a d-dimensional interacting classical theory

$$(\mathcal{O}(\mathcal{E}), Q, \{-,-\})$$

we want to try to solve

$$QI + \frac{1}{2}\{I, I\} = 0.$$

This is naive!

1) $\{-,-\}$ is not defined on $\mathcal{O}(\mathcal{E})$

2) We don't want an arbitrary functional I

1.3.2 Local functionals

Idea: A functional F is *local* if its k^{th} homogeneous component f_k is of the form

$$\phi \mapsto \int_M (D_1\phi)\cdots(D_k\phi)\mathrm{dvol}$$

for linear differential operators D_i.

Definition 1.7.

$$\mathcal{O}_{loc}(\mathcal{E}) := \mathrm{Dens}_M \otimes_{\mathcal{D}_M} \mathcal{O}(\mathcal{J}(\mathcal{E}))$$

where

- $\mathcal{D}_M$: differential operators on M
- $\mathcal{J}(\mathcal{E})$: ∞-jets
- $\mathcal{O}(\mathcal{J}(\mathcal{E})) = P\,\mathrm{Diff}(\mathcal{E}, C^\infty_M)$ (polydifferential operators)
- $\phi \mapsto F(\phi)$, $F(\phi)(p)$ depends only on ∞-jets

If M is compact,

$$\begin{aligned} \mathcal{O}_{loc}(\mathcal{E}) &\xhookrightarrow{i} \mathcal{O}(\mathcal{E}) \\ \mathcal{L} &\longmapsto i(\mathcal{L}) \end{aligned}$$

where

$$i(\mathcal{L})(\phi) = \int_M \mathcal{L}(\phi).$$

Remark 1.10. The bracket was *not* previously well-defined. The bracket is now well-defined on *local* functionals. So now we want to solve $(\mathcal{O}_{loc}(\mathcal{E}), Q + \{I, -\} = \{S, -\} = \iota_{dS}, \{-,-\})$.

For a d-dimensional interacting *quantum*

$$(\mathcal{O}_{loc}(\mathcal{E})[[\hbar]], Q + \hbar\Delta, \{-,-\})$$

we want to try to solve

$$QI + \frac{1}{2}\{I, I\} + \hbar\Delta I = 0$$

or (equivalently)

$$(Q + \hbar\Delta)e^{I/\hbar} = 0.$$

Problem: Δ is not defined even on $\mathcal{O}_{loc}(\mathcal{E})$! This is a problem of physical origin (related to UV divergences).

1.3.3 Definition of Quantum Field Theory (after Costello)

In the 0-dimensional case,

$$(V, \omega) \quad (-1)\text{-shifted symplectic}$$

leads to a Poisson bivector

$$K \in (\operatorname{Sym}^2 V)[1]$$

and the BV differential Δ is given by contracting K. We then obtain $\{-,-\}$ from Δ (or from K – it is precisely the Poisson bracket associated to the bivector).

In a d-dimensional theory, starting from $(\mathcal{E}, \omega)$, we obtain

$$K_0 \in \underbrace{(\overline{\mathcal{E}}\hat{\otimes}\overline{\mathcal{E}})_{S_2}}_{\cong \operatorname{Sym}^2(\overline{\mathcal{E}})}[1]$$

such that

$$\omega(K_0(x, y), \phi(x)) = \phi(y).$$

Here, $\overline{E}$ are *distributional* sections of $\mathcal{E}$.

But! "Δ_0" = contracting with K_0 isn't defined. (Problem? Distributions don't multiply.[9])

Idea to fix: *effective field theory* – mollify K_0!

Example 1.12. For scalar field theory, there exists a heat kernel regularization K_t such that

$$\omega(K_t(x, y), \phi(x)) = (e^{-tD}\phi)(y),$$

and K_t is smooth for $t > 0$.

Return to our general setting $(\mathcal{E}, Q)$.

Definition 1.8. A *gauge-fixing operator*

$$Q^{GF} : \mathcal{E} \to \mathcal{E}$$

is a differential operator of cohomological degree -1 such that

$$[Q, Q^{GF}] = D$$

is a generalized Laplacian.

Example 1.13 (Scalar field theory)**.**

$$C^\infty(M) \rightleftarrows \Omega^n(M)$$

$$Q = (D + m^2), \qquad Q^{GF} = \mathrm{id}$$

[9] If their support is not disjoint.

Example 1.14 (Chern-Simons theory).

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M)$$

$$Q = d, \qquad Q^{GF} = d^*$$

Note that the space of possible gauge fixings is the space of metrics (to determine the adjoint to d).

Remark 1.11. The main results depend on gauge-fixing in terms of homotopy data. In practice the space of gauge-fixings is contractible (see Example 1.14).

Now, Q^{GF} yields $D = [Q, Q^{GF}]$, and there is a corresponding operator K_L such that

$$\omega(K_L(x, y), \phi(x)) = (e^{-LD}\phi)(y),$$

and we use $-K_L$ to define the BV-Laplacian Δ_L (and the corresponding bracket $\{-, -\}_L$).

Now recall: We wanted $I \in \mathcal{O}_{loc}(\mathcal{E})$ satisfying

$$QI + \frac{1}{2}\{I, I\} + \hbar\Delta I = 0,$$

but we realised that this doesn't make sense. So we introduced this scaling/mollification and try to solve the *QME at scale* L

$$QI + \frac{1}{2}\{I, I\}_L + \hbar\Delta_L I = 0$$

for $I \in (\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta, \{-, -\}_L)$.

Remark 1.12. We are no longer working in the UV, but at finite length scale – correspondingly we do not expect I to be *local*; instead it will depend on a finite length scale.

What we are aiming for is

$$\{I[L]\} \quad \text{effective action}$$

such that $I[L]$ solves the QME at scale L.

Question: How are K_L and K_0 related?

Define a propagator

$$P(\epsilon, L) := \int_\epsilon^L (Q^{GF} \otimes 1) K_t \, dt.$$

Proposition 1.3. *$P(\epsilon, L)$ gives a cochain homotopy with respect to Q between K_ϵ and K_L.*

Proof sketch. We have:

- K_t is the kernel for e^{-tD}
- $P(\epsilon, L)$ is the kernel for $\int Q^{GF} e^{-tD}$

So

$$\begin{aligned}\left[Q, \int Q^{GF} e^{-tD}\right] &= \int [Q, Q^{GF}] e^{-tD} \\ &= \int_\epsilon^L D e^{-tD} = e^{-\epsilon D} - e^{-LD}\end{aligned}$$

and hence

$$(Q \otimes 1 + 1 \otimes Q)P = K_\epsilon - K_L.$$

□

Remark 1.13. In the above proposition:

- $P \in \mathrm{Sym}^2(\mathcal{E})$
- $K \in \mathrm{Sym}^2(\mathcal{E})[1]$

Define $\partial_{P(\epsilon,L)}$ as contraction with $P(\epsilon, L)$; then

$$[\partial_{P(\epsilon,L)}, Q] = \Delta_L - \Delta_\epsilon$$

with $\Delta_L = \partial_{K_L}$, $\Delta_\epsilon = \partial_{K_\epsilon}$.

Remark 1.14. All of the scale dependent operators we have defined are appropriately equivariant with respect to the scaling action; so one expects the scale dependent QME to also be "appropriately equivariant".

Definition 1.9. A *homotopy renormalization group flow* is a map

$$W(P(\epsilon, L), -) : \mathcal{O}(\mathcal{E})[[\hbar]] \to \mathcal{O}(\mathcal{E})[[\hbar]]$$

defined by

$$W(P(\epsilon, L), I) := e^{t\partial_{P(\epsilon,L)}} e^{I/\hbar}.$$

Remark 1.15. Useful way to think about this is in terms of Feynman diagrams (contraction of propagators – keep in mind that now the propagators depend on the length scale).

Proposition 1.4. *$I \in \mathcal{O}(\mathcal{E})[[\hbar]]$ satisfies QME at ϵ if and only if $W(P(\epsilon, L), I)$ satisfies QME at L.*

Proof. Proof as exercise. Need to combine the fact

$$[\partial_{P(\epsilon,L)}, Q] = \Delta_L - \Delta_\epsilon$$

with the expression

$$W(P(\epsilon, L), I) := e^{t\partial_{P(\epsilon,L)}} e^{I/\hbar}.$$

□

Suppose we have $I[\epsilon]$ which solves the QME at scale ϵ. Then we know how to obtain a solution of the QME at scale L – we just take

$$I[L] = W(P(\epsilon, L), I[\epsilon]).$$

So why did we bother introducing all of these scale dependencies?

Main point: "We wanted $I \in \mathcal{O}_{loc}(\mathcal{E})[[\hbar]]$ solving QME". So we need to make sure that we don't forget about locality!

Definition 1.10. A *pre-quantum field theory* is $\{I[L]\}$ such that

(1) $I[L] = W(P(\epsilon, L), I[\epsilon])$

(2) $I[L]$ becomes local as $L \to 0$

The following theorem is due to Costello.

Theorem 1.5. *There exists a non-canonical bijection*

$$\{\mathcal{O}^+_{loc}(\mathcal{E})[[\hbar]]\} \leftrightarrow \{\textit{pre-theories}\}$$

where the "+" superscript indicates terms of cubic order and higher (interaction terms). The choice of bijection is given by a choice of "renormalization scheme" (roughly a specification of how to select out the "singular part" of a function and introduce counterterms).

Definition 1.11. A *quantum field theory* is a pre-quantum field theory such that each $I[L]$ satisfies the QME at L.

Remark 1.16. Solving the QME is a difficult problem, and has obstructions (which can be studied with usual obstruction theory methods – suppose a solution $I^{(n)}[L]$ is defined modulo $\hbar^n$ and determine the class that obstructs extending this to $\hbar^{n+1}$).

Computing $I[\infty]$ can give interesting results – i.e. non-trivial invariants:

- Grady-Gwilliam: $\hat{A}$-genus from 1d topological QM
- Costello: Witten genus from $\beta - \gamma$ system

1.4 Lecture 4 (Kevin Costello)

1.4.1 Towards interactions

Recall the setup for the scalar field: $\varphi \in C^\infty(M)$, $\int \varphi D \varphi$. We're changing notation now however – $D = \sum \partial_{x_i}^2$ is the usual Laplacian (not the one with nonnegative spectrum).

If $U \subseteq M$, $f_i, g_j \in C_c^\infty(U)$,

$$\mathrm{Obs}^q(U) = \text{ cochain complex spanned by } \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_{g_1}^\star \cdots \mathcal{O}_{g_m}^\star \text{ in degree } -m$$

The differential on this complex is

$$d_{\mathrm{Obs}^q(U)} = \Delta_{BV} + \frac{1}{\hbar} Q$$

where Δ_{BV} contracts one f with one g,

$$\Delta_{BV}(\mathcal{O}_f \mathcal{O}_g^\star) = \int_U fg,$$

i.e. Δ_{BV} is "divergence with respect to the Lebesgue measure".

Q is a derivation,

$$Q\mathcal{O}_g^\star = \mathcal{O}_{Dg}$$

where D is the Laplacian. So Q is the Lie derivative of a vector field on $S(\varphi) = \int \varphi D \varphi$.

In this interacting case,

$$S(\varphi) = \int \varphi D \varphi + \int \varphi^4$$

we need to add a term

$$\{I, -\}, \quad I = \int \varphi^4.$$

The extra term is the Lie derivative of a vector field on the function

$$\varphi \mapsto \varphi^4.$$

Then

$$\{I, \mathcal{O}_g^\star\} = \int_{x \in M} g(x) \varphi(x)^3.$$

Problem! There is a technical but very important problem at this stage. The function

$$\varphi \mapsto \int g \varphi^3$$

is *not* in the class of functions we considered. It's not of the form

$$\int_{M^3} f_1(x_1)f_2(x_2)f_3(x_3)\varphi(x_1)\varphi(x_2)\varphi(x_3)$$

for $f_i \in C_c^\infty(M)$. Instead,

$$\int g\varphi^3 = \int \underbrace{g(x_1)\delta_{x_1=x_2=x_3}}_{\text{This is a distribution!}} \varphi(x_1)\varphi(x_2)\varphi(x_3).$$

Tentative Solution: Enlarge our allowed functions to include things like

$$\varphi \mapsto \int D(x_1, \dots, x_n)\varphi(x_1)\cdots\varphi(x_n)$$

where D is a distribution on M^n with compact support.

New Problem! The BV Laplacian is ill-defined on this class of "functions". Taking $f, g \in D_c(U)$, we have

$$\Delta_{BV}(\mathcal{O}_f\mathcal{O}_g^\star) = \text{``}\int f(x)g(x)\text{''}$$

which is ill-defined, since we cannot multiply distributions.

Summary:

- In the free theory, the "functional measure" was defined on a class of distributions.
- As soon as we introduce interactions, we are required to integrate over products of observables coming from this class of "functions" – and distributions don't multiply.

Actual Solution:

- Use observables which are distributions.
- Mollify Δ_{BV} to Δ_ϵ where

$$\Delta_\epsilon(\mathcal{O}_f\mathcal{O}_g^\star) = \int K_\epsilon(x_1, x_2)f(x_1)g(x_2)$$

  such that K_ϵ is smooth and the expression reproduces Δ_{BV} as $\epsilon \to 0$.

We saw that

$$\Delta_0 - \Delta_\epsilon = [Q, \partial_{P_0^\epsilon}]$$

where

$$P_0^\epsilon = \int_0^\epsilon K_t$$

so that on Q-cohomology, this mollification does nothing. Here, $\partial_{P_0^\epsilon}$ contracts two $\mathcal{O}_f$'s by

$$\partial_{P_0^\epsilon}(\mathcal{O}_{f_1}\mathcal{O}_{f_2}) \to \int f_1(x_1)f_2(x_2)P_0^\epsilon(x_1, x_2).$$

As different scales, we have the relation between mollifications

$$\Delta_\epsilon - \Delta_L = [Q, \partial_{P_\epsilon^L}].$$

The differentials

$$\frac{1}{\hbar}Q + \Delta_\epsilon$$

are up to homotopy independent of ϵ.

What about if we ad an interacting term? Then the differential

$$\frac{1}{\hbar}Q + \Delta_\epsilon + \{I[\epsilon], -\}_\epsilon$$

is up to homotopy independent of ϵ as long as $I[\epsilon]$ satisfies "homotopical RG-flow" (and $I[\epsilon]$ must satisfy the QME to ensure that the differential squares to zero).

- **Fact:** algebras for interacting theories can be constructed once we have $\{I[L]\}$ satisfying the axioms from Phil's talk.
- Solutions $\{I[L]\}$ to these axioms can be found by obstruction theories.
- In fancy language (e.g. Lurie's paper on deformation theory), the solutions form a "formal moduli problem".
- In the case of free scalars, solutions $\{I[L]\}$ to these axioms are in noncanonical[10] bijection with Lagrangians

  $$\sum \hbar^i \int P_i(\varphi, \partial\varphi, \partial^2\varphi, \ldots)$$

  (a series in $\hbar$ of integrals over polynomials in the fields and their derivatives).

1.4.2 Renormalizability

Problem: There seem to be too many theories.

Example 1.15. $\int_{\mathbb{R}^4} \varphi^{10}$ should be a "bad" interaction; so why do we include it?

$\mathbb{R}_{>0}$ acts on $\mathbb{R}^n$. If $\mathcal{F}$ is a factorisation algebra on $\mathbb{R}^n$ then so is its pullback $\lambda^*\mathcal{F}$.[11] So $\mathbb{R}_{>0}$ acts on the set of factorisation algebras on $\mathbb{R}^n$.

Definition 1.12. This action is called the *renormalisation group flow (RG flow).*

$\mathcal{F}$ is a fixed point if

$$\lambda^*\mathcal{F} \cong \mathcal{F}$$

(this is data).

Example 1.16. The factorisation algebra of a free massless scalar field is a fixed point: if $\varphi \in C^\infty(\mathbb{R}^n)$, make $\lambda \in \mathbb{R}_{>0}$ act on φ by

$$\varphi(x) \to \lambda^{n/2-1}\varphi(\lambda x),$$

then $\int \varphi D\varphi$ is scale invariant.

If λ acts on $f \in C_c^\infty(U)$ by

$$f(x) \to \lambda^{n/2+1} f(\lambda x)$$

and on g by

$$g(x) \to \lambda^{n/2-1} g(\lambda x)$$

then d on $\mathrm{Obs}^q(U)$ spanned by $\mathcal{O}_{f_1}\cdots\mathcal{O}_{f_n}\mathcal{O}^\star_{g_1}\cdots\mathcal{O}^\star_{g_m}$ is scale invariant.

$$d = \Delta_{BV} + \frac{1}{\hbar}Q$$

commutes with the $\mathbb{R}_{>0}$-action.

[10] Depending on a renormalisation scheme.

[11] This is a new factorisation algebra for each scale λ – factorisation algebras (unlike sheaves) only pull back under local isomorphisms. Another way to discuss this would be to consider it as a factorisation algebra valued in sheaves on $\mathbb{R}_{>0}$.

What about deformations?

$$\mathcal{M} = \{\text{moduli of translation invariant factorisation algebras on } \mathbb{R}^m\}$$
$$T_{free}\mathcal{M} = \{\text{Lagrangians}\}$$

$\mathbb{R}_{>0}$ acts on $T_{free}\mathcal{M}$ (tangent space at the free theory). We can compute this action (skipping small subtleties) by asking how a Lagrangian $\mathcal{L}(\varphi)$ changes under

$$\varphi(x) \to \lambda^{n/2-1}\varphi(\lambda x).$$

In dimension 4,

$$\int_{\mathbb{R}^4} \underbrace{\varphi^n}_{\text{wt } n} \underbrace{dx_1 \cdots dx_4}_{\text{wt } -4}$$

transforms by λ^{n-4}. If we do some small calculations and consider $SO(4)$-invariant Lagrangians, then the tangent space looks like Figure 1.

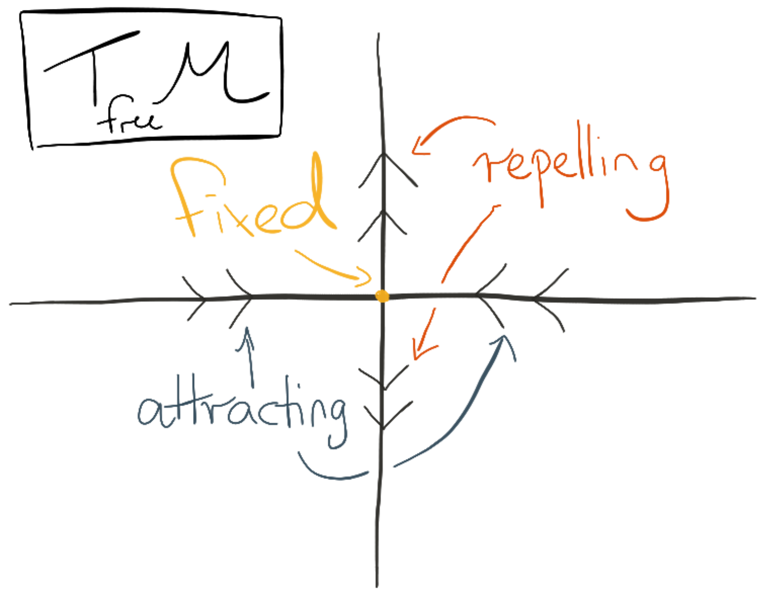


Figure 1: Tangent space to free theory with leading order RG-flow attracting and repelling directions.

Terminology: If we are flowing to large scales (i.e. flowing to the IR) then

- Repelling directions = Relevant
- Attracting directions = Irrelevant
- Fixed points = Marginal

For the 4-dimensional free theory:

- Marginal: $\int \varphi^4$
- Irrelevant: lots of them, e.g. $\int \varphi^n$ for $n > 4$
- Relevant: Finitely many – $\int \varphi$, $\int \varphi^2$, $\int \varphi^3$

So, even though the space of theories is ∞-dimensional, if we look at things from far away only finitely many parameters matter.

Conversely, at small scales, only marginal and relevant interactions can have good behaviour.

1.4.3 Corrections to the flow

Consider

$$\int \varphi D \varphi + c \int \varphi^4.$$

We claimed that to leading order in c this is scale invariant. Does this extend beyond leading order? **No:**

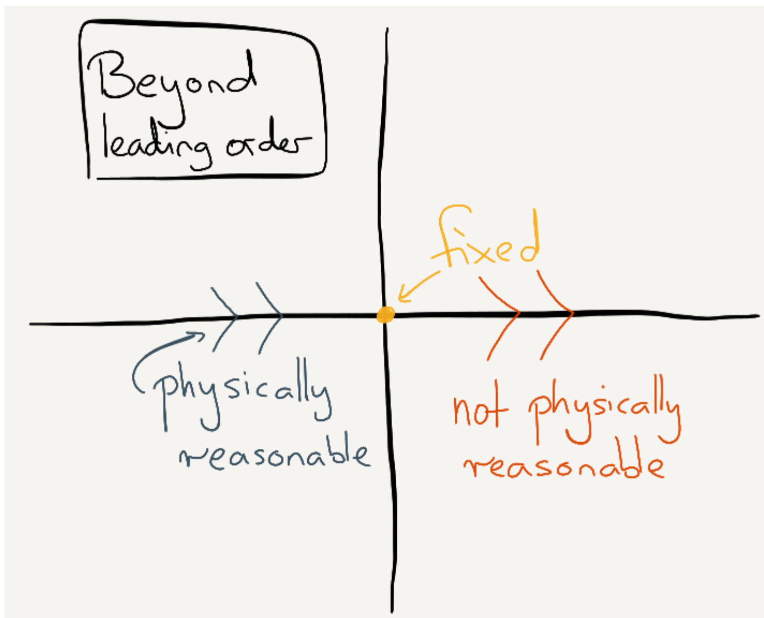


Figure 2: Subleading order corrections to the RG-flow.

There is an RG flow trajectory in the space of theories with coordinate c, where the vector field which generates the flow looks like

$$c^2 \frac{d}{dc} + O(c^3) \frac{d}{dc}.$$

The effect of the subleading term is to turn one of the first-order attracting directions into a repelling direction (Figure 2). Only one of these directions (the attracting direction which flows to a fixed point) is physically reasonable. I.e.

- $c > 0$ "good" theory (unitary)
- $c < 0$ "bad"
- Sign is such that in the IR, $c > 0$ flows down to $c = 0$.

Remark 1.17 (Yang-Mills Theory). An important discovery from the 1960s – the opposite sign exhibits attracting behaviour. I.e. the "good" coupling constant flows to 0 in the UV (*asymptotic freedom*).

How to compute.

There are two methods of computation:

(1) Understand the leading order corrections to the factorisation algebra as we deform away from a free theory.

(2) Compute with the $I[L]$'s.

Notation: R_λ = how $\lambda \in \mathbb{R}_{>0}$ acts on $\varphi \in C^\infty(\mathbb{R}^4)$.

Want to define an action on $I[L](\varphi)$. Define

$$R_\lambda(I[L])(\varphi) = I[\lambda^2 L](R_\lambda \varphi),$$
$$K_L = e^{-\frac{\|x_1 - x_2\|^2}{L}} L^{-n/2}$$

Check: If $I[L]$ satisfies the axioms, so does $R_\lambda(I)[L]$.

Computation:

- Start with $\int \varphi D \varphi + c \int \varphi^4$
- Build $I[L](c)$. (Family of scale dependent interactions depending on the parameter c.)
- $R_\lambda(I)[L](c) = I[L](c + \hbar c^2 \log \lambda + O(c^3))$.

Naively $I[L] = \lim_{\epsilon \to 0} W(P_\epsilon^L, I)$, which can be expressed as in Figure 3, where $I = c \int \varphi^4$.

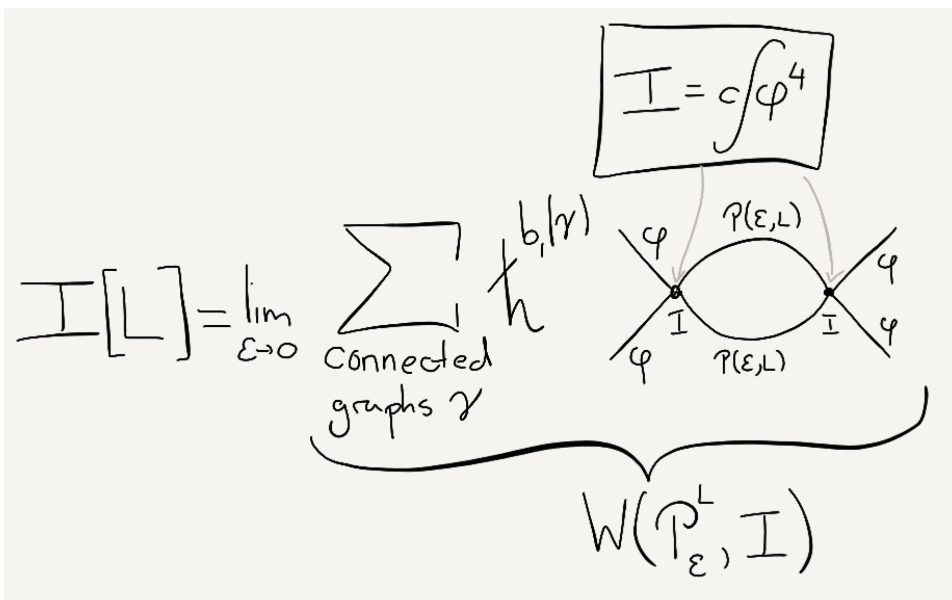


Figure 3: Naive calculation of $I[L]$ in φ^4 theory.

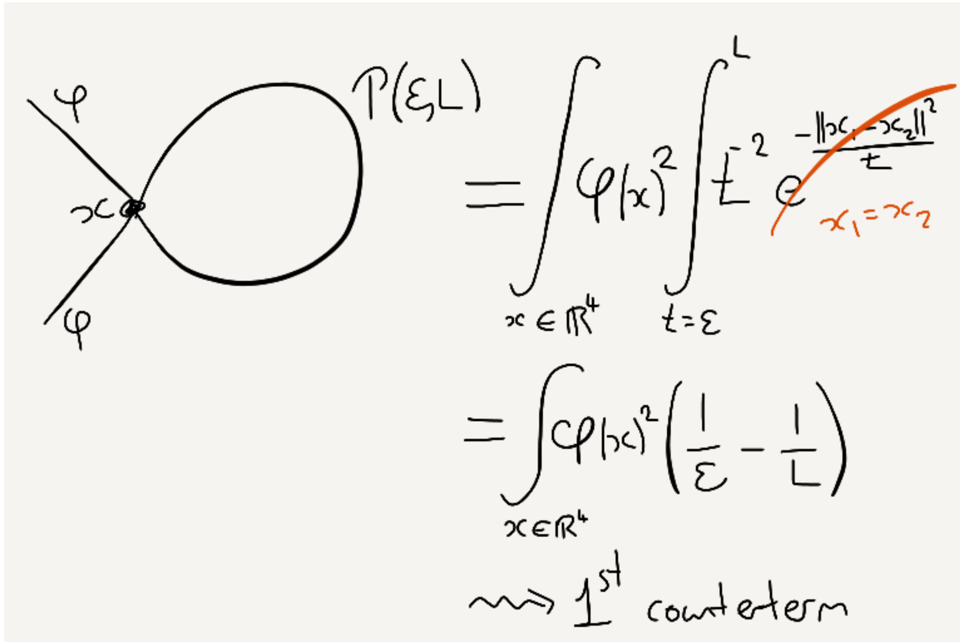
Sometimes this limit doesn't exist, so we introduce ϵ-dependent terms

$$I \to I - I^{CT}(\epsilon)$$

called *counter-terms*. By studying these we'll see the flow.

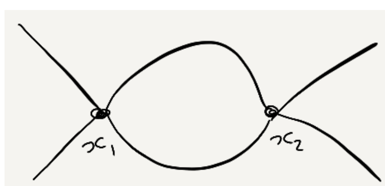
- *Tree diagrams:* These are fine.

- *1-loop:* There is a "baby bird" diagram,



So the 1st counterterm is: $\hbar\frac{1}{\epsilon}\int\varphi(x)^2$.

And there is a "hard candy" diagram,



$$= \int_{x_1,x_2\in\mathbb{R}^4} \varphi(x_1)^2\varphi(x_2)^2 P_\epsilon^L(x_1,x_2)^2$$

This is equal to

$$\int_{t_1,t_2=\epsilon}^{L}\int_{x_1,x_2}\varphi(x_1)^2\varphi(x_2)^2 t_1^{-2}t_2^{-2} r^{\left(-\frac{1}{t_1}-\frac{1}{t_2}\right)\|x_1-x_2\|^2};$$

integrating over $x_1 - x_2$, we get

$$\int_{t_1,t_2=\epsilon}^{L}\int_x \varphi(x)^4 t_1^{-2}t_2^{-2}\left(\frac{t_1t_2}{t_1+t_2}\right)^2 dt_1dt_2 \sim \log\epsilon\int\varphi^4.$$

When we scale everything:

$$I[L] = \lim_{\epsilon\to 0} W(P_\epsilon^L, I - I^{CT}(\epsilon)),$$
$$R_\lambda I[L] = \lim_{\epsilon\to 0} W(P_{\lambda^2\epsilon}^{\lambda^2 L}, I - R_\lambda I^{CT}(\lambda^2\epsilon))$$

So if we scale everything,

$$\epsilon^{-1}\int\varphi^2$$

is fixed. So

$$\log\epsilon\int\varphi^4 \to \log\epsilon\int\varphi^4 + 2\log\lambda\int\varphi^4$$

Conclude: If we scale

$$I^{CT} \to I^{CT} + \hbar \log \lambda \int \varphi^4$$

this means

$$I \to I - \hbar \log \lambda \int \varphi^4.$$

1.5 Lecture 5 (Kevin Costello)

Recall that last lecture we saw: given a factorisation algebra $\mathcal{F}$ on $\mathbb{R}^n$, there is a scaling action by $\lambda \in \mathbb{R}^\times$ on $\mathbb{R}^n$,

$$\lambda(x) = \lambda x$$

under which we can pull back to obtain a new factorisation algebra, $\lambda^* \mathcal{F}$.

This gives a flow on the space of factorisation algebras: the *RG-flow*.

Last time, we computed this for φ^4 theory. It was quite an involved computation.

Today: More direct computation, using Poisson version of factorisation algebras.

1.5.1 Classical observables

Consider the *classical observables* of some theory, $\mathrm{Obs}^{cl}(U)$, $U \subset \mathbb{R}^n$, defined by

$$\mathrm{Obs}^{cl}(U) = \frac{\{\text{functions on fields on } U\}}{\text{equations of motion}}$$

If $x \in U$, we want to consider

$$\mathrm{Obs}^{cl}_x \in \bigcap_{V \ni x} \mathrm{Obs}^{cl}(V);$$

in his talk today, Tudor called these *local operators* and used the notation Ops.

For a free scalar field theory on $\mathbb{R}^n$,

$$\mathcal{O}_f(\varphi) = \int_U f\varphi, \qquad f \in C^\infty_c(U).$$

- We allow f to be a distrobution.
- If f is supported at $x \in U$ then
$$f = (\text{derivatives}) \partial^{i_1}_{x_1} \cdots \partial^{i_n}_{x_n} \delta_x.$$

Set

$$d\mathcal{O}^\star_g = \mathcal{O}_{\Delta g}.$$

Then g is also supported at x, and we find at the level of cohomology that

$$\left(\sum \partial^2_{x_i}\right) \partial^{i_1}_{x_1} \cdots \partial^{i_n}_{x_n} \delta = 0.$$

Defining

$$D_0 = \mathbb{R}[\partial_{x_1}, \ldots, \partial_{x_n}]$$

we have found that the cohomology of local observables is

$$\mathrm{Sym}\left(D_0 / \sum \partial^2_{x_i}\right).$$

Remark 1.18. Recall that at the quantum level,

$$d^q \mathcal{O}_g^\star = \hbar^{-1} \mathcal{O}_{\Delta g},$$
$$d^q(\mathcal{O}_f \mathcal{O}_g^\star) = 1 \int fg$$

and we have that

$$d := d^{cl} = \lim_{\hbar \to 0} \hbar d^q.$$

Alternate description.

$$\mathrm{Obs}_0^{cl} = \mathrm{Sym}\left(D_0 / \sum \partial_{x_i}^2\right)$$

This is a commutative algebra with n commuting derivations. It is generated by 1 element,

$$\mathcal{O}_\delta : \varphi \mapsto \varphi(0)$$

subject to the relation

$$\left(\sum \partial_{x_i}^2\right) \mathcal{O}_\delta = 0.$$

In an *interacting* theory, e.g. φ^4, it's easy to see that Obs_0^{cl} is generated in this sense by $\mathcal{O}$ with the relation that

$$\sum \partial_{x_i}^2 \mathcal{O} + \mathcal{O}^3 = 0.$$

Today, we will show that Obs_0^{cl} has a "Poisson bracket".[12]

1.5.2 Reminder on Deformation Quantisation

If we have a classical mechanical system, with a commutative algebra A of operators, then A has a Poisson bracket defined as follows.

Let $\hat{A}$ be the non-commutative quantum algbera, defined modulo $\hbar^2$ by the following rule: if $\alpha, \beta \in A$, let $\hat{\alpha}, \hat{\beta}$ be lifts to $\hat{A}$. Then

$$\{\alpha, \beta\} = \lim_{\hbar \to 0} \frac{1}{\hbar} [\hat{\alpha}, \hat{\beta}].$$

Note:

- $[\hat{\alpha}, \hat{\beta}] = 0$ modulo $\hbar$
- This does not depend on our choice of lifts of α and β.

In the factorisation algebra story we have (Figure 4)

$$[\hat{\alpha}, \hat{\beta}] = \hat{\alpha}(0)\hat{\beta}(\epsilon) - \hat{\alpha}(0)\hat{\beta}(-\epsilon).$$

This commutator is the obstruction to $\hat{\alpha}(0)\hat{\beta}(\epsilon)$ being a continuous function of ϵ.

If there's no Hamiltonian,

$$\frac{\partial}{\partial \epsilon} \hat{\beta}(\epsilon) = 0,$$

so $\hat{\alpha}(0) \cdot \hat{\beta}(\epsilon)$ is constant in the region $\epsilon \neq 0$. So the two-dimensional space of such functions is spanned by

$$\hat{\alpha}(0)\hat{\beta}(\epsilon) = c_1(\text{indept. of } \epsilon) + \underbrace{c_2(\delta_{\epsilon>0} - \delta_{\epsilon<0})}_{\text{continuity obstruction}}$$

[12] Beware: scare quotes.

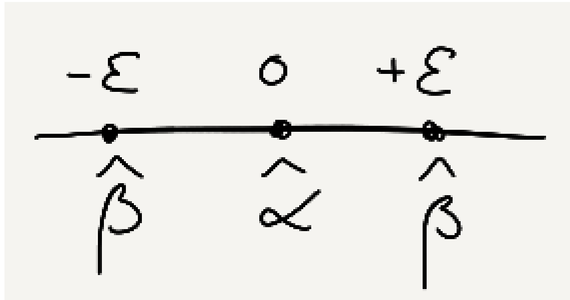


Figure 4: In the $\epsilon \to 0$ limit this calculates $[\hat{\alpha}, \hat{\beta}]$.

and we have made explicit that the obstruction to being continuous is the commutator.

To write this down in general, we need some notation:

$$\begin{array}{ccl} C^\omega(\mathbb{R}^n \setminus 0) & = & \{\text{real analytic functions}\} \\ \uparrow & & \\ C^\omega_+(\mathbb{R}^n \setminus 0) & = & \{\text{real analytic functions which extend to a continuous function on } \mathbb{R}^n\} \end{array}$$

E.g. $r\log(r) \in C^\omega_+(\mathbb{R}^n \setminus 0)$.

Theorem 1.6. *Consider any classical field theory. Let* Obs^{cl}_0 *be the point observables (taking cohomology). Then there exists a map*

$$\{-,-\}_{OPE} : \mathrm{Obs}^{cl}_0 \otimes \mathrm{Obs}^{cl}_0 \to \mathrm{Obs}^{cl}_0 \otimes \left(\frac{C^\omega(\mathbb{R}^n \setminus 0)}{C^\omega_+(\mathbb{R}^n \setminus 0)}\right)$$

$\{-,-\}_{OPE}$ *is a derivation in each factor, and a map of modules for the Lie algebra* $\mathbb{R}^n$ *which also acts*[13] *on* $\frac{C^\omega(\mathbb{R}^n \setminus 0)}{C^\omega_+(\mathbb{R}^n \setminus 0)}$.

The bracket of Theorem 1.6 is defined as follows:

If Obs^q are the observables of a quantization defined mod $\hbar^2$, take lifts of $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Obs}^{cl}_0$, to $\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2 \in \mathrm{Obs}^q_0$ (see Figure 5)

$$\begin{aligned} \hat{\mathcal{O}}_1(0) &\in \mathrm{Obs}^q(D(0,r)), \quad \forall r \\ \hat{\mathcal{O}}_2(x) &\in \mathrm{Obs}^q(D(x,r)), \quad \forall r \\ \hat{\mathcal{O}}_1(0)\hat{\mathcal{O}}_2(x) &\in \mathrm{Obs}^q(D(0,s)), \quad \forall x \text{ not too close to } 0,\ |x| < s - r. \end{aligned}$$

Better: For all $x \neq 0$, $|x| < s$,

$$\hat{\mathcal{O}}_1(0)\hat{\mathcal{O}}_2(x)$$

extends across $x = 0$ modulo $\hbar$.

Can show: Failure to extend across $x = 0$ is in

$$\hbar \frac{C^\omega(\mathbb{R}^n \setminus 0)}{C^\omega_+(\mathbb{R}^n \setminus 0)} \otimes \mathrm{Obs}^{cl}_0;$$

so define

$$\{\mathcal{O}_1, \mathcal{O}_2\}_{OPE} = \lim_{\hbar \to 0} \frac{1}{\hbar}\Big(\text{obstruction to extending } \hat{\mathcal{O}}_1(0)\hat{\mathcal{O}}_2(x) \text{ across } x = 0\Big)$$

[13]This action may be defined with some non-standard signs, so be wary of potential sign issues in the future.

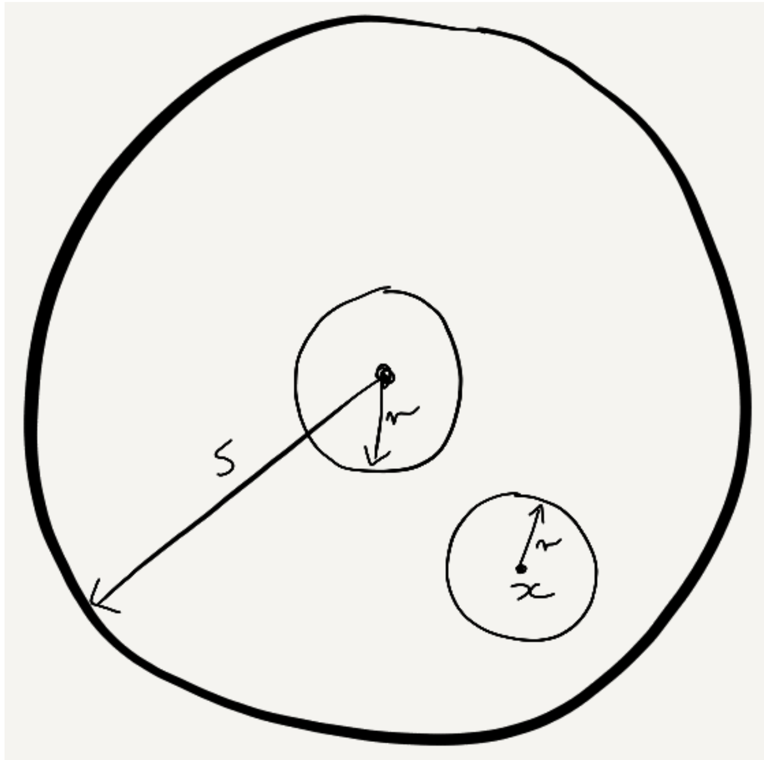


Figure 5: Radii of convergence for lifts of local operators.

Remark 1.19. For holomorphic theories, the range of $\{-,-\}$ is

$$\frac{H^*_{\bar{\partial}}(\mathbb{C}^n \setminus 0)}{H^*_{\bar{\partial}}(\mathbb{C}^n)} \otimes \mathrm{Obs}^{cl}_0 \simeq H^*_c(\mathbb{C}^n, \mathcal{O}) \otimes \mathrm{Obs}^{cl}_0,$$

but in general there is not such a nice description.

Example 1.17. Consider a free scalar field theory on $\mathbb{R}^n$. Field is $\varphi \in C^\infty(\mathbb{R}^n)$. Let $\mathcal{O} \in \mathrm{Obs}^{cl}_0$ be the observable

$$\mathcal{O} : \varphi \mapsto \varphi(0)$$

i.e. $\mathcal{O} = \mathcal{O}_{\delta_0}$. Then we can compute that

$$\{\mathcal{O}, \mathcal{O}\}_{OPE} = 1 \cdot G(x)$$

where $G(x) \in C^\omega(\mathbb{R}^n \setminus 0)$ is the Green's function for the Laplacian, $\Delta G(x) = \delta_{x=0}$.

Proof: Consider $\mathcal{O}_{\delta_0} \cdot \mathcal{O}_{\delta_x}$. By the defining property of the Green's function we can rewrite the δ-function at zero as

$$\mathcal{O}_{\delta_0} = \hbar d^q(\mathcal{O}^\star_G).$$

Now,

$$\begin{aligned} \hbar d^q(\mathcal{O}^\star_G \mathcal{O}_{\delta_x}) &= \mathcal{O}_{\delta_0}\mathcal{O}_{\delta_x} + \hbar \int G\delta_x \\ &= \mathcal{O}_{\delta_0}\mathcal{O}_{\delta_x} + \hbar G(x) \end{aligned}$$

Therefore in cohomology,

$$\mathcal{O}_{\delta_0} \cdot \mathcal{O}_{\delta_x} = -\hbar G(x).$$

Example 1.18 (1-dimension)**.** In 1-dimension, the structure of $\{-,-\}_{OPE}$ is the ordinary Poisson bracket. Take complex fermions ψ_i, ψ^i, action $\int \psi_i \partial_x \psi^i$. Define

$$\mathcal{O}^i(\psi_i, \psi^i) = \psi_i(0)$$
$$\mathcal{O}_i(\psi_i, \psi^i) = \psi^i(0)$$

Then

$$\{\mathcal{O}^i, \mathcal{O}_j\}_{OPE} = \delta^i_j(\delta_{|x|>0})$$

Key point:

$$\frac{\partial}{\partial x}\delta_{|x|>0} = \delta_{x=0}.$$

$\delta_{|x|>0}$ is the Green's function, same argument as before applies.

Example 1.19 (2d Chiral Theory)**.** For 2d chiral theories, $\{-,-\}_{OPE}$ gives us a Poisson vertex algebra,

$$\{-,-\}_{OPE} : \mathrm{Obs}^{cl}_0 \otimes \mathrm{Obs}^{cl}_0 \to \mathrm{Obs}^{cl}_0 \otimes \left(\frac{\mathrm{Hol}(\mathbb{C}^\times)}{\mathrm{Hol}(\mathbb{C})}.\right)$$

For complex fermions

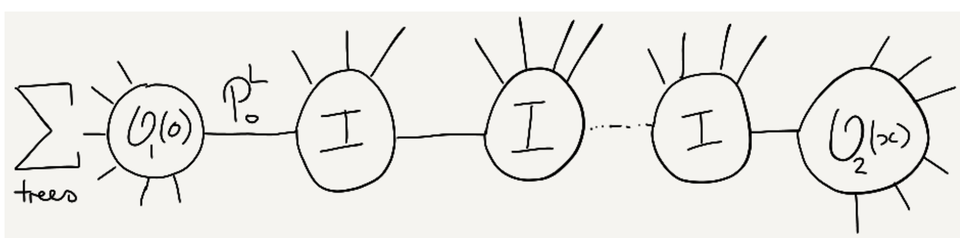
$$\int \psi^i \bar{\partial} \psi_i,$$

then because $\bar{\partial}\frac{1}{z} = \delta_{z=0}$, we have

$$\{\mathcal{O}^i, \mathcal{O}_j\} = \delta^i_j \frac{1}{z}.$$

1.5.3 Interacting Theories

Consider the φ^4 theory on $\mathbb{R}^4$. In general, for *any* theory, there is a formula for $\{-,-\}_{OPE}$ given entirely in terms of classical data. If $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Obs}^{cl}_0$, then

$\{\mathcal{O}_1, \mathcal{O}_2\}_{OPE} =$ 

– i.e. as a sum over trees of a particular form.

- In a given theory, only finitely many terms contribute.
- Answer is independent of L – changing L changes things by a quantity that is regular at $x = 0$.

In φ^4 theory on $\mathbb{R}^4$, and with the operator

$$\mathcal{O} : \varphi \mapsto \varphi(0),$$

we can compute $\{\mathcal{O}, \mathcal{O}\}_{OPE}$. We only need to consider the diagrams shown in Figure 6 – if there are more than two vertices in the middle of the diagram the integral converges (so doesn't contribute).

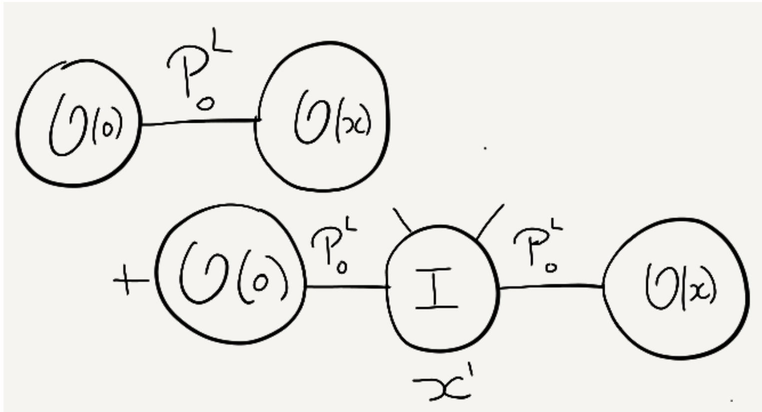


Figure 6: Terms contributing to the classical OPE bracket $\{\mathcal{O}, \mathcal{O}\}_{OPE}$.

$$
\begin{aligned}
(\text{first diagram in Figure 6}) &= \int_0^L K_t(0,x)dt \\
&= \int t^{-2} e^{-\|x\|^2/t} dt \\
&= \underbrace{\|x\|^{-2}}_{\text{Green's function}} \quad +(\text{terms continuous at } x=0) \\
(\text{second diagram in Figure 6}) &= \int_{x' \in \mathbb{R}^4} \int_{t_1,t_2=0}^{L} K_{t_1}(0,x')K_{t_2}(x',x)\varphi(x')^2
\end{aligned}
$$

You can show that the second diagram is

$$\log(\|x\|)\varphi(0)^2 + (\text{terms continuous at } x = 0).$$

Sketch: Write

$$
\begin{aligned}
\varphi(x') &= \varphi(0) + \varphi_1(x') \\
\varphi_1(x') &= 0 \text{ at } x' = 0
\end{aligned}
$$

Expand $\varphi(x')^2$: terms involving $\varphi_1(x')$ are *more* convergent, and so they don't contribute. We are left with

$$\varphi(0)^2 \int_{t_1,t_2} \int_{x'} K_{t_1}(0,x')K_2(x',x) = \varphi(0)^2 \int_{t_1,t_2} K_{t_1+t_2}(0,x) = \varphi(0)^2 \log \|x\| + (\text{regular})$$

In sum:

$$\{\mathcal{O}, \mathcal{O}\}_{OPE} = \frac{1}{\|x\|^2} + \mathcal{O}^2 \log \|x\| \in \mathrm{Obs}_0^{cl} \otimes \left(\frac{C^\omega(\mathcal{R}^4 \setminus 0)}{C_+^\omega(\mathbb{R}^4)} \right)$$

Problem 5. *Consider chiral complex fermions ψ_i, ψ^i. Then we have*

$$\{\mathcal{O}_i, \mathcal{O}^j\} \sim \delta_i^j \frac{1}{z}$$

(classical limit of chiral vertex algebra). If we have anti-chiral fermions, $\bar{\mathcal{O}}_i$, $\bar{\mathcal{O}}^j$,

$$\{\bar{\mathcal{O}}_i, \bar{\mathcal{O}}^j\} = \delta_i^j \frac{1}{\bar{z}}.$$

If chiral + anti-chiral fermions are deformed by an interaction

$$\psi_i \psi^j \bar{\psi}_k \bar{\psi}^l M_{ik}^{jl},$$

then the one loop OPE[14] *is deformed by (Figure 7)*

$$\{\psi_i, \psi^j\} = M_{ik}^{jl} \bar{\psi}_k \bar{\psi}^l \frac{\bar{z}}{z} \log |z|.$$

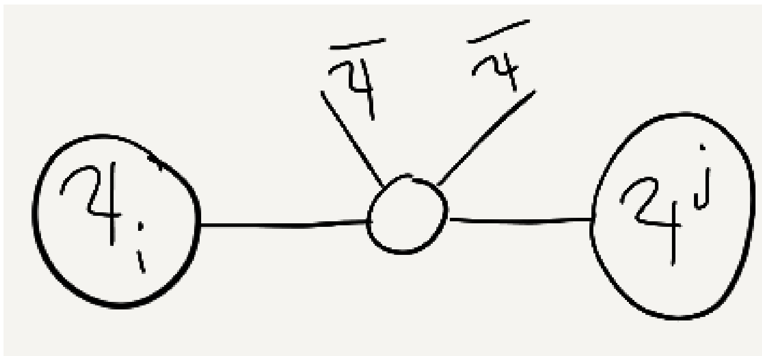


Figure 7: A 1-loop correction to the OPE $\mathcal{O}_i \mathcal{O}^j$.

1.5.4 RG flow and scale invariance

In the φ^4 theory, is there an $\mathbb{R}_{>0}$ on the fields that preserves all the structures?

$$\mathrm{Obs}_0^{cl} = \text{ algebra with 4 commuting derivations, generated by } \mathcal{O}, \text{ subject to } \sum \partial_{x_i}^2 \mathcal{O} + \mathcal{O}^3 = 0.$$

The $\mathbb{R}_{>0}$ should give $\mathcal{O}$ weight 1. But,

$$\{\mathcal{O}, \mathcal{O}\} = \underbrace{\frac{1}{\|x\|^2}}_{\text{good!}} + \underbrace{\mathcal{O}^2 \log \|x\|}_{\text{bad!}}$$

since

$$\log \|x\| \to \log \|x\| + \log \lambda.$$

Upshot: We've sketched why the quantum factorization algebra cannot be a fixed point of the $\mathbb{R}_{>0}$ action.

(This is something we have shown before, but now we have done it purely by studying structures on factorisation algebras.)

1.6 Lecture 6 (Philsang Yoo)

Today's topic: Noether's Theorem (as formulated by Costello-Gwilliam).

[14] I.e. to first order in $\hbar$.

1.6.1 Statement of Noether's Theorem

Suppose that $\mathcal{L}$ is a symmetry of the space of fields $\mathcal{E}$;[15] then there is a map

$$U^{BD}_{\ltimes}\mathcal{L} \to \mathrm{Obs}^q$$

where the left hand side is some twisted version of an enveloping algebra.

1.6.2 First: Classical, 0-dimensional, unshifted

I.e. Hamiltonian mechanics. Let (X, ω) be a symplectic manifold. Then a symmetry is a Lie algebra map

$$\mathfrak{g} \to \mathrm{Symp}\,\mathrm{Vect}(X).$$

Then Noether's theorem asks if there is a corresponding map to observables $\mathfrak{g} \to \mathcal{O}(X)$, i.e. can we find a lift

$$\begin{tikzcd} & (\mathcal{O}(X), \{-,-\}) \arrow[d] & f \arrow[d, mapsto] & ? \\ \mathfrak{g} \arrow[ur, dashed] \arrow[r] & \mathrm{Symp}\,\mathrm{Vect}(X) & \{f,-\} & \end{tikzcd}$$

We have a class in $H^1(X)$ as an obstruction – there is however always a lift from a central extension

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, equal] & \hat{\mathfrak{g}} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O}(X) \arrow[r] & \mathrm{Symp}\,\mathrm{Vect}(X) \end{tikzcd}$$

Recall the adjoint functors

$$\begin{tikzcd} \mathrm{Ass} \arrow[d, shift left, "\mathrm{forget}"] & \mathrm{Pois} \arrow[d, shift left, "\mathrm{forget}"] \\ \mathrm{Lie} \arrow[u, shift left, "U"] & \mathrm{Lie} \arrow[u, shift left, "\mathrm{Sym}"] \end{tikzcd}$$

So given a map $\mathfrak{g} \to \mathcal{O}(X)$ as a Lie algebra we obtain a map of Poisson algebras

$$\mathrm{Sym}(\mathfrak{g}) \to \mathcal{O}(X).$$

1.6.3 Now: Classical, 0-dimensional

I.e. classical field theory. So instead of X, we have

$$(\mathcal{E}, \omega) \quad \text{(-1)-shifted symplectic.}$$

Now

$$\mathfrak{g} \to \mathrm{Symp}\,\mathrm{Vect}(\mathcal{E}) = \mathrm{Ham}\,\mathrm{Vect}(\mathcal{E}) = \mathcal{O}_{red}(\mathcal{E})[-1];$$

replace $\mathcal{O}(X)$ by $\mathcal{O}(\mathcal{E})$. Because $\mathcal{E}$ is (-1)-shifted symplectic, $(\mathcal{O}(\mathcal{E})[-1], \{-,-\})$ is a dgla, and Noether's theorem asks about a lift

$$\begin{tikzcd} & (\mathcal{O}(\mathcal{E})[-1], \{-,-\}) \arrow[d] \\ \mathfrak{g} \arrow[ur, dashed] \arrow[r] & \mathrm{Symp}\,\mathrm{Vect}(\mathcal{E}) \end{tikzcd}$$

[15] Something like: L_∞-algebra acting on a shifted symplectic space.

The adjunction we are now interested in is

$$\begin{tikzcd} \mathbb{P}_0 \arrow[d, shift left, "(-)[-1]"] \\ \mathrm{Lie} \arrow[u, shift left, "\mathrm{Sym}(-[-1])"] \end{tikzcd}$$

and so such a lift gives us a map $\mathrm{Sym}(\mathfrak{g}[1]) \to \mathcal{O}(\mathcal{E})$. There is an obstruction $\alpha \in H^1(\mathfrak{g})$ corresponding to an L_∞-algebra central extension,

$$0 \to \mathbb{C}_c[-1] \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

and so by applying the enveloping algebra adjunction construction we obtain a map[16]

$$U_\alpha^{\mathfrak{g}} \to \mathcal{O}(\mathcal{E}).$$

Concretely we can think of the enveloping algebra construction as

$$U_\alpha^{P_0}(\mathfrak{g}) = U^{P_0}(\hat{g})_{c=1}.$$

1.6.4 Now again: Quantum, 0-dimensional

Goal: Given $\mathfrak{g} \to \mathrm{Symp\,Vect}(\mathcal{E})$ (or its quantisation), we want a quantisation of $\mathfrak{g} \to \mathcal{O}(\mathcal{E})[-1]$.

Recall: This corresponds to – given an action on fields, lift to an action on observables.

One thing we have learned is that

$$(\mathcal{O}(\mathcal{E})[-1], Q + \{I, -\})$$

is the deformation-obstruction complex for the theory $\mathcal{E}$. Order-by-order in $\hbar$ we are trying to solve the QME; there is an obstruction whose vanishing implies that such a lift is possible.

Recall:[17]

$$\mathcal{O}(\mathcal{E}) = \mathcal{O}(B\mathcal{M}) = C^\bullet(\mathcal{M})$$

where $\mathcal{M} = \mathcal{E}[-1]$.[18] Including the data of the action functional, we have

$$\begin{tikzcd} \mathcal{O}(\mathcal{E}) \arrow[r, equal] & C^\bullet(\mathcal{M}) \\ \{S, -\} \arrow[r, leftrightarrow] & d_{CE} \end{tikzcd}$$

and $(C^\bullet(\mathcal{M}), d_{CE})$ corresponds to an L_∞-structure on $\mathcal{M}$.

Recall from Exercise 4 of the first day:

- A map of L_∞ algebras $\mathfrak{g}_1 \to \mathfrak{g}_2$ is equivalent to,
- a map of cdgas $C^\bullet(\mathfrak{g}_2) \to C^\bullet(\mathfrak{g}_1)$ is equivalent to
- a Maurer-Cartan element in $C^\bullet_{red}(\mathfrak{g}_1) \otimes \mathfrak{g}_2$.[19]

So: the lifting question we are now asking about is

$$\begin{tikzcd} & \mathcal{O}(\mathcal{E})[-1] = C^\bullet(\mathcal{M})[-1] \arrow[d] \\ \mathfrak{g} \arrow[r] \arrow[ur, dashed] & \mathrm{Symp\,Vect}(\mathcal{E}) = C^\bullet_{red}(\mathcal{M})[-1] \end{tikzcd}$$

[16]Corresponding to a map *out of the central extension* – a map out of $\mathfrak{g}$ exists only if $\alpha = 0$.
[17]Some of this recollection is from the week 1 exercises.
[18]$B\mathcal{M}$ is just notation, i.e. we define it by the property $\mathcal{O}(B\mathcal{M}) = C^\bullet(\mathcal{M})$. Recall also that $C^\bullet(\mathfrak{g}) = \widehat{\mathrm{Sym}}(\mathfrak{g}^*[-1]$.
[19]Reduced cochains are the kernel of the augmentation map.

There's a lot going on here: consider the following diagram which defines the objects Inner Act and Act,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C^\bullet_{red}(\mathfrak{g})[-1] \arrow[r, equal] \arrow[d] & C^\bullet_{red}(\mathfrak{g})[-1] \arrow[d] & & \\
0 \arrow[r] & \mathrm{Inner\,Act}(\mathfrak{g}, \mathcal{M}) \arrow[r] \arrow[d] & C^\bullet_{red}(\mathfrak{g} \oplus \mathcal{M})[-1] \arrow[r] \arrow[d] & C^\bullet_{red}(\mathcal{M})[-1] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Act}(\mathfrak{g}, \mathcal{M}) \arrow[r] \arrow[d] & \frac{C^\bullet_{red}(\mathfrak{g} \oplus \mathcal{M})[-1]}{C^\bullet_{red}(\mathfrak{g})[-1]} \arrow[r] \arrow[d] & C^\bullet_{red}(\mathcal{M})[-1] \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The left hand column is an extension with obstruction class living in $H^1(C^\bullet_{red}(\mathfrak{g}))$, and our lifting question is about whether we can lift lifting an action to an inner action.[20]

Proposition 1.7. *Let $\mathcal{M}$ be a classical field theory.*

- *An action of $\mathfrak{g}$ on $\mathcal{M}$ is a MC element $S^{\mathfrak{g}} \in \mathrm{Act}(\mathfrak{g}, \mathcal{M})$; in particular it is $S^{\mathfrak{g}} \in C^\bullet_{red}(\mathfrak{g} \oplus \mathcal{M})$ such that*

$$(d_{\mathfrak{g}} + d_{\mathcal{M}})S^{\mathfrak{g}} + \frac{1}{2}[S^{\mathfrak{g}}, S^{\mathfrak{g}}] \in C^\bullet_{red}(\mathfrak{g}),$$

  or, $S^{tot} = S + S^{\mathfrak{g}}$ such that

$$d_{\mathfrak{g}} S^{tot} + \frac{1}{2}[S^{tot}, S^{tot}] = 0$$

  in $\mathrm{Act}(\mathfrak{g}, \mathcal{M})$.

- *An inner action of $\mathfrak{g}$ on $\mathcal{M}$ is a MC element $S^{\mathfrak{g}} \in \mathrm{Inner\,Act}(\mathfrak{g}, \mathcal{M})$.*
- *Given an action of $\mathfrak{g}$ on $\mathcal{M}$ there is an obstruction class in $H^1(C^\bullet_{red}(\mathfrak{g}))$, and the action extends to an inner action if and only if the obstruction class vanishes.*

1.6.5 d-dimensional classical case

In the d-dimensional case, what happens?

- $\mathrm{Symp\,Vect}(\mathcal{E})$ remains $\mathrm{Symp\,Vect}(\mathcal{E}) = \mathcal{O}_{loc,red}(\mathcal{E})[-1]$
- $\mathcal{O}(\mathcal{E})$ becomes Obs

So we are interested in the following problem:

$$\begin{tikzcd}
 & \mathrm{Obs} \\
\mathfrak{g} \arrow[r] \arrow[ur, dashed] & \mathrm{Symp\,Vect}(\mathcal{E})
\end{tikzcd}$$

Note that in the d-dimensional case there is not a natural map $\mathrm{Obs} \to \mathrm{Symp\,Vect}(\mathcal{E})$, so the problem of understanding how to lift a symmetry of fields to a symmetry of observables is somewhat subtle.

Now: we have to replace $\mathfrak{g}$ with a *local DG Lie algebra* $\mathcal{L}$, i.e. $\mathcal{L}$ is a sheaf of sections of a bundle together with differential operators d, l_2.[21]

[20] I'm almost certainly misunderstanding this.
[21] Satisfying various conditions, etc.

Example 1.20 (G-bundles)**.** Let Σ be a Riemann surface, $\mathfrak{g}$ a Lie algebra. Then we take

$$\mathcal{L}_\Sigma : U \mapsto \Omega^{0,\bullet}(U) \otimes \mathfrak{g}, \bar{\partial}, [-,-]).$$

Example 1.21 (Complex structure)**.** If X is a complex manifold, set

$$\mathcal{L}^{hol}(X) = (\Omega^{0,\bullet}(X, T_X), \bar{\partial}, [-,-]).$$

Fact: There exists a bijection between k-shifted local central extensions of a local L_∞-algebra $\mathcal{L}$ and $H^{k+2}(C^\bullet_{loc,red}(\mathcal{L}))$.

Theorem 1.8 (Classical Noether Theorem)**.** *Suppose that $\mathcal{L}$ acts on $\mathcal{M}$ and suppose that $\alpha \in H^1(C^\bullet_{loc,red}(\mathcal{L}))$ is the obstruction to making it inner, defining a central extension $\hat{\mathcal{L}}$. Then there is a map $U^{\mathbb{P}_0}_\alpha(\mathcal{L}) \to \mathrm{Obs}^{cl}$.*

1.6.6 Noether theorem

Upshot: A defomration of a theory arises from a symmetry in cohomological degree 1.

Example 1.22 (Free scalar field theory on M)**.** Given

$$\phi \in C^\infty(M) \xrightarrow{D} \Omega^d(M)[-1] \ni \psi$$

Introduce $\mathcal{L} = \Omega^d(M)[-1]$. Then

$$\begin{array}{ccc} \Omega^d_c(U)[-1] & \longrightarrow & \mathrm{Obs}^{cl}(U)[-1] \\ \alpha & \longmapsto & J(\alpha) \end{array}$$

where $J(\alpha)$ is the classical observable

$$J(\alpha) : \phi \mapsto \int \phi\alpha.$$

Phil now presents the following statement: **Global symmetry is local symmetry.**

Given $\mathfrak{g}$ a Lie algebra, can construct $\Omega^\bullet_M \otimes \mathfrak{g}$. In some sense this is the same amount of information – the de Rham complex is a resolution of the constant sheaf – and this is the sort of thing that Phil will mean by the statement "global symmetry is local symmetry".

Proposition 1.9. *If $\mathcal{M}$ is in degrees 1,2, then the action factors through $\Omega^{\leq 1}_M \otimes \mathfrak{g}$.*

Proof. $\mathrm{Act}(\Omega^\bullet_M \otimes \mathfrak{g}, \mathcal{M}) = \mathcal{O}_{loc,red}(\Omega^\bullet_M, \mathfrak{g}[1] \oplus \mathcal{M}[1])[-1]$. Missed the rest of this argument :-(□

Example 1.23 (Free scalar theory on a surface Σ)**.** If

$$\mathcal{E} = \left(\Omega^{0,0} \otimes \mathbb{C}^n \xrightarrow{\partial\bar{\partial}} \Omega^{1,1} \otimes \mathbb{C}^n[-1] \right)$$

then there is an action of $\mathfrak{so}(n)$ on $\mathcal{E}$ ("$\mathfrak{so}(n)$ flavour symmetry"). So $\Omega^\bullet \otimes \mathfrak{so}(n)$ acts on $\mathcal{E}$ and we can write down an action functional on fields from $\mathcal{O}(B(\mathcal{M} \oplus \mathcal{L}))$, i.e. from

$$\begin{array}{cccccc} & 0 & & 1 & & 2 \\ \mathcal{L} & c \in \Omega^0 \otimes \mathfrak{so}(n) & \xrightarrow{d} & A \in \Omega^1 \otimes \mathfrak{so}(n) & \xrightarrow{d} & \Omega^2 \otimes \mathfrak{so}(n) \\ \mathcal{M} & & & \phi \in \Omega^{0,0} \otimes \mathbb{C}^n & \longrightarrow & \psi \in \Omega^{1,1} \otimes \mathbb{C}^n \end{array}$$

as

$$S^{\mathcal{L}}(A, c, \phi, \psi) = \int \phi_i c_{ij} \psi_j + \frac{1}{2} \int \bar{\partial}\phi_i A^{1,0}_{ij} \phi_j + \frac{1}{2} \int A^{0,1}_{ij} \phi) j \partial \phi_i + \frac{1}{2} \int A^{0,1}_{ij} \phi_j A^{1,0}_{ik} \phi_k.$$

Definitely didn't understand what was precisely meant here – hoping that the exercises illuminate.

Results in the quantum setting: $U_{\alpha}^{\mathbb{P}_0} \to U^{BD}$.

Example 1.24.
- $\Omega_{\Sigma}^{0,\bullet} \otimes \mathfrak{g}$ yields a Kac-Moody vertex algebra.
- $\Omega_{\Sigma}^{0,\bullet}(\Sigma, T_{\Sigma})$ yields a Virasoro vertex algebra.

Example 1.25 (B. Williams)**.** $\Omega_{\Sigma}^{0,\bullet}(\Sigma, T_{\Sigma})$ acts on the $\beta\gamma$ system with V of dimension n. This yields a map $\mathrm{Vir}_{c=2n} \to \mathrm{Obs}^q_{\beta\gamma}$.

1.7 Lecture 7 (Kevin Costello)

Today's topic: **More on Noether's Theorem.**

1.7.1 Factorization Envelope

If

- $\mathfrak{g}$ is a Lie algebra, we can produce
- $U\mathfrak{g}$ an associative algebra.

Similarly, if one consults Beilinson-Drinfeld, there is a similar statement that if

- $\mathfrak{g}$ is a Lie* algebra/Vertex Lie algebra, we can construct
- $U^{ch}\mathfrak{g}$ the "chiral envelope/vertex algebra envelope", a vertex algebra.

What happens in the factorisation algebra world?

Suppose that $\mathcal{L}$ is a sheaf of dlgas on M.[22] Then we can produce

$$U^{fact}(\mathcal{L})$$

a *factorisation algebra*. The encompasses $U\mathfrak{g}$ and $U^{ch}\mathfrak{g}$ as special cases, and also higher Kac-Moody algebras [Williams, Kapranov et al.].

Example 1.26. We take

$$\mathcal{L} = \Omega^*_{\mathbb{R}} \otimes \mathfrak{g} \text{ sheaf on } \mathbb{R},$$
$$\mathcal{L} = \Omega^{0,*}_{\mathbb{C}} \otimes \mathfrak{g} \text{ sheaf on } \mathbb{C}.$$

We'll consider the factorization envelopes below.

Factorization envelope:

$$U^{fact}(\mathcal{L})(U) = C_*(\mathcal{L}_c(U))$$

where C_* are Lie algebra chains, and the c subscript denotes compactly supported sections.

Lemma 1.10. *If $\mathcal{L} = \Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$ then $H^*(U^{fact}(\mathcal{L})) = U\mathfrak{g}$ as an associative algebra.*

[22] Technical assumption: $\mathcal{L}(U) = C^\infty(U, L)$ for L a graded vector bundle, and d, $[-,-]$ are differential operators. Don't worry too much about this technical assumption

Proof. $C_*(\mathcal{L}_c((a,b))) = \mathrm{Sym}^*(\Omega^*_c(a,b)\otimes\mathfrak{g}[1])$. If $X\in\mathfrak{g}$ let $X_0 = X\otimes f(t)dt$ where f is a bump function around 0, and set

$$X_\epsilon = X\otimes f(t-\epsilon)dt$$

which is supported around ϵ. Then $[X,Y]$ is the cohomology class of

$$X_0\cdot Y_\epsilon - X_0\cdot Y_{-\epsilon}.$$

There exists a function $g(t)$ such that $g=1$ near 0 and

$$dg = f(t-\epsilon)dt - f(t+\epsilon)dt.$$

Take

$$d(X_0\cdot(Y\otimes g)) = \underbrace{X_0\cdot(Y\otimes dg)}_{\text{comes from de Rham differential}} + \underbrace{[X_0, Y\otimes g]}_{\text{comes from Lie algebra chains differential}}$$

So $[X,Y]=$ cohomology class of $X_0Y\epsilon - X_0Y_{-\epsilon}$ which is equal to the second term. So

$$[X,Y] = [X,Y]f(t)g(t) = [X,Y]_0$$

so there's a homomorphism. Checking that it is an isomorphism is an exercise. □

For the second example we'll allow ourselves the luxury of allowing distributions.

Lemma 1.11. *For $\mathcal{L} = \Omega^{0,*}_{\mathbb{C}}\otimes\mathfrak{g}$, $H^*(U^{fact}(\mathcal{L}))$ is a VOA which is a level 0 Kac-Moody algebra.*

Proof. Note that for the disc D,

$$H^*_{\bar\partial,c}(D)\otimes\mathfrak{g}\sim z^{-1}\mathfrak{g}[z^{-1}][-1],$$

spanned by δ_0 and its derivatives. Then

$$\begin{aligned} H^*(U^{fact}(\mathcal{L})(D)) &= H^*(\mathrm{Sym}^*(\Omega^{0,*}_c(D)\otimes\mathfrak{g}[1])) \\ &= \mathrm{Sym}^*(z^{-1}\mathfrak{g}[z^{-1}]) \\ &= \text{ Vacuum module for Kac-Moody} \end{aligned}$$

If $X\in\mathfrak{g}$ set $X_0 = X\otimes\delta_0$. We want to compute the cohomology class of $X_0\cdot Y_{z_0}$. To compute this, note that

$$\delta_0 = \bar\partial\left(\frac{1}{z}\right),$$

so that

$$X\otimes\delta_0 = \bar\partial\left(X\otimes\frac{1}{z}\right).$$

Since $\frac{1}{z}$ isn't compactly supported, choose a function f which is 1 near 0 and vanishes far away from 0. Then

$$X\otimes\delta_0 = \bar\partial\left(X\otimes\frac{f}{z}\right) + (\text{stuff supported away from } 0)$$

So

$$(X\otimes\delta_0)\cdot(Y\otimes\delta_{z_0}) = d\left(\left(X\otimes\frac{f}{z}\right)\cdot(Y\otimes\delta_{z_0})\right) - [X,Y]\frac{f(z_0)}{z_0}$$

Terms of the form $X\frac{\bar\partial f}{z}Y\delta_{z_0}$ vanish. But now taking z_0 small, we can replace $f(z_0)$ by 1, and so

$$(X\delta_0)(Y\delta_{z_0})\sim\frac{1}{z_0}[X,Y].$$

□

1.7.2 Central Extensions

Suppose that $\mathcal{L}$ is a sheaf of dglas.

Definition 1.13. A *1-shifted cocycle* is a cochain map

$$\alpha : \mathcal{L}_c \times \mathcal{L}_c \to \mathbb{C} \cdot c[-1].$$

Example 1.27. If $\mathcal{L} = \Omega^{0,*}_\Sigma \otimes \mathfrak{g}$, there's a 1-shifted cocycle

$$\alpha(l_1, l_2) = \int \mathrm{Tr}_R \, l_1 \partial l_2.$$

Why is this shifted? It takes one thing in degree 0 and one thing in degree 1 and gives you a number – hence the shift by 1.

Given such a cocycle we can form a central extension

$$0 \to \mathbb{C} \cdot c[-1] \to \widehat{\mathcal{L}_c} \to \mathcal{L}_c \to 0$$

and define the *twisted factorization envelope* by

$$U \to C_*(\widehat{\mathcal{L}_c}(U)) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1};$$

the central element in degree 0, and we specialize to $c = 1$.

Lemma 1.12. *The twisted factorization envelope of $\Omega^{0,*}_\Sigma \otimes \mathfrak{g}$ is the Kac-Moody at level given by the central extension.*

Remark 1.20. This lemma shouldn't be surprising, since there's a term of the form in Example 1.27 in the definition of the Kac-Moody.

Remark 1.21. There is also a notion of central extension for non-compactly supported section.

Example 1.28. On $\mathbb{R}^n$ define a sheaf of dglas by

$$\mathcal{L} = \begin{pmatrix} \mathcal{C}^\infty_{\mathbb{R}^n} & \xrightarrow{\Delta} & \mathcal{C}^\infty_{\mathbb{R}^n} \\ 0 & & 1 \end{pmatrix}$$

with $[-,-] \equiv 0$. Denote degree 0 elements by φ and degree 1 elements by $\varphi^\star$. Then $\mathcal{L}$ has a shifted by 1 central extension

$$\alpha(\varphi, \varphi^\star) = \int \varphi \varphi^\star.$$

Lemma 1.13. *The twisted factorization envelope of this is the quantum observables of the free scalar.*

Proof. We have

$$U^\alpha(\mathcal{L}) = \mathrm{Sym}^* \begin{pmatrix} \mathcal{O}^\star_g & & \mathcal{O}_f \\ C^\infty_c(U) & \xrightarrow{\Delta} & C^\infty_c(U) \\ -1 & & 0 \end{pmatrix},$$

and the central extension gives

$$d(\mathcal{O}^\star_g \mathcal{O}_f) = \int fg.$$

□

1.7.3 Noether's Theorem

Theorem 1.14. *If we have a sheaf $\mathcal{L}$ of dglas which acts on a QFT, then there exists a cocycle*[23] α*, dependent on* $\hbar$*, and a map of factorzation algebras*

$$U^*(\mathcal{L}) \to \mathrm{Obs}^q.$$

Upshot: Given a symmetry we have a collection of operators which we have more control over (since they come explicitly from the twisted factorization envelope).

Example 1.29. If

$$\mathcal{L} = \begin{pmatrix} \gamma & & \gamma^\star \\ \mathcal{C}^\infty_{\mathbb{R}^n} & \xrightarrow{\Delta} & \mathcal{C}^\infty_{\mathbb{R}^n} \\ 0 & & 1 \end{pmatrix}$$

then $\mathcal{L}$ acts on a free scalar field theory[24] by translation. If φ and $\varphi^\star$ are the fields of the scalar field theory[25], then

$$\varphi \to \varphi + \gamma$$
$$\varphi^\star \to \varphi^\star + \gamma^\star$$

is an action. As, e.g.

$$H^0(\mathcal{L}(U)) = \{\gamma, \Delta\gamma = 0\},$$

if φ satisfies the EOM ($\Delta\varphi = 0$) then so does $\varphi + \gamma$. Therefore there exists a cocycle α such that

$$U^\alpha_{fact}(\mathcal{L}) \to \mathrm{Obs}^q;$$

the cocycle is of course $\int \gamma\gamma^\star$.

Remark 1.22. This "explains" why free field theories have such a simple factorization algebra – they have lots of symmetries.

Example 1.30. Consider topological quantum mechanics, fields p_i, q^i, Lagrangian $\sum \int p_i dq^i$. If $1 \leq i \leq n$ then $\mathfrak{gl}_n$ acts.

But there is something better we can say:

$$\mathcal{L} = \Omega^*_{\mathbb{R}} \otimes \mathfrak{gl}_n \text{ acts.}$$

To see this, use BV:

$$p_i \rightsquigarrow P_i \in \Omega^*(\mathbb{R}) \otimes \mathbb{R}^n$$
$$q^i \rightsquigarrow Q^i \in \Omega^*(\mathbb{R}) \otimes (\mathbb{R}^n)^\vee$$

$\mathcal{L}$ acts on $\Omega^*(\mathbb{R}) \otimes \mathbb{R}^n / \Omega^*(\mathbb{R}) \otimes (\mathbb{R}^n)^\vee$ by the fundamental/antifundamental representations.

The way that Phil explained this is that the action is a functional that depends on $l \in \mathcal{L}[1]$ and P, Q. The function is

$$S(P_i, l^j_i, Q^j) = \int P_i dQ^i + \int P_i l^i_j Q^j.$$

This is

$$\omega(P, l \cdot Q)$$

[23] In general this is some L_∞ version of the cocycle we discussed above.

[24] I.e. acts on itself.

[25] Important to distinguish the fields of the theory from the symmetry generators, even though they come from "the same" sheaf of dglas.

where ω is an odd symplectic pairing and $\cdot$ denotes the action.

What does Noether's theorem say in this case? We will have a map

$$U^{fact}(\mathcal{L}) \to \mathrm{Obs}^q .$$

At the level of cohomology,[26]

$$\begin{aligned} U^{fact}(\mathcal{L}) &= U\mathfrak{gl}_n \\ \mathrm{Obs}^q &= \mathrm{Diff}(\mathbb{R}^n) \end{aligned}$$

and the map is the obvious one.[27]

Fact: If $l \in \mathcal{L}_c(U)$ and $S(P,Q,l)$ is the action then the map

$$\mathrm{Sym}^*(\mathcal{L}_c(U)[1]) \supseteq \mathcal{L}_c(U)[1] \to \mathrm{Obs}^q(U)$$

is

$$l \mapsto \left(\frac{\partial}{\partial l} S(P,Q,l)\right)_{l=0} .$$

Concretely: Take the term linear in $\mathcal{L}$, and insert $l \in \mathcal{L}_c(U)$ into $S(P,Q,l)$. If $l = E)_j \otimes f(t)dt$ for $f \in C_c^\infty((a,b))$, then the element of $\mathrm{Obs}^q((a,b))$ is

$$\int P_i(t)Q^j(t)f(t)dt.$$

This is the *Noether charge*.

At the level of cohomology, in $\mathrm{Diff}(\mathbb{R}^n)$ this is $p_i q^j$. Up to some annoying stuff these give a homomorphism

$$U(\mathfrak{gl}_n) \to \mathrm{Diff}(\mathbb{R}^n).$$

Example 1.31 ($\beta - \gamma$ system)**.** Let R be a representation of $\mathfrak{g}$,

$$B \in \Omega^{1,*}(\mathbb{C}, R^*), \qquad \Gamma \in \Omega^{0,*}(\mathbb{C}, R),$$

and

$$\mathcal{L} = \Omega^{0,*}(\mathbb{C}) \otimes \mathfrak{g}.$$

Evidently, $\mathcal{L}$ acts. The action S coupling $B, \Gamma, \mathcal{L}$ is

$$S(B,\Gamma,l) = \int B\bar{\partial}_l\Gamma = \int B\bar{\partial}\Gamma + \int Bl\Gamma.$$

The map

$$U^{\alpha}_{fact}(\mathcal{L}) \to \mathrm{Obs}^q$$

is

$$X \in \mathfrak{g}, \quad X \otimes \delta_{z=0} \mapsto \langle B, X \cdot \Gamma\rangle(0).$$

This is the usual Noether charge, J^a, a an index for a basis of $\mathfrak{g}$. There's a central extension here – we won't go into this – and so there is a map

$$\mathcal{KM}_{\mathfrak{g}} \to \mathrm{Obs}^q .$$

[26] In this case the cocycle is trivial – there is an argument using the fact that it should be $\mathfrak{gl}_n$-invariant.

[27] I.e. induced by the action of GL_n on $\mathbb{R}^n$.

1.7.4 Recall: Classical bracket

Last time, we considered point observables Obs_0^{cl}, and we showed the existence of a bracket

$$\{-,-\} : \mathrm{Obs}_0^{cl} \otimes \mathrm{Obs}_0^{cl} \to \mathrm{Obs}_0^{cl} \otimes \left(\frac{C^\omega(\mathbb{R}^n \setminus 0)}{C_+^\omega(\mathbb{R}^n)} \right)$$

Example 1.32. In the $\beta - \gamma$ system,

$$\mathcal{O}_\beta^i : \beta \mapsto \beta(0)$$
$$\mathcal{O}_{\gamma,i} : \gamma \mapsto \gamma(0)$$

and

$$\{\mathcal{O}_\beta^i, \mathcal{O}_{\gamma,j}\} = \delta_j^i \frac{1}{z}.$$

For $U_{fact}(\mathcal{L})$ there is a similar limit. We take distributions $l \in \overline{\mathcal{L}}_0$ supported at 0. These have a bracket

$$\overline{\mathcal{L}}_0 \otimes \overline{\mathcal{L}}_0 \to \overline{\mathcal{L}}_0 \otimes \frac{C^\omega(\mathbb{R}^n \setminus 0)}{C_+^\omega(\mathbb{R}^n)}$$

defined in the same way

Example 1.33. Concretely, for Kac-Moody,

$$H^*(\underbrace{\overline{\Omega}_0^{0,*}}_{\text{supp at 0}} \otimes \mathfrak{g}) = z^{-1}\mathfrak{g}[z^{-1}]$$

spanned by $X \otimes \delta_0$, $X \otimes \delta_0'$, etc., and

$$\{X \otimes \delta_0, Y \otimes \delta_0\} = [X,Y]\frac{1}{z}.$$

Example 1.34. For $\mathfrak{gl}_n$ acting on the $\beta - \gamma$ system, $\mathcal{O}_\beta^i$, $\mathcal{O}_{\gamma,j}$ have a bracket like the $\mathfrak{gl}_n$ Kac-Moody:

$$\{\mathcal{O}_\beta^i \mathcal{O}_{\gamma,j}, \mathcal{O}_\beta^k \mathcal{O}_\gamma, l\} = \mathcal{O}_\beta^i \mathcal{O}_{\gamma,l} \delta_j^k \frac{1}{z}.$$

1.7.5 Spending some time on Tudor's questions

Tudor asks what happens in higher dimensions. We're going to explore this a little.

Question 1. *If a Lie algebra* $\mathfrak{g}$ *acts on a theory, what do we do?*

Answer 1.1. This is equivalent to $\Omega_M^* \otimes \mathfrak{g}$ acting. So we get a homomorphism

$$U_{fact}^\alpha(\Omega_M^* \otimes \mathfrak{g}) \to \mathrm{Obs}^q.$$

But there's something funny about this answer. If $D \subseteq \mathbb{R}^n$ is a disc, then $H_c^*(D)$ is in degree n. So,

$$H^*(U_{fact}(\Omega_{\mathbb{R}^n}^* \otimes \mathfrak{g})(D)) = \mathrm{Sym}^*(\mathfrak{g}[1-n]).$$

If $n > 1$, there are **no** degree 0 elements, except for the identity. In general, we build a local operator of degree $n-1$ from a symmetry.

Doesn't this contradict what Noether said? No – because she wasn't interested in local operators, she was interested in something that you can integrate over a codimension one submanifold, and that's a priori something a bit different.

But: If $M = N \times \mathbb{R}$ with N compact and we take $U_{fact}(\Omega^*_M \otimes \mathfrak{g})(N \times (-\epsilon, \epsilon))$, we get something like

$$\mathrm{Sym}^* \left(H^*_c(N \times (-\epsilon, \epsilon))[1] \otimes \mathfrak{g} \right) = \mathrm{Sym}^*(H^*(N) \otimes \mathfrak{g})$$

(cancellation of shifts). In fact we get

$$U(H^*(N) \otimes \mathfrak{g}).$$

This is good! In degree 0, this is $U\mathfrak{g}$.

Fact: If we have a theory which is not a gauge theory, so that the observables are in degrees ≤ 0, then an action of $\Omega^*_{\mathbb{R}^n} \otimes \mathfrak{g}$ factors through $\Omega^{\leq 1}_{\mathbb{R}^n} \otimes \mathfrak{g}$. So in this situation, we can get a local Noether charge J.

One last point for the experts: For $n = 1$, we say that in degree zero we found the universal enveloping algebra. You probably won't be surprised to find out that for $n > 1$ one finds the E_n universal enveloping algebra. So if you have a symmetry, you obtain a map to observables from an E_n factorisation algebra.

2 Supersymmetric Quantum Mechanics and All That

2.1 Lecture 1 (Mathew Bullimore)

Good references with lots of examples:

- Mirror Symmetry, Hort et al, Ch 10
- Dirichlet branes and Mirror Symmetry, Aspinwall et al, Ch 3.1

Since these are so good, we're going to try and cover some aspects that aren't covered by these references.

2.1.1 Motivation

Want to study SQM, i.e. SUSY QFT in $d = 1$. So we're working on $M = \mathbb{R}$, coordinatised by a parameter τ.

This is useful for QFT in dimensions $d > 1$ – taking a theory on $\mathbb{R}_\tau \times M_{d-1}$ and reducing certain questions to a problem in SQM on the line $\mathbb{R}_\tau$.

Example 2.1 ($d = 2$)**.** Boundary conditions in 2d SUSY QFT form a category. Given two boundary conditions B_1 and B_2 meeting at a juncture, the space of BC changing local operators at the juncture gives $\mathrm{Hom}(B_1, B_2)$.

Now, there is a state-operator map: can think of a theory on and interval I tiems $\mathbb{R}$, with $B_1^\vee$ on one boundary and B_2 on the other boundary. Under certain conditions, can argue that the size of the interval can be shrunk to zero – the corresponding theory is a 1d QFT on the "squashed together" boundary, and so we can turn some 2d questions into SQM questions in this way.

2.1.2 Quantum Mechanics

We have the following objects in a QM theory:

A. **States:** elements (rays) in a complex Hilbert space Ω, $\langle -, - \rangle : \Omega \times \Omega \to \mathbb{C}$.

B. **Operators:** linear operators $A : \Omega \to \Omega$, which generate an associative algebra $\mathcal{A}$.

Putting these together we can consider **measurements**:

- Measurable quantities are *self-adjoint* operators A.
- Possible outcomes of a measurement are $\mathrm{Spec}(A) \subset \mathbb{R}$ (the spectrum – eigenvalues – are real due to the self-adjoint assumption).

In particular we are interested in **time evolution** of a system:[28]

- Euclidean time τ.
- distinguished self-adjoint operator H ("Hamiltonian")

[28] We will work in a Euclidean framework rather than Lorentzian. This won't really play a role going forward in this lecture series, but as the rest of the workshop will be Euclidean focused it seems reasonable to make this assumption now and avoid confusion later.

– *Schrodinger picture:* states evolve,

$$\partial_\tau \psi = -H \cdot \psi, \quad \psi \in \Omega$$

– *Heisenberg picture:* operators evolve,

$$\partial_\tau A = [H, A], \quad A \in \mathcal{A}$$

Definition 2.1. An operator A is conserved if $[H, A] = 0$.

In particular: we won't be starting with a classical theory and then "quantizing" – the quantum theory is fundamental, and so we just start there.

Example 2.2 (Particle on S^1)**.** The theory of a particle on a circle has:

$$\Omega = L^2(S^1, \mathbb{C}), \qquad \langle f, g\rangle = \int_0^{2\pi} \overline{f(\theta)} g(\theta)\, d\theta$$

There is a self-adjoint operator called the *momentum operator*,

$$\begin{aligned} p &= -i\frac{\partial}{\partial\theta}, \\ \mathrm{Spec}(p) &= \mathbb{Z}. \\ \phi_n(\theta) &= \frac{1}{\sqrt{2\pi}} e^{in\theta},\ n \in \mathbb{Z} && \text{(eigenfunctions)} \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} H &= \frac{p^2}{2} = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2}, \\ \mathrm{Spec}(H) &= \left\{\frac{n^2}{2},\ n \in \mathbb{Z}\right\} \end{aligned}$$

We have that $[H, p] = 0$, so p is *conserved*.

Example 2.3 (Riemannian σ-model)**.** Start with X a compact smooth Riemannian manifold. If we were to start with a classical theory, we would begin by writing down an action functional with input given by maps from $\mathbb{R}_\tau$ to X.

We won't do that – the quantum theory is fundamental – so let's just write down the data for the quantum theory:

$$\begin{aligned} \Omega &= L^2(X, \mathbb{C}) \\ \langle f, g\rangle &= \int_X \bar{f} g\, \mathrm{dvol}_X \end{aligned}$$

The algebra of operators $\mathcal{A}$ is the algebra of differential operators on X:

- Smooth functions $f : X \to \mathbb{C}$ act by
$$A_f : \psi \mapsto f\psi,$$
and this provides a commutative subalgebra of $\mathcal{A}$ (self-adjoint if f is a real valued function).
- Vector fields V act by
$$A_V : \psi \mapsto -iV[\psi] = -iV^i \frac{\partial\psi}{\partial X^i}$$
and if V is real then again this will be self adjoint.

The Hamiltonian in this theory is $H = \Delta$, the Laplacian on X; consequently the time evolution equation is the heat equation on X:

$$\partial_\tau \psi = -\Delta \psi$$

If V is a real vector field that generates an isometry of X, then

$$[H, A_V] = 0$$

and we have another conserved quantity (again interpretable as momentum).

2.1.3 Supersymmetric Quantum Mechanics

This is quantum mechanics with *additional structure*:

- The Hilbert space must be $\mathbb{Z}/2$-graded,

$$\Omega = \Omega_e \oplus \Omega_o \qquad \text{(even and odd parts).}$$

- There exist odd operators Q and $Q^\dagger$ which satisfy[29]

$$\begin{aligned} \{Q, Q\} &= 0 \\ \{Q, Q^\dagger\} &= H \\ \{Q^\dagger, Q^\dagger\} &= 0 \end{aligned}$$

Immediate consequence: $[H, Q] = [H, Q^\dagger] = 0$, so $Q, Q^\dagger$ are conserved. In the physics literature such odd conserved charges are called *supercharges*.

Also note: Whatever Q is, $Q^\dagger$ is its adjoint. (As we required H to be self-adjoint.)

Remark 2.1. This is really $\mathcal{N} = 2$ SQM – there are variants with more or fewer supercharges. We'll focus on $\mathcal{N} = 2$ today.

This algebra has some outer automorphisms:

$$O(2) = U(1) \otimes \mathbb{Z}/2$$

- The $U(1)$ is the *fermion number/R-charge*:

	Q	$Q^\dagger$	H
weights	$+1$	-1	0

- The $\mathbb{Z}/2$ is *charge conjugation*,

$$Q \leftrightarrow Q^\dagger$$

This is a symmetry (of the theory) if it lifts to an action on the Hilbert space Ω. How can this lift?

For fermion number:

[29] Square brackets are commutators, curly brackets are anticommutators. I.e.

$$\begin{aligned} [A, B] &= AB - BA \\ \{A, B\} &= AB + BA \end{aligned}$$

- Require a self-adjoint operator F satisfying

$$\begin{aligned}
[F, Q] &= Q \\
[F, Q^\dagger] &= -Q^\dagger \\
[F, H] &= 0
\end{aligned}$$

i.e. F generates the R-symmetry action on $Q, Q^\dagger, H$.

- This promotes Ω to a $\mathbb{Z}$-graded Hilbert space,

$$\begin{aligned}
\Omega &= \bigoplus_{j \in \mathbb{Z}} \Omega^j, \\
\Omega^j &= \{\psi \in \Omega \,|\, F\psi = j\psi\}
\end{aligned}$$

- Note that the original $\mathbb{Z}/2$-grading is recovered by considering the operator $(-1)^F$; so

$$\Omega_{e/o} = \bigoplus_{j \text{ even/odd}} \Omega^j.$$

- We can unwind the action of Q and $Q^\dagger$ componentwise:

$$\begin{tikzcd}
{} & \Omega^{j-1} \arrow[l, bend left, "Q^\dagger"] \arrow[r, bend left, "Q"] & \Omega^j \arrow[l, bend left, "Q^\dagger"] \arrow[r, bend left, "Q"] & {}
\end{tikzcd}$$

and the H operator acts preserving each graded component. Note: If we forget about $Q^\dagger$ we have a cochain complex.

For charge conjugation: Require a unitary operator

$$C : \Omega^j \to \Omega^{-j}$$

satisfying

$$\begin{aligned}
CQC^{-1} &= Q^\dagger \\
CQ^\dagger C^{-1} &= Q.
\end{aligned}$$

If we have both of these structures: The Ω is promoted to a representation of $O(2)$.

Example 2.4 (Riemannian σ-model)**.** Again we have X a compact smooth Riemannian manifold. Now $\Omega = \Omega^*(X, \mathbb{C})$, and the $\mathbb{Z}$-grading is given by

$$\begin{aligned}
\Omega^j &= \Omega^j(X, \mathbb{C}) \\
\langle f, g \rangle &= \int_X \bar{f} \wedge \star g
\end{aligned}$$

The supercharges and Hamiltonian are

$$\begin{aligned}
Q &= d \\
Q^\dagger &= d^\dagger \\
H &= \{Q, Q^\dagger\} = \{d, d^\dagger\} = \Delta
\end{aligned}$$

The $O(2)$-representation structure is given by:

- The fermion number operator is *naively*

$$F_{naive} = \sum_i dx^i \iota_{\frac{\partial}{\partial x^i}} = \text{ form degree}$$

- Charge conjugation is given by the Hodge star

$$\star : \Omega^j \to \Omega^{m-j}$$

  where $m = \dim(X)$.

- So in fact we need to *shift* our naive fermion number operator:

$$F = \frac{1}{2}\{dx^i, \iota_{\partial/\partial x^i}\} = \sum_i dx^i \iota_{\partial/\partial x^i} - \frac{m}{2}$$

With these definitions $(F, \star)$ give the $O(2)$-representation structure on Ω.

Remark 2.2. Q is independent of the metric on X.

Example 2.5 (Hermitian σ-model)**.** Take as input

- X a compact hermitian manifold
- E a holomorphic vector bundle equipped with a hermitian metric

Define the graded Hilbert space by

$$\Omega^j = \Omega^{0,j}(X, E)$$
$$\langle f, g \rangle = \int_X \bar{f} \wedge \star g$$

Inplicit in the formula for $\langle -, - \rangle$ is a contraction involving the hermitian metric on E.

Now: take

$$Q = \bar{\partial}_E,$$
$$Q^\dagger = \bar{\partial}_E^\dagger$$

and so

$$H = \{\bar{\partial}_E, \bar{\partial}_E^\dagger\} = \frac{1}{2}\Delta_{\bar{\partial}_E}, \qquad \text{(Dolbeault Laplacian)}$$

The fermion number operator is

$$F_{naive} = \sum_i d\bar{z}^i \iota_{\frac{\partial}{\partial \bar{z}^i}} = \text{ form degree}$$

So far as MB can see, there is no charge conjugation ($\mathbb{Z}/2$) operator in this theory.

Remark 2.3. Q is independent of the hermitian metrics on X and E.

2.1.4 Spectrum of the Hamiltonian

An important question in any quantum mechanical theory: *What is the spectrum of the Hamiltonian?*

In QM the spectrum must be real (self-adjointness of H). In SQM:

$$\mathrm{Spec}(H) \subseteq \mathbb{R}_{\geq 0}$$

Why?

Proof. Given an eigenfunction

$$H \cdot \psi = E\psi,$$

and we can evaluate to find

$$\langle \psi, H\psi \rangle = E\|\psi\|^2.$$

But now using the Q-operators,

$$\langle \psi, H\psi \rangle = \langle \psi, \{Q, Q^\dagger\}\psi \rangle = \|Q\psi\|^2 + \|Q^\dagger\psi\|^2 \geq 0.$$

Hence $E \geq 0$. □

The states with lowest possible energy ($E = 0$) are important for out theory, and we give them a name.

Definition 2.2. The states which saturate this bound are called *SUSY ground states*.

Remark 2.4. In a non-SUSY theory we have to put in by hand the requirement that the spectrum of H is bounded below (and then we can shift by a constant to make the lower bound 0). In a SUSY theory, we get this for free.

Observe that

$$\mathcal{H} := \ker(H) = \ker(Q) \cap \ker(Q^\dagger) \subset \Omega.$$

Remark 2.5. Note that $\mathcal{H}$ could be empty – in that situation we say that the system "breaks supersymmetry".

Another assumption that is usually made in physics is tha the spectrum is *gapped*; i.e. the non-zero energies do not limit to zero (Figure 8).[30]

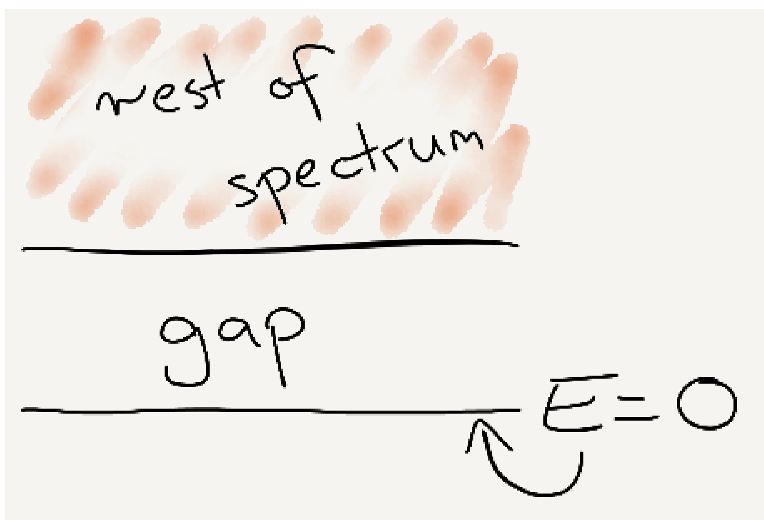


Figure 8: A theory with an energy gap has a finite difference between the zero energy and first excited modes.

Example 2.6. A discrete spectrum

$$\mathrm{Spec}(H) = \{0 = E_0 < E_1 < E_2 < \cdots\}$$

is gapped.

We expect that there is an orthogonal decomposition

$$\Omega = \mathcal{H} \oplus \mathrm{im}\ (Q) \oplus \mathrm{im}\ (Q^\dagger)$$

This is "proved" in two steps:

[30]This assumption allows us to make the orthogonal decomposition argument below.

1) (Hard) $\Omega = \ker(H) \oplus \text{im }(H)$ (orthogonal decomposition)

2) (Easy) For $\psi \in \text{im }(H)$,

$$\psi = H\omega = \{Q, Q^\dagger\}\omega = \underbrace{Q(Q^\dagger\omega)}_{\text{im }(Q)} + \underbrace{Q^\dagger(Q\omega)}_{\text{im }(Q^\dagger)}$$

Remark 2.6. If you don't like the gapped assumption, just assume the existence of this decomposition instead.

Example 2.7 (Riemannian σ-model)**.** We have

$$\mathcal{H}^j = \ker \Delta|_{\Omega^j} = \text{Harm}^j(X, \mathbb{C})$$
$$\Omega = \mathcal{H} \oplus \text{im }(d) \oplus \text{im }(d^\dagger)$$

(Harm^j is harmonic j-forms).

Example 2.8 (Hermitian σ-model)**.** We have

$$\mathcal{H}^j = \text{Harm}^{0,j}_{\Delta_{\bar{\partial}_E}}(X, \mathbb{C})$$
$$\Omega^j = \mathcal{H}^j \oplus \text{im }(\bar{\partial}_E) \oplus \text{im }(\bar{\partial}_E^\dagger)$$

2.1.5 Next time and questions (lecture 2)

- Remember that Q was independent of "metric" structures.
- So we'll look at homological structures associated to the supercharge Q.

Question 2. *Are Riemannian/Hermitian σ-models supposed to be quantisations of a classical theory?*

Answer 2.1. Yes for Riemannian; yes for Hermitian but making that one precise is a little more subtle. Note that the SUSY classical action would have many more fields than the non-SUSY classical action.

Question 3. *How is this lecture related to the factorisation algebra lecture?*

Answer 2.2. Can consider these examples as quantisations of classcial systems as per Philsang's talk; be careful – the homological structures that arise in these talks do not originate with the BV formalism

2.2 Lecture 2 (Mathew Bullimore)

2.2.1 Reminder

We have:

- Hilbert space $\Omega = \Omega_e \oplus \Omega_o$
- Operators $\mathcal{A}$
- $\{Q, Q\} = 0$
- $\{Q, Q^\dagger\} = H$
- $\{Q^\dagger, Q^\dagger\} = 0$

Assume:

- fermion number symmetry F

- orthogonal decomposition: $\Omega = \ker(H) \oplus \mathrm{Im}(Q) \oplus \mathrm{Im}(Q^\dagger)$

Today: Want to throw out some of this structure and see what happens – specifically, forget the Hilbert space structure, remember only one of the supercharges, and try to understand the algebraic structures that arise.

Notation: From now on we will discard the "curly bracket" notation for anticommutators and simply write

$$[A, B] = AB - (-1)^{F(A)F(B)} BA.$$

2.2.2 Operators Revisited

Recall that there is a fermion number symmetry,[31] generated by a self-adjoint operator F. So:

- $\mathcal{A}$ is $\mathbb{Z}$-graded:

$$\mathcal{A} = \bigoplus_{j \in \mathbb{Z}} \mathcal{A}^j,$$
$$\mathcal{A}^j = \{A \in \mathcal{A} \,|\, [F, A] = jA\}$$

- Q is degree 1:

$$[Q, -] : \mathcal{A}^j \to \mathcal{A}^{j+1}$$

- Compatible with product:

$$[Q, AB] = [Q, A]B + (-1)^{F(A)} A[Q, B].$$

Upshot: $(\mathcal{A}^\bullet, [Q, -])$ is a DG-algebra.

2.2.3 States Revisited

We have:

- Ω is $\mathbb{Z}$-graded:

$$\Omega = \bigoplus_{j \in \mathbb{Z}} \Omega^i,$$
$$\Omega^i = \{\psi \in \Omega \,|\, F\psi = j\psi\}$$

- Q provides a differential:

$$\cdots \longrightarrow \Omega^i \xrightarrow{Q} \Omega^{i+1} \longrightarrow \cdots$$

- Differentials are compatible:

$$Q(A \cdot \psi) = [Q, A] \cdot \psi + (-1)^{F(A)} A(Q\psi)$$

Upshot: $(\Omega^\bullet, Q)$ is a DG-module for $(\mathcal{A}^\bullet, [Q, -])$.

Next we will be interested in studying what happens on the level of cohomology – this is of interest if one wishes to understand e.g. SUSY ground states.

[31] Examples where fermion numnber symmetry is broken: harmonic oscillator; equivariant SQM.

2.2.4 Operators: cohomology of Q

A *BPS operator* is an operator A such that $[Q, A] = 0$.

Remark 2.7. Might more accurately wish to call this a $\frac{1}{2}$-*BPS operator*, since it commutes with half of the supercharges.

A BPS operator A yields a class
$$[A] \in H^\bullet(\mathcal{A}) =: \mathcal{O}^\bullet$$

Transfer of structure: $\mathcal{O}^\bullet$ inherits an A_∞-structure:
$$\mu_n : \mathcal{O}^{\otimes n} \to \mathcal{O}, \qquad \text{degree } 2 - n$$

These operations must obey a collection of relations. Since $\mathcal{O}$ arises as the cohomology of $\mathcal{A}$ we has some more information on the operations:

- $\mu_1 = 0$ (vanishing differential)
- μ_2 is inherited from the product on $\mathcal{A}$,
$$[A_1] \cdot [A_2] = [A_1 A_2]$$
- μ_n, $n \geq 3$ may not be trivial! (Massey products.)

This is unique up to quasi-isomorphism – if one remembers the structure given by $Q^\dagger$ one can uniquely fix a representative.

Example 2.9 (Triple Massey product)**.** Suppose we have 3 BPS operators A_1, A_2, A_3,
$$[A_1 A_2] = 0, \qquad\qquad [A_2 A_3] = 0.$$

So
$$A_1 A_2 = [Q, B_{12}], \qquad\qquad A_2 A_3 = [Q, B_{23}].$$

Define
$$A_{123} = B_{12} A_3 - (-1)^{F(A_1)} A_1 B_{23}.$$

Then $[Q, A_{123}] = 0$, so A_{123} is *also* BPS.

Remark 2.8. The example above shows how the higher operations should arise. One must be careful however – there were choices involved in constructing A_{123} (the B-operators), and different choices of B-operators may lead to different cohomology classes. This problem is related to the idea fact that $\mathcal{O}$ was unique only up to quasi-isomorphism.

Time Dependence: in Heisenberg picture
$$\partial_\tau A = [H, A] = [Q, [Q^\dagger, A]]$$

if A is BPS. So: $[A]$ is independent of τ. Would like to integrate this equation: "descent".

Finite version: Given $A \in \mathcal{A}^j$, its descendent is
$$A^{(1)} := [Q^\dagger, A] d\tau.$$

This is an $\mathcal{A}^{j-1}$-valued 1-form on $\mathbb{R}_\tau$. It has the property that
$$[Q, A^{(1)}] = d_\tau A.$$

Hence,

$$A(\tau_2) - A(\tau_1) = \left[Q, \int_{\tau_1}^{\tau_2} A^{(1)}(\tau)\right].$$

Research Idea: Use descent to construct higher products. What does this mean?

Take a product of BPS operators at different times

$$A_1(\tau_1) \cdots A_n(\tau_n)$$

and via descent, construct k-forms on one of the following "configuration spaces"

$$\mathrm{Conf}_n(\mathbb{R}) := \begin{cases} \{x_1 < \cdots < x_n\}, & \text{or,} \\ \{x_1 < \cdots < x_n\}/\mathbb{R}, & \text{or,} \\ \{x_1 < \cdots < x_n\}/\mathbb{R} \times \mathbb{R}_+, \end{cases}$$

where in the last definition the first $\mathbb{R}$ is translations and the second $\mathbb{R}_+$ is scalings.

Proposal: Construct μ_n by integrating $(n-2)$-form descendants.

2.2.5 States: cohomology of Q

$[H, Q] = 0$, so we can decompose $(\Omega^\bullet, Q)$ into eigenspaces of H. Let's assume for simplicity that the spectrum of H is discrete

$$\mathrm{Spec}(H) = \{0 = E_0 < E_1 < \cdots\}.$$

So we get a decomposition:

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \Omega^i_{(1)} & \xrightarrow{Q} & \Omega^{i+1}_{(1)} & & (E_1) \\ \Omega^i_{(0)} & \xrightarrow{Q} & \Omega^{i+1}_{(0)} & & (E_0) \end{array}$$

The cohomology vanishes for $n > 0$: Given $\psi \in \Omega_{(n)}$,

$$[Q, Q^\dagger]\psi = E_n \psi;$$

if moreover $Q\psi = 0$, then

$$\psi = Q\left(\frac{Q^\dagger \psi}{E_n}\right).$$

Claim: $H^j(\Omega^\bullet, Q) \simeq \ker H|_{\Omega^j_{(0)}} \simeq \mathcal{H}^j$, the SUSY ground states.

Example 2.10. For Riemannian X,

$$\mathcal{H}^i = \mathrm{Harm}^i(X, \mathbb{C}) \simeq H^i_{\mathrm{dR}}(X, \mathbb{C}).$$

Example 2.11. For Hermitian (X, E),

$$\mathcal{H}^i = \mathrm{Harm}^{0,i}_{\Delta_{\bar\partial_E}}(X, \mathbb{C}) \simeq H^{0,i}_{\bar\partial}(X, E).$$

Transfer of structure: $\mathcal{H}^\bullet$ inherits an A_∞-module structure for $\mathcal{O}^\bullet$:

$$\nu_n : \mathcal{O}^{\otimes n} \otimes \mathcal{H} \to \mathcal{H}, \qquad \text{degree } 1-n$$

Due to the way we obtained this module, we know:

- $\nu_0 = 0$.
- ν_1 is induced from the module action $\mathcal{A} \times \Omega \to \Omega$.
- There could be higher Massey product style operations.

Example 2.12. For Riemannian X, operators should be the cohomology of $X \times X$ with convolution product. Would be interesting to see if there are higher Massey products.

2.2.6 Flavour Symmetry

To produce some interesting examples, let's introduce the notion of a *G-flavour symmetry*: G is a compact connected Lie group, and

- Ω is a unitary representation of G,
- the action commutes with Q, $Q^\dagger$, H.

The *infinitesimal action* is described by:

- Self-adjoint $J_a \in \mathcal{A}^0$, $a = 1, \dots, \dim(G)$
- $[J_a, J_b] = i f_{ab}^c J_c$
- $[J_a, Q] = [J_a, Q^\dagger] = 0$

This data means that $\mathfrak{g}$ acts on $(\Omega^\bullet, Q)$ by cochain maps; i.e. $(\Omega^\bullet, Q)$ is a DG-module of $\mathfrak{g}$.

Example 2.13 (Riemannian X)**.** X has an isometry generated by a real vector field V:

$$G = U(1), \qquad\qquad J = -i\mathcal{L}_V.$$

Example 2.14 (Hermitian (X, E))**.** X has an isometry generated by a real vector field that lifts to an equivariant action on E preserving the hermitian metric.

2.2.7 Flavour Action on Cohomology

By our assumption that $[Q, J] = 0$, J defines a cohomology class $[J] \in \mathcal{O}^0$. There are two cases:

(1) $[J] = 0$. Then $J = [Q, I]$ for some I, and $\mathcal{H}^\bullet$ is a trivial module: $J \cdot \psi = [Q, I] \cdot \psi = Q \cdot (I \cdot \psi)$, which is trivial in cohomology.

(2) $[J] \neq 0$. Then $\mathcal{H}^\bullet$ may be a non-trivial module.

Example 2.15 (Riemannian X)**.** $J = -i\mathcal{L}_V = -i\{d, \iota_V\} = \{Q, I\}$ for $I = i\iota_V$. So we are in case (1), and de Rham cohomology classes are not charged under this flavour symmetry.

Example 2.16 (Hermitian (X, E))**.** $\mathcal{L}_V = \{J, \iota_{V^{0,1}}\} + \mathcal{L}_{V^{1,0}}$, and the second term places us in the situation of case (2).

Example 2.17. Let's take a specific Hermitian (X, E): let

$$X = \mathbb{CP}^1, \qquad\qquad E = \mathcal{O}(n), \quad n \geq 0.$$

This has an action of $\mathfrak{g} = \mathfrak{su}(2)$ (by rotations of the sphere). Now

$$\mathcal{H}^j = H^{0,j}_{\bar\partial}(\mathbb{CP}^1, \mathcal{O}(n)).$$

Borel-Weil-Bott tells us that this is precisely the $(n+1)$-dimensional representation of $\mathfrak{su}(2)$, and indeed this is the module structure induced the rotation $\mathfrak{su}(2)$-action.

2.2.8 Homological G-action

Work with: Beem, Ben-Zvi, Dimofte, Neitzke; with the "prime-mover" of the project being Tudor.

Claim: In case (1), $\mathcal{H}^\bullet$ is an A_∞-module for $H_\bullet(G)$.

For concreteness, let's focus on $G = U(1)$. The idea is to perform descent *on the group* G.

- $\psi \in \Omega^j$, $Q\psi = 0$
- Promote to an Ω^j-valued function on $G = U(1)$:

  $$\psi^{(0)} := e^{i\theta J}\psi$$

  This now depends on an angle θ; hence it lives on $U(1)$.
- Want to descend this operator: construct an Ω^{j-1}-valued 1-form on $G = U(1)$. Use that we are in case (1), so that $J = [Q, I]$, and define

  $$\psi^{(1)} := (I\psi^{(0)})d\theta.$$

  This obeys a descent equation on $U(1)$:

  $$Q\psi^{(1)} = d_{U(1)}\psi^{(0)}$$
- Since we now have a 1-form, we can integrate over 1-cycles in the group. Take $\gamma \in C_1(G)$ and define

  $$\gamma \cdot \psi := \int_\gamma \psi^{(1)}.$$

  Then

  $$Q\int_\gamma \psi^{(1)} = \int_\gamma d_{U(1)}\psi^{(0)} = \int_{\partial\gamma}\psi^{(0)} = 0 \quad \text{if } \partial\gamma = 0.$$
- So this descends to a map

  $$H_1(G) \times \mathcal{H}^j \to \mathcal{H}^{j-1}.$$

Of course, this action *could* be trivial. Let's give an example where it is not:

Example 2.18 (Particle on a circle)**.** $X = S^1$ and $G = U(1)$ acting by rotations. Coordinatise X and G by the angles ϕ and θ respectively. To vector field on X generating the symmetry is $V = \frac{\partial}{\partial\phi}$. The SUSY ground states are

$$\mathcal{H}^i = H^i_{\mathrm{dR}}(S^1, \mathbb{C}) = \begin{cases} \mathbb{C}\cdot 1, & i = 0, \\ \mathbb{C}\cdot d\theta, & i = 1 \end{cases}$$

So we at least have the *potential* for a non-trivial map

$$H_1(G) \times \mathcal{H}^1 \to \mathcal{H}^0.$$

Start with $\psi = d\phi$. Then

$$\begin{aligned}\psi^{(0)} &= e^{i\theta J} d\phi \\ &= e^{\theta \mathcal{L}_V} d\phi \\ &= d\phi\end{aligned}$$

Then

$$\begin{aligned}\psi^{(1)} &= (I \cdot \psi^{(0)}) d\theta \\ &= (\iota_V \psi^{(0)}) d\theta \\ &= d\theta\end{aligned}$$

and so

$$\gamma \cdot \psi = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1,$$

which is non-vanishing.

Problem 6. *$X = S^3$ has a $G = U(1)$-action given by rotating the fibres of the Hopf fibration. Show that there is a non-trivial A_∞ operation*

$$\nu_2(\gamma, \gamma, dvol_{S^3}) = 1.$$

2.3 Lecture 3: Superpotentials (Mathew Bullimore)

Idea: Superpotentials are "deformations" of SQM – indeed, they will be *flat* deformations that we will use to simplify certain computations.

2.3.1 Deformations

We are going to deform our supercharge,

$$Q \to Q + x,\ x \in \mathcal{A}^1.$$

We demand that the deformed supercharge still square to zero – in order for this to hold, x must obey the Maurer-Cartan equation for $\mathcal{A}$ (c.f. Davide's lectures):

$$\begin{gathered}[Q + x, Q + x] = 0, \text{ i.e.} \\ [Q, x] + \frac{1}{2}[x, x] = 0.\end{gathered}$$

There are two types of *finite* deformations:

(1) Complex parameter $u \in U$,

$$\partial_{\bar{u}} Q(u) = 0.$$

- $(\Omega^\bullet, Q)$ will become a complex of holomorphic vector bundles on U.
- Passing to cohomology, $\mathcal{H}^\bullet$ will become a coherent sheaf on U.

Example 2.19. Hermitian (X, E) with $Q = \bar{\partial}_E$. Then we can deform the complex structure of E,

$$\bar{\partial}_E \to \bar{\partial}_E + \bar{a}$$

and the MC equation is

$$\bar{\partial}_E \bar{a} + \bar{a} \wedge \bar{a} = 0.$$

Example 2.20 (E, J-type superpotentials)**.** Hermitian (X, E); now promote E to a $\mathbb{Z}$-graded bundle

$$E = \bigoplus_{p\in\mathbb{Z}} E^p$$

and define a new fermion number operator

$$F = \text{ form degree } + p.$$

Deform

$$Q = \bar{\partial}_E + \delta$$

where δ is a holomorphic differental on E of degree 1.[32] Then $\mathcal{H}^\bullet$ is the hypercohomology of this complex.

(2) Real parameters $\lambda \in \Lambda$,

$$\partial_\lambda Q(\lambda) = [Q(\lambda), h]$$

where $h \in \mathcal{A}^0$ is self-adjoint of degree zero.

- $Q(\lambda) = Q_0 + \lambda[Q_0, h] + \cdots$; satisfies MC equation.
- *flat* deformation
- $\mathcal{H}^\bullet$ = local system on Λ

As we deform the system, we must make sure that our theory remains gapped.

Example 2.21. If we deform SQM over $\mathbb{R}$, we may need to take $\Lambda = \mathbb{R} \setminus \{0\}$ since at 0 the theory develops new massless degrees of freedom. Then the sort of thing we might expect is

- A space of SUSY ground states $\mathcal{H}_-$ on $\mathbb{R}_{<0}$.
- A potentially very different space of SUSY ground states $\mathcal{H}_+$ on $\mathbb{R}_{>0}$.
- Possibly: wall-crossing phenomena that we can describe as we pass over 0.

The two main models we will discuss in today's lecture are:

Example 2.22 (Riemannian X)**.** Given a Morse function $h : X \to \mathbb{R}$, deform

$$Q = e^{-\lambda h} d e^{\lambda h} = d + \lambda dh \wedge -$$

Example 2.23 (Hermitian (X, E))**.** Suppose that (X, E) has a $U(1)$ flavour symmetry, and further assume that

- X is Kähler
- $U(1)$ has isolated fixed points
- $h : X \to \mathbb{R}$ is the moment map for $U(1)$

Then we deform

$$Q = e^{-\lambda h}\bar{\partial}_E e^{\lambda h} = \bar{\partial}_E + \lambda\bar{\partial}h \wedge -$$

Idea:

- Send $\lambda \to \infty$.
- Exploit flatness to get a useful description of $\mathcal{H}$.

[32] Another way of saying this: we used the new grading to form a bicomplex with vertical differential given by δ (up to signs); we then took the total complex of this bicomplex.

2.3.2 Riemannian Model (Witten)

Choose a Morse function h. We obtain a potential

$$V \sim \frac{1}{2}\lambda^2 \|dh\|^2,$$

and as $\lambda \to \infty$ the system localises to the critical points of h.

Schematically: We have a critical point $p \in X$ of a potential, and we want to scale the potential to make it vey steep in a small neighbourhood of p.

Local model: Simple harmonic oscillator (SHO).

$$\begin{aligned} X &\simeq \mathbb{R}, \\ h(x) &= h(0) + \frac{\lambda}{2} h''(O) x^2 + \cdots \\ & h(0) + \frac{\omega}{2} x^2 \end{aligned}$$

So

$$\begin{aligned} Q &= \left(\frac{d}{dx} + \omega x\right) dx \\ Q^\dagger &= \left(-\frac{d}{dx} + \omega x\right) \iota_{\frac{d}{dx}} \end{aligned}$$

Then $\mathcal{H} = \ker(Q) \cap \ker Q^\dagger$, and there is a single state:

$$\Psi_p = \begin{cases} e^{-\frac{\omega x^2}{2}}, & \omega > 0, \\ e^{\frac{\omega x^2}{2}} dx, & \omega < 0. \end{cases}$$

At a general critical point p:

- Morse index: n_p
- 1 state Ψ_p: n_p-form

So a first (perturbative) approximation to the space of SUSY ground states is

$$\mathcal{H}^i_{pert} = \bigoplus_{p \,:\, n_p = i} \mathbb{C}\Psi_p;$$

indeed, these are exactly the perturbative ground states.

But – there are nonperturbative effects as well: *Instanton corrections.*

Remark 2.9. One way that we can see that there must be nonperturbative effects is that our deformation is supposed to be flat. A different choice of Morse function could lead to a very different critical point structure, however – if we are to obtain the same end result, then, there must be some nonperturbative corrections we need to take into account.

In this situation, the instanton corrections arise precisely from the gradient flow equations connecting two critical points

$$\frac{dx^i}{d\tau} = g^{ij}\partial_j h.$$

This leads to a further differential on the complex of perturbative ground states. Contributions can only come from critical points with Morse index differing by 1, and so the differential looks like

$$\delta\Psi_p = \sum_{p' : n_{p'} = n_p + 1} n_{pp'} \Psi_{p'}$$

where $n_{pp'}$ is a signed count of the number of gradient flow lines $p \to p'$.

Example 2.24 (Particle on S^1)**.** Take as Morse function the height function h. This has two critical points: p at the bottom of the circle with $n_p = 0$ and p' at the top of the circle with $n_{p'} = 1$. So we obtain perturbative ground states

- Ψ_p: 0-form
- $\Psi_{p'}$: 1-form

The differential is $\delta\Psi_p = n_{pp'}\Psi_{p'}$; in this case there are two gradient flow lines (up each side of the circle), and the have cancelling contributions, so $n_{pp'} = 1 - 1 = 0$. Hence we have reproduced the cohomology of the circle.

2.3.3 Hermitian Model (X, E)

We make the assumptions:

- X Kähler
- $U(1)$ action on X with isolated fixed points
- $h : X \to \mathbb{R}$ the moment map for $U(1)$ on X

As $\lambda \to \infty$ we localise to the critical points of h – these are precisely the fixed points of the $U(1)$ action. Already we are seeing hints of a localisation procedure.

Local model: Complex SHO.

- $X = \mathbb{C}$
- $U(1)$ action by rotations
- $h = \omega|z|^2$

Then

$$Q = (\partial_{\bar{z}} + \omega z)d\bar{z}$$
$$Q^\dagger = (-\partial_z + \omega\bar{z})\iota_{\frac{\partial}{\partial\bar{z}}}$$

In the real case we had only a single ground state corresponding to a critical point. In the complex case, we will have an entire Fock space of normalisable SUSY ground states:

$$\Psi_p^{(n)} = \begin{cases} z^n e^{-\omega|z|^2}, & \omega > 0, \quad (\mathcal{F}_+) \\ \bar{z}^n e^{\omega|z|^2} d\bar{z}, & \omega < 0, \quad (\mathcal{F}_-) \end{cases} \qquad n \geq 0.$$

We obtain tensor products of copies of $\mathcal{F}_\pm$ at critical points according to their Morse index:

$$\mathcal{H}^j_{pert} = \bigoplus_{p\,:\,n_p=j} \left(\bigotimes_{a=1}^{n-n_p} \mathcal{F}_+ \right) \left(\bigotimes_{b=1}^{n_p} \mathcal{F}_- \right)$$

Again there must be instanton corrections, arising from solutions to the equation

$$\frac{dz^i}{d\tau} = g^{i\bar{j}} \partial_{\bar{j}} h.$$

Note that in this case the corrections must be more drastic – for instance if our manifold is compact it has finite dimensional cohomology, while the space of perturbative ground states is always infinite dimensional.

In principle we could try and solve this equation, and analyse how the solutions correct the perturbative answer. This is difficult however, so instead we are going to consider a more algebraic approach (reference: Frenkel-Loseu-Nekrasov).

- **Step 1:** Conjugation.

  Whenever we have a state, define an "in" and an "out" version as follows

$$\begin{aligned} \Psi_{in} &:= e^{\lambda h} \Psi, \\ \Psi_{out} &:= e^{-\lambda h} \Psi, \\ \tilde{\mathcal{O}} &:= e^{\lambda h} \mathcal{O} e^{-\lambda h}. \end{aligned}$$

  We have a pairing (induced from the pairing on our original Hilbert space)

$$\langle -, - \rangle : \Omega_{out} \otimes \Omega_{in} \to \mathbb{C}$$

  and as such we can restrict attention to just one type of state. Let us focus on the "in" states, and simply write $\Psi_p^{(n)} \equiv (\Psi_p^{(n)})_{in}$.

- **Step 2:** $\lambda \to \infty$ strictly.

  In the local model,

$$\lambda \to \infty : \Psi_p^{(n)} \to \begin{cases} z^n, & \omega \to +\infty, \\ \frac{(-1)^n}{n!} \bar{\partial} \left(\frac{1}{z^{n+1}} \right), & \omega \to -\infty. \end{cases}$$

  Note that the $\omega \to -\infty$ terms correspond to derivatives of δ-functions.

- **Step 3:** Compute instanton corrections!

Example 2.25 (($X = \mathbb{CP}^1, E = \mathcal{O}$))**.** Our $U(1)$ action is rotation, our moment map h is the height funciton, and we have critical points:

- p at the south pole, with $n_p = 0$; give local coordinate z
- p' at the north pole, with $n_{p'} = 1$: give local coordinate w

The perturbative states are

$$\begin{aligned} \Psi_p^{(n)} &= z^n, \\ \Psi_{p'}^{(n)} &= \bar{\partial} \frac{1}{w^{n+1}} \end{aligned}$$

for $n \geq 0$. Now,

$$\begin{aligned} Q\Psi_p^{(n)} &= \bar{\partial}(z^n) \\ &= \bar{\partial}\left(\frac{1}{w^n}\right) \\ &= \begin{cases} 0, & n = 0, \\ \Psi_{p'}^{(n-1)}, & n > 0. \end{cases} \end{aligned}$$

So we are left with a single remaining 0-form state: $\Psi_p^{(0)}$. This agrees with the Dolbeault cohomology of $\mathbb{CP}^1$.

Example 2.26 **($(X = \mathbb{CP}^1, E = \mathcal{O}(m))$).** The difference between this and the previous example is that we must include the transition function:

$$\begin{aligned} Q\Psi_p^{(n)} &= \bar{\partial}(z^n) \\ &= w^m \bar{\partial}\frac{1}{w^{n+1}} \\ &= \begin{cases} 0, & n \leq m, \\ \Psi_{p'}^{(n-m-1)}, & n > m. \end{cases} \end{aligned}$$

The remaining states are

$$1, z, z^2, \ldots, z^m,$$

which agrees with the Dolbeault cohomology $H^{0,\bullet}(\mathbb{CP}^1, \mathcal{O}(m))$.

2.3.4 Grothendieck-Cousin Complex

In the $\mathbb{CP}^1$ example, flow "lines" run from the south pole p to the north pole p'. Really, however, we should think of these flows as flows of $\mathbb{CP}^1$'s connecting fixed points. I.e. we have ascending manifolds

$$\begin{aligned} X_p &= \mathbb{CP}^1 \setminus \{p'\} \\ X_{p'} &= \{p'\} \end{aligned}$$

Note that $X_{p'} \subset \overline{X_p}$.

Then we can interpret the perturbative ground states as *local* cohomology:

$$\begin{aligned} \mathcal{H}^0_{pert} &= \mathbb{C}[z] \\ &= H^0(X_p, E) \\ &= X^0_{X_p}(E) \\ \mathcal{H}^1_{pert} &= \mathbb{C}[w, w^{-1}]/\mathbb{C}[w] &&= H^1_{X_{p'}}(E) \end{aligned}$$

The differential that was described in the previous section is then precisely the Grothendieck-Cousin operator on local cohomology

$$\delta : H^0_{X_p}(E) \to H^1_{X_{p'}}(E).$$

General picture:

- ascending manifolds $X_p \simeq \mathbb{C}^{n-n_p}$: BB decomposition of X
- assume stratification:

$$\overline{X_p} \setminus X_p = \bigcup_{p' < p} X_{p'}$$

- perturbative ground states
$$\mathcal{H}^i_{pert} = \bigoplus_{p\,:\,n_p=i} H^i_{X_p}(E)$$
- differential: Grothendieck-Cousin operators

2.3.5 Geometric Representation Theory

Recall the $\mathbb{CP}^1$ example. The SUSY ground states transform in a representation of the flavour symmetry – for $\mathbb{CP}^1$ we have that $\mathfrak{g} = \mathfrak{su}(2)$ is the flavour symmetry, and

$$\mathcal{H}^j = \mathcal{H}^{0,j}(\mathbb{CP}^1; \mathcal{O}(m)) = V_m$$

where V_n is the $(n+1)$-dimensional representation of $\mathfrak{su}(2)$. We have:

- the instanton complex is
$$\delta : H^0_{X_p}(E) \to H^1_{X_{p'}}(E);$$
- $H^0_{X_p}(E)$, $H^1_{X_{p'}}(E)$ are Verma modules;
- so we've really just given BGG resolution of V_m.

General story: Due to Kempf.

- $X = G/B$
- h the moment map for $U(1) \subset T \subset B$
- critical points $\{w \in W\}$
- X_p are Schubert cells
- choose E to be the line bundle labelled by highest dominant weight λ
- the instanton complex is BGG resolution of V_λ

3 Boundary Conditions and Extended Defects

3.1 Lecture 1 (Davide Gaiotto)

3.1.1 QFT and Local Operators

Want to begin by giving a *physical* definition of a QFT.

Idea: a QFT is determined by a collection of correlation functions.

- Take a manifold X.
- Choose a collection of points $\{p_i\} \subset X$.
- To each point there is a collection of observables Obs_p.
- Then given $\mathcal{O}_i(p_i) \in \mathrm{Obs}_{p_i}$ there should be a correlation function
$$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_m(p_m) \rangle.$$
This gives a map
$$\mathrm{Obs}_{p_1} \otimes \cdots \otimes \mathrm{Obs}_{p_m} \to \mathbb{C}$$
satisfying some axioms. Importantly – $p_i \neq p_j$.
- These should be compatible in a precise sense – given observables at a collection of points, should be able to replace by a convergent series of operators at a single point lying within a common radius of convergence (Figure 9). So there are *OPE maps* that replace m-point correlation functions with 1-point functions:
$$\mathrm{OPE} : \langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_m(p_m) \rangle \to \langle \tilde{\mathcal{O}}(p) \rangle$$
I.e. a map $\mathrm{OPE} : \mathrm{Obs}_{p_1} \otimes \cdots \otimes \mathrm{Obs}_{p_m} \to \mathrm{Obs}_p$.[33]
- The algebraic structure that this gives rise to is a *factorisation algebra.*

Then we say that a QFT *is* this collection of observables and OPEs, subject to consistency conditions involving some possible extra data. (E.g. there might be prescribed nontrivial global 1-point functions.)

Remark 3.1. Often interested in topological or holomorphic theories, where the OPEs are easier to get a handle on.

Given an operator, we can consider displacing/translating it – this gives rise for each $\mathcal{O} \in \mathrm{Obs}$ to a derivative $\partial\mathcal{O} \in \mathrm{Obs}$.

Remark 3.2. Want the space of local operators to vary continuously as we move around our manifold.

Remark 3.3. We haven't yet described conditions on the manifold X. Usually we at least want a metric, to describe when points are "near" or "far" from each other.

QFT on flat space.

We're going to start by considering QFT on flat $\mathbb{R}^d$; such theories can often then be transferred to other classes of manifolds (as well as being interesting in their own right.)

Some extra assumptions we will make:

[33] Like a flat connection – but a slightly silly one; if you say it in words, it is something like, "the derivative of an operator is an operator called 'the derivative of the operator"'.

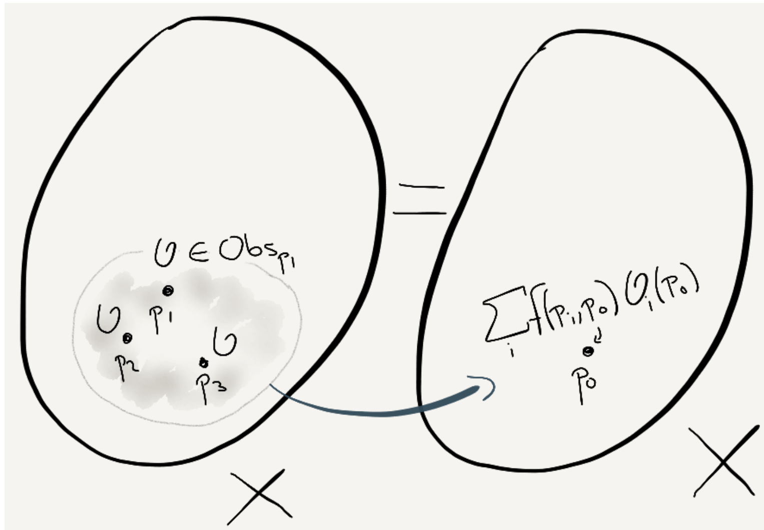


Figure 9: The OPE expresses a collection of operators at different points as a sum of operators at a single point.

- The QFT is translation invariant. (I.e. the space of local operators is the same everywhere, identified via translation.)
- The QFT is rotationally invariant. (We might not use this assumption?)

Problem 7 (Free scalar in 3d)**.** *Take the action*

$$S = \int \Phi(x)(-\Delta^2 + m^2)\Phi(x)$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}$*. Some correlation functions are*

$$\langle \Phi(x_1) \cdots \Phi(x_n) \rangle = \int D\Phi e^{-\frac{i}{\hbar} S[\Phi]} \Phi(x_1) \cdots \Phi(x_n).$$

So $\Phi \in \mathrm{Ops}$*.* **Exercise is to determine what else must be in** Ops **in order to have a well defined QFT.**

The is a map between theories defined at different energies (scales),

$$RG_\Lambda : T \to T_\Lambda$$

called *RG-flow.* The operation on spacetime is scaling by Λ, and for the correlation functions the requirement is

$$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_n(p_n) \rangle_{T_\Lambda} = \langle \mathcal{O}_1(\Lambda p_1) \cdots \mathcal{O}_n(\Lambda p_n) \rangle_T.$$

Remark 3.4. You may have to redefine your fields under RG-flow.

So: solving RG-flow at arbitrarily high energies (small Λ) is essentially solving your theory.

Remark 3.5. By repeated Taylor expansions, you can determine the OPE of two operators at finite distance apart by the UV OPE (i.e. move them closer together by a series of small "UV allowed" distances, then when they are close enough take the UV OPE).

An assumption we like to make: that RG-flow makes sense as $\Lambda \to \infty$. The theory we obtain in this limit will be scale invariant (perhaps conformal).

Remark 3.6. If you have enough computational power, you could obtain (in principle) a QFT as the continuum limit of a discretised theory.

This is all that Davide wants to tell us about the **definition** *of a QFT.*

3.1.2 Defects and defect OPE

Now suppose that we have a manifold M_d and a submanifold $L_k \subset M_d$. Away from L_k we have the same spaces Obs_p as before, and can take *bulk OPEs*

$$\mathrm{Obs}_{p_1} \otimes \cdots \otimes \mathrm{Obs}_{p_m} \to \mathrm{Obs}_p$$

But now if you choose a point $p_1^D \in L_k$ we obtain a different space of *defect operators* $\mathrm{Obs}^D_{p_1^D}$ (Figure 10), and when we take OPEs we may only "push" points onto the defect (Figure 11), i.e. we have *defect OPEs*

$$\mathrm{Obs}_{p_1} \otimes \cdots \otimes \mathrm{Obs}_{p_m} \otimes \mathrm{Obs}^D_{p_1^D} \otimes \cdots \otimes \mathrm{Obs}^D_{p_n^D} \to \mathrm{Obs}^D_{p^D}$$

Remark 3.7. Mathematically this is something like a "factorisation module".

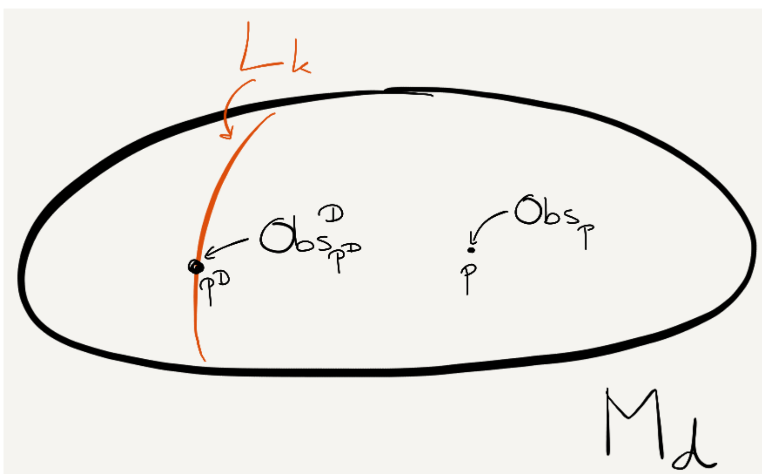


Figure 10: Distinction between bulk and defect observables.

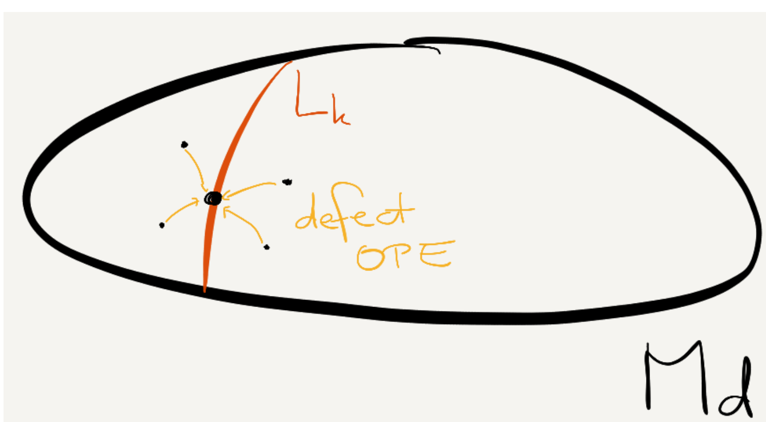


Figure 11: Local operators may be pushed onto the defect via OPE, but not vice-versa.

Of course if we choose a *point* as a defect, we are placing a factorisation module at a point. We declare the following axiom:

"True QFT" Axiom: Obs is the *only* 0-dimensional factorisation Obs-module.

Remark 3.8.

Problem 8 (Free gauge field in 3d)**.** *A* $U(1)$ *connection on* $\mathbb{R}^3$*,*

$$S = \sum_{i,j} \int ((dA)_{ij})^2,$$

$F = dA$*. Then we need to also include* monopole operators*; defect operators important in gauge theory. (Is the exercise to figure out what these must be?)*

Perturbative defects.

These take the form of lower-dimensional QFTs embedded into the original theory

$$S = \int dx^d \, L[\Phi] + \int dx^k \, L^D[\Phi|_{L_k}, \Phi^D]$$

where $\Phi : \mathbb{R}^d \to ?$ and $\Phi^D : \mathbb{R}^k \to ?$

Remark 3.9. Note that the defect theory involves bulk fields coupled to the boundary theory.

Remark 3.10. This intuition is useful, but dangerous. There is no way to take an arbitrary defect and write it as a lower-dimensional QFT embedded in the original QFT. (This is related to the observation of Remark 3.9.)

Monoidal structure from RG-flow.

Suppose you take two flat defects D_1 and D_2, living over a flat base space (i.e. of the form $D_i \times \mathbb{R}^k \subset \mathbb{R}^{d-k} \times \mathbb{R}^k$ where D_i is a linear subspace). Further suppose that the bulk theory is *scale invariant*.

Then applying RG_∞ in the $\mathbb{R}^k$ directions, we may "fuse" these defects to obtain a new defect $D_1 \circ D_2$. This gives a monoidal structure on the collection of such defects (Figure 12).

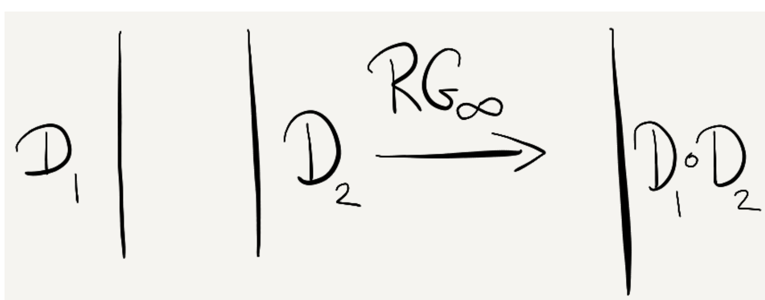


Figure 12: RG-flow inducing a monoidal structure on the category of lines.

Remark 3.11. This means, for instance, that even a topological field theory with *no* local operators can be nontrivial (e.g. due to the presence of line operators).

Remark 3.12. There are other options also – one could have two defects that under RG-flow fuse to give two defects meeting at a junction (Figure 13).

3.1.3 Line defects

If one wants to consider 1d defects, one option would be to couple a theory of quantum mechanics on a line L to the fields in the bulk theory. Define the Hamiltonian of the quantum mechanical theory by choosing a

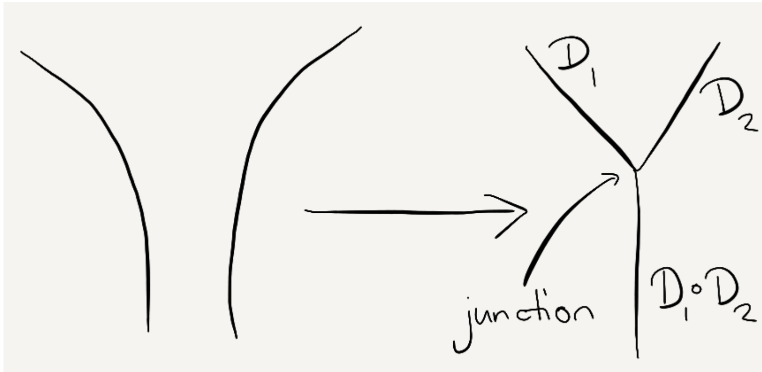


Figure 13: More complicated fusion of lines resulting in a junction.

(finite dimensional) vector space V and a collection of $M_i \in \mathrm{End}(V)$, and setting

$$H_D(t) = \sum_i M_i \mathcal{O}_i|_L(t; x^1 = x^2 = 0)$$

(we are taking our line to be $L = \{x^1 = x^2 = 0\} \subset \mathbb{R}^3$).

Example 3.1 (3d Free Scalar)**.** Recall the action is

$$S = \int \phi(-\Delta + m^2)\phi d^3\vec{x}.$$

An easy way to modify the action is by taking

$$S = \int \phi(-\Delta + m^2)\phi d^3\vec{x} + g_1 \int_{-\infty}^{\infty} \phi(x_1, 0, 0) dx_1;$$

this gives a free field theory with a free defect. Let's calculate a defect 1-point function $\langle \phi(x) \rangle_D$. Can represent this diagrammatically (Figure 14), with the result that[34]

$$(-\Delta + m^2)\langle \phi(x) \rangle_D = g_1 \delta(x_2)\delta(x_3) = g_1 K_0(|x|^m)$$

(Exercise: check this).

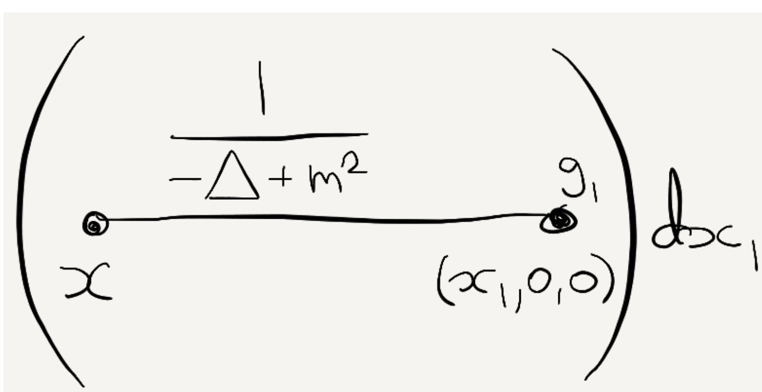


Figure 14: Defect 1-point function.

[34]Note that without the defect, $\langle \phi(x) \rangle = 0$.

Example 3.2 (σ-model)**.** Fields are maps $\Phi : \mathbb{R}^d \to M$; a family of quantum mechanical theories on M is a vector bundle V on M equipped with a "Hamiltonian" given by a connection on V. Then we obtain line defects on $\mathbb{R}^d$ by pulling back this connection (the connection gives us a way to do parallel transport, and hence integrate along a line).

Remark 3.13*.* One could argue that this is the way that one produces *all* line defects in a σ-model.

Example 3.3 (Free Gauge Field)**.** Let A be a $U(1)$ connection on $\mathbb{R}^d$,

$$S = \int \sum_{i,j} (F_{ij})^2 \, d^d\vec{x},$$

$F = dA$. This S is gauge invariant, i.e. invariant under shifts $A \to A + d\lambda$.

The easiest way to create a line defect is then to pull back this connection to a line and then take

$$S = \int \sum_{i,j} (F_{ij})^2 \, d^d\vec{x} + q \underbrace{\int_{-\infty}^{\infty} A_{x_1}(x_1, 0, \ldots, 0) dx_1}_{\text{parallel transport of connection}} .$$

For a scalar field the parameter g_1 was continuous – if one is careful one finds that the parameter q must be quantised. So we have a discrete set of choices we can make. By considering closed loops $\mathcal{L}$, one finds that these correspond to representations of our group R, as the defect term becomes

$$\mathrm{tr}_R \left(P \exp \int_{\mathcal{L}} A \right)$$

Problem 9. *Consider a Wilson line in 4d $U(1)$ gauge theory, and compute correlation functions perturbatively.*

3.1.4 Boundaries

Given a codimension 1 defect $L \subset M$, that separates M into two disjoint regions, we can consider placing *different* theories on either side of the defect (Figure 15 top). In particular, we can consider placing our theory on a manifold with boundary – we then need to make some choices to determine how our theory ends at this boundary (Figure 15 bottom).

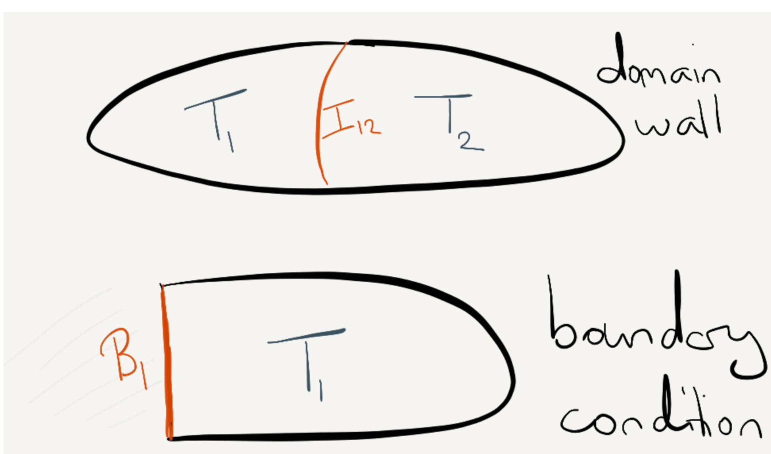


Figure 15: Domain wall separating theories (top); boundary condition for a single theory (bottom).

Example 3.4. In the free scalar field we have

$$(-\Delta + m^2)\langle\phi(x_1)\phi(x_2)\rangle = \delta(x_1 - x_2);$$

we want to choose boundary conditions such that this operator is still invertible.

For classical boundary conditions we require:

- Elliptic boundary condition
- Local bounday condition

Example 3.5 (Scalar field BCs)**.** Consider $\phi : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$.

- **Dirichlet BC:** Set $\phi|_{\mathbb{R}^2\times\{0\}} = 0$. Then

  $$\partial_\perp \phi|_{\{0\}} \in \mathrm{Obs}^D$$

  where $\partial_\perp$ is the normal derivative.

- **Neumann BC:** Set $\partial_\perp \phi|_{\{0\}} = 0$. Then

  $$\phi|_{\{0\}} \in \mathrm{Obs}^D .$$

- **Enriched Neumann BC:** Require

  $$\partial_\perp \phi|_{\{0\}} = \frac{\delta L}{\delta \phi|_{\{0\}}}.$$

  This comes from the defect action

  $$\int \phi(-\Delta + m^2)\phi \, d^d\vec{x} + \int L(\phi^D, \phi|_{\{0\}}).$$

Characterising all possible BCs for the scalar field is a nontrivial problem!

Example 3.6 ($U(1)$ Gauge Theory)**.** Have a $U(1)$-connection A.

- **Dirichlet BC:** Set $A|_{\mathbb{R}^2\times\{0\}} = 0$ (or alternatively a fixed a_0). Then

  $$F_\perp|_{\{0\}} \in \mathrm{Obs}^D .$$

- **Neumann BC:** Set $F_\perp|_{\{0\}} = 0$.

- **Enriched Neumann BC:** Require

  $$F_\perp|_{\{0\}} = \frac{\delta L}{\delta A|_{\{0\}}}.$$

  This comes from the defect action

  $$\int F^2 \, d^d\vec{x} + \int L(\phi^D, A|_{\{0\}}).$$

Remark 3.14. If you have a theory with a G-symmetry (and presumably a 1-form field) but without gauge symmetry, it is interesting to ask whether it might arise as a boundary condition for an honest G-gauge theory one dimension higher.

3.2 Lecture 2 (Davide Gaiotto)

Topic for today: 1d topological defects, mostly in the context of 2d (homological) topological QFT.

Typical physical theory: local operators live in (super) vector spaces.

Theory obtained by twisting: involves passing to Q-cohomology, and so naturally obtain DG vector spaces, homological machinery.

How does a typical topological QFT arise? Start with a QFT with a mass gap, and study the theory at extremely low energies. So:

Definition 3.1. A "physical" TFT is one where the operators satisfy $\partial\mathcal{O} \equiv 0$.

Definition 3.2. A "homological" TFT is one where the operators satisfy

$$\partial\mathcal{O}_i = \{Q, \tilde{\mathcal{O}}_i\}$$

for an operator satisfying $\{Q^\dagger, Q\} = \partial$.

The "physical" setup is fairly well understood mathematically and physically.

The "homological" setup is less well understood and richer structurally. Davide will lecture on this today – perhaps the audience can explain it to him.[35]

3.2.1 Physical versus homological comparison

The physical TFT setup: since $\partial\mathcal{O} = 0$ we obtain an actual *algebra* of local operators with an associative composition law, and a *category* of line defects where $\mathrm{Hom}(L_1, L_2) = \mathrm{Obs}_{L_2}^{L_1}$.

The homological TFT setup: cannot just collide operators willy-nilly. Indeed, there can be divergences when one brings operators together, requiring one to work at a finite cutoff distance. The operators may not even be closed! Instead now one has to consider A_∞-algebras (quasi-isomorphic to E_1-algebras) of OPEs, and instead of a category of line defects on obtains an A_∞*-category* of line defects.

3.2.2 Deforming SUSY quantum mechanics

Recall the setup SUSY data for $\mathcal{N} = 2$ SQM:

$$Q^2 = 0, \qquad (Q^\dagger)^2 = 0, \qquad \{Q, Q^\dagger\} = H \quad (\partial_t).$$

To deform this, we can deform our supercharge: $Q \to Q + x$. This will be square-zero if

$$Qx + x^2 = 0$$

i.e x solves the Maurer-Cartan equation for the dg-algebra A.

Remark 3.15. There is a corresponding deformation of $Q^\dagger$ – we suppress this in what follows.

The Hamiltonian is deformed:

$$H \to H + \{Q^\dagger, x\}$$

The action is deformed:

$$S \to S + \int_{-\infty}^{\infty} dt\, \{Q^\dagger, x\}(t).$$

[35] *audience laughs*

The correlation functions are deformed:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle_x = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \cdots e^{\int_{-\infty}^{\infty} dt \{\mathcal{O}^\dagger, x\}(t)} \rangle$$

Example 3.7. Consider first when there are no operator insertions:

$$\begin{aligned} \langle 1 \rangle_x &= \langle e^{\int_{-\infty}^{\infty} dt \{\mathcal{O}^\dagger, x\}(t)} \rangle \\ &= \langle 1 + \int_{-\infty}^{\infty} \{\cdots\}(t_1) dt_1 + \int_{-\infty}^{\infty} \int_{-\infty}^{t_1} \{\cdots\}(t_1)\{\cdots\}(t_2) dt_1 dt_2 \cdots \rangle \end{aligned}$$

This expression is the *path ordered exponential* (it comes up a lot – it's important). When is this expression BRST-invariant?

$$\begin{aligned} \delta_{BRST} \langle \cdots \rangle &= \langle 0 + \int_{-\infty}^{\infty} \partial_{t_1} x(t_1) + \iint \partial_{t_1} x(t_1)\{\cdots\}(t_2) + \iint \{\cdots\}(t_1) \partial_{t_2} x(t_2) \cdots \rangle \\ &= \cdots \end{aligned}$$

One would have to consult the video of this talk for the rest of the calculation, together with an explanation. The left hand side of the board is being kept pristine for important statements; the right hand side is being used for annoying calculations – of which there will be plenty – and so is erased rather more frequently than one might otherwise prefer.

3.2.3 A_∞-algebra

We have a collection of operations

$$\mu_n : A^{\otimes n} \to A[2-n].$$

Observe that the shift allows us to write down a reasonable equation, the *MC-equation*

$$\mu_1(x) + \mu_2(x, x) + \mu_3(x, x, x) + \cdots = 0 \tag{3.1}$$

Remark 3.16. One might worry about convergence of the right hand side. We won't for now, and Davide comments that he doesn't know the precise mathematical condition under which this makes sense.

We can construct deformations of an A_∞-algebra from a solution of the MC-equation:

$$\begin{aligned} A^{(x)} &= A \\ \mu_1^x(-) &= \mu_1(-) + \mu_2(-, x) + \mu_2(x, -) + \mu_3(-, x, x) + \cdots \\ \mu_2^x(-, -) &= \mu_2(-, -) + \mu_3(x, -, -) + \mu_3(-, x, -) + \mu_3(-, -, x) + \cdots \end{aligned}$$

and so on.

Slogan: MC-equation gives a perturbative description of the deformations of an A_∞-algebra.

An A_∞-morphism $\varphi : A \to B$ is a collection of morphisms

$$\varphi_n : A^{\otimes n} \to B[1-n],$$

and as one might expect there is a notion of A_∞ quasi-isomorphim. From a solution to the MC-equations for A and a quasi-isomorphism, x_A, one can define a solution to the MC-equation for B:

$$\sum_n \varphi_n(x_A^{\otimes n}) = x_B$$

Would like a sharp version of the following statement: "The solution to the MC-equation x_A is a topological effective action for the theory."

Remark 3.17. Be careful! Quasi-isomorphism is not an "innocent" process from the physical point of view. E.g. one can give the cohomology an A_∞-structure (with trivial differential), and passing to cohomology is then a quasi-isomorphism – but one may lose physical information in this passage.

One illustrative comment on this: we shouldn't integrate away massless degrees of freedom. So if under quasi-isomorphism one loses an operator of mass dimension, I dunno, 2034, we don't worry too much about that. But if we are studying a massless scalar field theory and under quasi-isomorphism we lose the scalar field itself – that would be a problem!

Now: in actuality we shouldn't allow operators to actually collide (divergences), and this needs to be taken into account in renormalisation of the expression

$$\langle P \exp \int_{-\infty}^{\infty} \{Q^\dagger, x\}(t) dt \rangle.$$

In fact in reality, operators aren't attached to points – instead we assign a space of operators to an *inteval*

$$(a, b) \to \mathrm{Obs}_{(a,b)}$$

and we will have operations

$$E_{\Gamma_n}(\cdots) : \mathrm{Obs}_{(0,L)}^{\otimes n} \to \mathrm{Obs}_{(0,L)}\,.$$

How does this work?

- Take n (ordered) operators attached to the interval $(0, L)$.
- Shrink the interval $(0, L)$ and embed n disjoint copies of the shrunken interval into $(0, L)$.
- Include the operators from the disjoint intervals into the original interval (factorisation structure).

Generally: given a chain $\Gamma_n \in C^*(\mathrm{Conf}_n)$,[36], can consider

$$[Q, E_{\Gamma_n}] = E_{\partial\Gamma_n}$$

and

$$\langle P \exp \int_{-\infty}^{\infty} \{Q^\dagger, x\}(t) dt \rangle = \sum_n E_{\Gamma_n}(x, x, \ldots, x)$$

for choices of $\Gamma_n \in C^n(\mathrm{Conf}_n)$. Then

$$\sum_n Q E_{\Gamma_n}(\cdots) = \sum_n E_{\Gamma_n}(x, x, \ldots, Qx, x \ldots, x) + \sum_n E_{\partial\Gamma_n}(x, x, \ldots, x).$$

Idea: Picking the cycles is a renormalisation scheme – how do the sizes of the segments change as we move along the line, etc. In general will keep the sizes of the segment of order L.

We need to actually *find* these chains Γ_n. Γ_1 has no boundary – so there is no problem. Γ_2 has boundary – so we need a way to describe what happens when multiple segments come close together[37] $M_2 \in C^0(\mathrm{Conf}^2_{[0,L]})$ giving a correction

$$\partial\Gamma_n = \Gamma_{n-1} \circ M_2;$$

but then one has to take into account corrections coming from the boundary of the boundary – an operator $M_3 \in C^1(\mathrm{Conf}^3_{[0,L]})$ which gives a further corrections:

$$\partial\Gamma_n = \Gamma_{n-1} \circ M_2 + \Gamma_{n-2} \circ M_3 + \cdots$$

What are we trying to do?

[36] Configuration space of n-points.
[37] A configuration space of *segments* now, not of points!

- Want to have a way to compose operators.
- But to do so need to actually decide where to *place* the operators.
- These choices need to be compatible with each other. E.g. the chain Γ_n has a boundary, and we want to relate the boundary of Γ_n to the choices of previous chains Γ_{n-k}; we do this through a collection of M_j, which are chains in the configuration spaces of j segments in $[0, L]$, that allow us to write

$$\partial\Gamma_n = \sum_k \Gamma_{n-k} \circ M_k$$

  and such that $\partial^2\Gamma_n = 0$.

3.2.4 Example: $W = \phi^3$ A-twisted LG-model

The line operators in this theory from an A_∞-category. Start with two lines L_1 and L_2:

- $\mathrm{End}(L_1) = \mathbb{C}$ (only the identity)
- $\mathrm{End}(L_2) = \mathbb{C}$ (only the identity)
- $\mathrm{Hom}(L_2, L_1) = 0$
- $\mathrm{Hom}(L_1, L_2) = \mathbb{C}$

The differentials are all trivial, the identity composes as the identity. All of the μ_n operations are "as zero as possible".

We can take direct sums of line defects

$$L_1, L_2 \to L_1 \oplus L_2$$

and to give a local operator between the direct sums of two lines is precisely giving a matrix of operators

$$\begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{pmatrix} : L_1 \oplus L_2 \to L_1 \oplus L_2$$

We can also take shifts of line defects $L \to L[1]$, and the corresponding local operators will be the local operators shifted in degree:

$$\mathrm{Obs}_{L_1, L_2[1]} = \mathrm{Obs}_{L_1, L_2}[1].$$

Consider $\tilde{L} = L_1 \oplus L_2[1]$. Then

$$\mathrm{End}(\tilde{L}) = \begin{pmatrix} \mathbb{C} & \mathbb{C}[1] \\ & \mathbb{C} \end{pmatrix}$$

Now given $x \in \mathrm{Hom}(L_1, L_2)[1] \in \mathrm{End}(\tilde{L})$, we can construct the line defect $L_3 = [\tilde{L}, x]$.

Problem 10. *Study properties of this line defect.*

3.2.5 MC-elements in A-infinity categories

Given $\mathcal{A}$ and A_∞-category, there is a *universal deformation* $MC[\mathcal{A}]$, a "bigger" A_∞-category that controls the deformations of the original $\mathcal{A}$.

Question: Can take a line defect and decompose it into direct sums of smaller and smaller line defects. Something like Karoubi completion?

3.2.6 IR gapped theories

If you have a theory which is gapped in the IR, then the line defects will be generated by an exceptional collection L_i with $\mathrm{End}(L_i) = \mathbb{C}$ and $\mathrm{Hom}(L_i, L_j) = 0$ if $i > j$.

Example 3.8 (B-model on $\mathbb{CP}^1$)**.** Has $\mathrm{Hom}(L_1, L_2) = \mathbb{C}^2$.

Question: Is there an obstruction to constructing a theory with $\mathrm{Hom}(L_1, L_2) = \mathbb{C}^3$?

3.2.7 IR gapped theories

Clarifications:

(1) This talk was entirely about homological topological 1-dimensional defects. So everything here was all about what was happening *on these defects*.

(2) Did it actually matter that the bulk theory was homologically topological? Or did we really only need homological topological invariance *in the direction of the line defects we wish to study*? Answer: Yep, this setup would work as well.

3.3 Lecture 3 (Tudor Dimofte)

Plan: Take intuition from a free field theory with a Lagrangian, bootstrap up to more interesting structures.

Today: Mostly about quantum mechanics – maybe SUSY QM at the end – so that we can form a chain of logic from fields and Lagrangians to algebraic structures.

Remark 3.18. One reason quantum mechanics is so important: computations in higher dimensional field theories will often be reduced to statements in a quantum mechanical theory (potentially with an infinite dimensional space of fields).

3.3.1 Bosonic Quantum Mechanics

A function $x(t)$ describing a particle moving on a line corresponds to the following setup:

- 1d QFT with spacetime $\mathbb{R}_t$ and target $\mathbb{R}_x$.
- Fields are

$$\{x(t)\} = C^\infty(\mathbb{R}_t) = \text{ sections of } \mathbb{R}_x \text{ bundle}$$

Remark 3.19. We work in Euclidean time – this won't matter for 1d QFT, but will matter in higher dimensions.

$$S = \int_{\mathbb{R}_t} dt\, (\partial_t x)^2, \qquad \text{(free)}$$

$$S = \int_{\mathbb{R}_t} dt\, \left(\dot{x}^2 + V(x)\right), \qquad \text{(potential)}$$

Problem 11. *Take everything Tudor says today and express it in the language from Phil and Kevin's lectures.*

The *equations of motion (EOM)* are

$$\delta S = 0$$

Since someone should do this once during this workshop, let's calculate this:

$$\begin{aligned}\delta S &= \int_{\mathbb{R}_t} dt\ (2\partial_t x \mathrm{pr}_t(\delta x) + V'(x)\delta x) \\ \text{(IBP)} &= \int_{\mathbb{R}^t} dt\ \underbrace{(-2\ddot{x} + V'(x))}_{\frac{\delta S}{\delta x(t)}=0}\delta x\end{aligned}$$

i.e. the EOM are

$$\ddot{x} = \frac{1}{2}V'(x)$$

Example 3.9. If $V(x) = x^2$ then $\ddot{x} = x$ which is solved by (Figure 16)

$$x(t) = ce^{\pm t}.$$

Note that these either decay to zero slowly or blow up to infinity quickly. This is because of our Euclidean

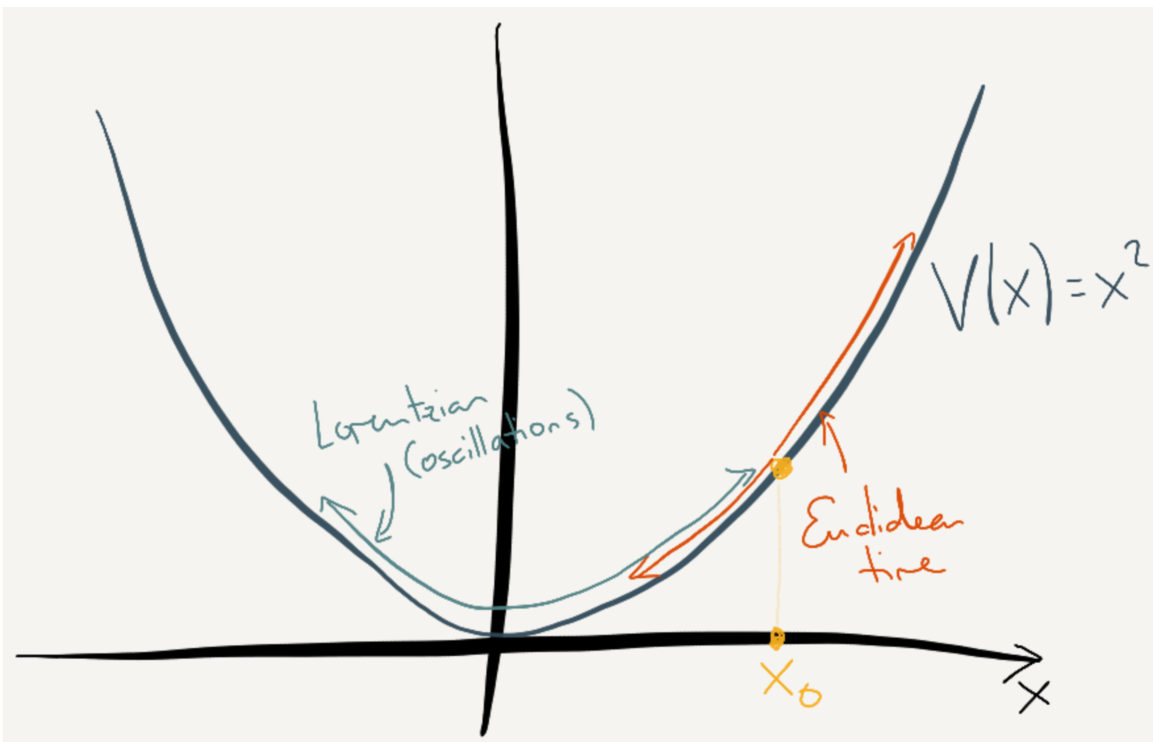


Figure 16: Quadratic potential with Euclidean and Lorentzian time solutions.

time convention – we have lost some intuition. If we restore Lorentzian time we find $x(t) = ce^{\pm it}$.

3.3.2 Operator perspective

Consider correlation functions

$$\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle = \int_{\text{Fields}} d\mu\, e^{-\frac{1}{\hbar}S}\mathcal{O}_1\cdots\mathcal{O}_n$$

where $\mathcal{O}_i \in C^\infty_{loc}(\text{Fields})$. Focus on functions on jets at a point (t),

$$\mathcal{O}(t_0) = [x(t_0)^{p_0}\dot{x}(t_0)^{p_1}\ddot{x}(t_0)^{p_2}\cdots],$$

c.f. Kevin's talk: these are of the form $\mathcal{O}_f$ for $f = \delta(t - t_0)$.

Simplify: Suffices to consider linear operators,

$$\mathcal{O} = x \qquad\qquad \mathcal{O} = \dot{x} \qquad\qquad \mathcal{O} = \ddot{x}$$

Can get arbitrary monomials from collisions (Figure 17). Furthermore can obtain commutation rules by

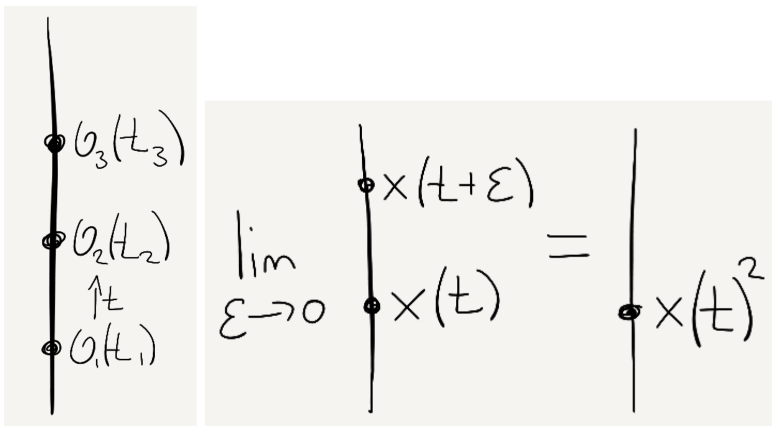


Figure 17: Schematic collision of local operators (left); colliding linear operators (right).

taking limits of differences (Figure 18):

$$\lim_{\epsilon\to 0}\langle\cdots(x(t)\dot{x}(t+\epsilon) - x(t+\epsilon)\dot{x}(t))\cdots\rangle = \langle\cdots(-\hbar)\cdots\rangle$$

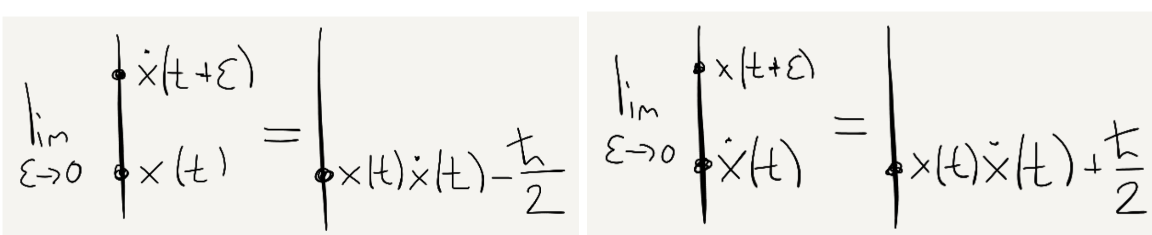


Figure 18: Calculating commutator of operators as a limit.

Also:

$$\begin{aligned} 0 &= \langle\cdots \underbrace{\frac{\delta S}{\delta x(t_0)}}_{(2\ddot{x}-V'(x))(t_0)} \cdots\rangle \\ &= \int d\mu\left(-\hbar\frac{\delta}{\delta x(t_0)}\right)\left[e^{-\frac{1}{\hbar}S}(\cdots)\right] \end{aligned}$$

Upshot: Can eliminate $\ddot{x}$ with EOM, so only need to consider $x, \dot{x}$.

3.3.3 Hilbert space

The Hilbert space $\mathcal{H}$ is a linear representation of the operator algebra. Physically, we obtain this as

- $\mathcal{H}$: geometric quantization of phase space (P, ω)
- $P = \{\text{solutions to EOM on } [0, \epsilon)\}$, a 0-shifted symplectic space.[38]

Example 3.10. In our example,

$$P = \mathbb{R}^2 = \{x(0), \dot{x}(0)\};$$

ω is indeuced by the action:

- consider S on $[0, \epsilon)$,
$$S = \int_0^\epsilon (\dot{x}^2 + V)dt$$
- first variation (restricted to EOM) produces boundary term at $t = 0$, which yields the Liouville 1-form θ on p:
$$\begin{aligned}\delta S &= \int_0^\epsilon (2\partial_t x \partial_t(\delta x) + V'(x)\delta x) \\ &= -2\dot{x}\delta x|_0 + (\cdots)|_\epsilon + \underbrace{\int \text{EOM}}_{=0},\end{aligned}$$
So
$$\begin{aligned}\theta &= -2\dot{x}(0)\delta(x(0)) \\ \omega &= \delta\theta = -2\delta\dot{x}(0) \wedge \delta x(0)\end{aligned}$$

Remark 3.20. Can often shortcut the above procedure: in local Darboux coordinates on P,

$$\begin{aligned}\omega &= \sum_{\text{Fields } \phi} \delta p_\phi \wedge \delta\phi, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}}\end{aligned}$$

where $S = \int dt\, L$ (integral of the Lagrangian).

What does it mean to geometrically quantise? We are taking

- $\mathcal{H} \simeq L^2$-sections of a $\mathbb{C}$-line bundle $\mathcal{L} \to P$
- with $c_1(\mathcal{L}) = \omega$,
- which are polarised (i.e. independent of half the coordinates).

For us: this is

$$L^2(\mathbb{R}) = \{f(x)\}$$

with

$$\begin{aligned}x &: f(x) \mapsto xf(x) \\ \dot{x} &: f(x) \mapsto \hbar\partial_x f(x)\end{aligned}$$

[38]We take a finite length interval, as we may not know about long-time solutions to the relevant PDEs. This finite length assumption is also what results in the unexpected 0-shifted symplectic space (where we might have expected a (-1)-shifted symplectic space from Phil's lectures).

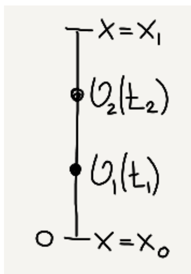
Remark 3.21. For those familiar with D-modules: if we think of phase space as the cotangent bundle to a space B, we are considering differential operators on B.

Example 3.11. $[\dot{x}, x] = \hbar$. This is pictured in Figure 18.

The Hilbert space $\mathcal{H}$ is roughly[39] functions on

$$\{\text{boundary conditions for path integral at } t = 0\}$$

Picture: suppose that $t \in [0, 1]$. Then we can intepret the following picture in the following ways:[40]



$$= \int_{\{x(t)\},\, x(0)=x_0,\, x(1)=x_1} d\mu\, e^{-\frac{1}{\hbar}S} \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)$$
$$= \langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\rangle_{x(0)=x_0,\, x(1)=x_1}$$

$$(\text{ALSO}) = \langle\langle \delta(x - x_1), \mathcal{O}_2(t_2)\mathcal{O}_1(t_1)\delta(x - x_0)\rangle\rangle_{\mathcal{H}}$$

This second expression doesn't really make sense – we can have it make sense by treating the δ-functions as half-densities and only ever integrating against two functions f and g:

$$\int dx_0\, f(x_0) \int dx_1\, g(x_1) \int_{\{x(t)\},\, x(0)=x_0,\, x(1)=x_1} d\mu\, e^{-\frac{1}{\hbar}S} \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)$$
$$= \iint (\cdots)\langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\rangle_{x(0)=x_0,\, x(1)=x_1}$$
$$(\text{ALSO}) = \langle\langle g(x), \mathcal{O}_2(t_2)\mathcal{O}_1(t_1)f(x)\rangle\rangle_{\mathcal{H}}$$

To interpret $\mathcal{O}(t_1)f(x)$, need time evolution.

In general:

$$H = L - \dot{x}\frac{\partial L}{\partial \dot{x}} = -\dot{x}^2 + V(x)$$

action on $\mathcal{H}$; i.e.

$$H = -\hbar^2 \left(\frac{\partial}{\partial x}\right)^2 + V(x).$$

Then we make sense of the expression by using H to time-evolve:

$$\langle\langle g(x), e^{-\frac{1}{\hbar}(1-t_2)H}\mathcal{O}_2 e^{-\frac{1}{\hbar}(t_2-t_1)H}\mathcal{O}_1 e^{-\frac{1}{\hbar}t_1 H} f(x)\rangle\rangle_{\mathcal{H}}$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ act as if they were acting at $t = 0$.

3.3.4 State-operator correspondence

Idea is that in general: Ops $\simeq \mathcal{H}$(on link of a point) (Figure 19).

[39] Up to some things that the physicists will know, and exactly in a topological theory.
[40] $\langle\langle -, -\rangle\rangle_{\mathcal{H}}$ is the inner product on the Hilbert space.

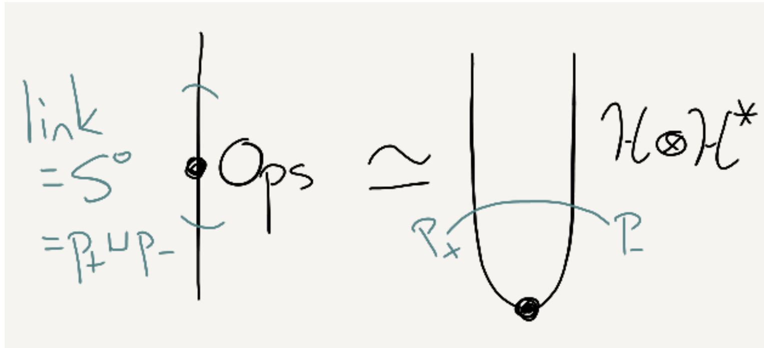


Figure 19: Local operators as states on the link of a point.

Sure: elements of $\mathcal{H} \otimes \mathcal{H}^*$ are the linear maps $\mathcal{H} \to \mathcal{H}$.

In the Hilbert space on

$$p_+ \sqcup p_- \simeq L^2(\mathbb{R} \times \mathbb{R}) = \{k(x, x')\}$$

the product comes from *convolution*:

$$k : f(x) \mapsto \int dx'\, k(x, x') f(x')$$

3.3.5 Fermionic Quantum Mechanics

Fermionic particles move on $\Pi\mathbb{R}$ (parity shifted line). The fields are

$$\psi(t) : \mathbb{R}_t \to \Pi\mathbb{R}$$

which anticommute:

$$\begin{aligned} \psi(t)\psi(t') &= -\psi(t')\psi(t) \\ (\psi(t))^2 &= 0 \end{aligned}$$

If we think of these as living in a spinor bundle, we'd like them to satisfy a Dirac-type equation. We could do this for a single fermion, but we're going to do this instead for two real fermions $\psi_1(t)$, $\psi_2(t)$. Then we can write down the action

$$\begin{aligned} S &= \int dt \psi_1 \partial_t \psi_2 \\ &= \int dt \psi_2 \partial_t \psi_1 \qquad \text{(IBP)} \end{aligned}$$

Alternatively: we can take one *complex* fermion $\psi : \mathbb{R}_t \to \Pi\mathbb{C}$ with complex conjugate field $\bar{\psi}$. I.e. we have fields

$$\text{Fields} = \{\psi(t), \bar{\psi}(t)\}$$

equipped with an involution

$$\dagger : (\psi, \bar{\psi})^\dagger = (\bar{\psi}, \psi).$$

The action is

$$S = \int \bar{\psi} \partial_t \psi dt$$

with EOM

$$\partial_t \psi = \partial_t \bar{\psi} = 0.$$

In Euclidean space we will have

$$(\partial_t)^\dagger = -\partial_t$$

and a product of fermions transforms as

$$(\psi\eta\gamma)^\dagger = \bar{\gamma}\bar{\eta}\bar{\psi}.$$

Problem 12. *Check that* $S^\dagger = S$.

Local operators?

Again suffices to only consider linear monomials (see Figure 20 for the nontrivial commutator),

$$\psi, \bar{\psi}, \qquad \underbrace{\dot{\psi}, \dot{\bar{\psi}}, \ldots}_{=0 \text{ (EOM eliminate)}} \quad .$$

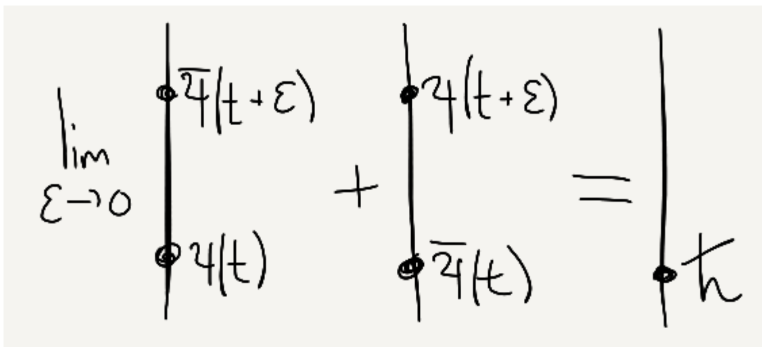


Figure 20: Fermion anticommutator.

Conjugates:

$$\frac{\partial L}{\partial \dot{\psi}} = \pm \bar{\psi} \qquad\qquad \frac{\partial L}{\partial \dot{\bar{\psi}}} = \pm \psi.$$

Phase space:

$$\begin{aligned} P &= \{\psi, \bar{\psi}\} \text{ at } 0 \\ \omega_P &= \delta\psi \wedge \delta\bar{\psi} \\ \mathcal{H} &= \{f(\psi)\} = \mathbb{C}[\psi] = \{a\psi + b\} = \mathbb{C}^2 \end{aligned}$$

This on $a\psi + b \in \mathbb{H}$ as the column vector $\begin{pmatrix} a \\ b \end{pmatrix}$. Then the operators ψ and $\bar{\psi}$ are represented as

$$\begin{aligned} \psi &: f \mapsto \psi f & &\leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \bar{\psi} &: f \mapsto \hbar \frac{\partial}{\partial \psi} f & &\leftrightarrow \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

and the commutation relation is

$$[\psi, \bar{\psi}] = \psi\bar{\psi} + \bar{\psi}\psi = \hbar \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now the state-operator correspondence is as shown in Figure 21.

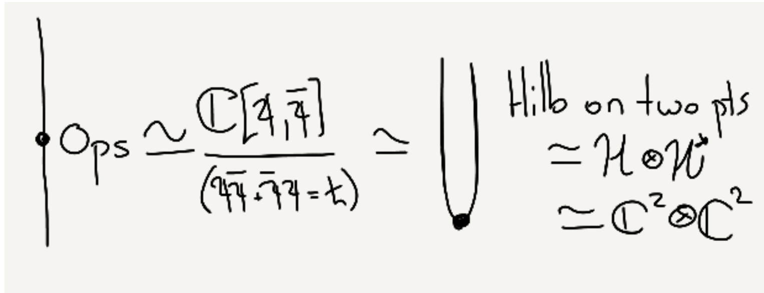


Figure 21: State-operator correspondence for a complex fermion.

3.3.6 1-dimensional $\mathcal{N}=2$ SUSY

Will use the superspace formalism:

$$\mathbb{R}_t \times (\Pi\mathbb{C})_{\theta,\bar{\theta}}.$$

There is a dagger involution

$$(t,\theta,\bar{\theta})^\dagger = (t,\bar{\theta},\theta).$$

A *bosonic superfield* is $\Phi \in C^\infty(\mathbb{R}\times\Pi\mathbb{C})$ can be expanded as

$$\Phi = A(t) + \theta\alpha(t) - \bar{\theta}\beta(t) + \theta\bar{\theta}B(t)$$

where the coefficient functions are even (bosonic). We could also have considered fermionic superfields, i.e. $\Phi \in \Pi C^\infty(\mathbb{R}\times\Pi\mathbb{C})$.

We have the following vector fields on superspace:

$$\begin{aligned} Q &= \partial_\theta - \bar{\theta}\partial_t & D &= \partial_\theta + \theta\partial_t \\ \bar{Q} &= -\partial_{\bar{\theta}} - \bar{\theta}\partial_t & \bar{D} &= -\partial_{\bar{\theta}} - \theta\partial_t \\ [Q,\bar{Q}] &= 2\partial_t & [D,\bar{D}] &= -2\partial_t \\ [Q's, D's] &= 0 \end{aligned}$$

These act on superfields, and so induce an action on components.

Remark 3.22. The $Q,\bar{Q}$ generate the $\mathcal{N}=2$ SUSY algebra.

Define

$$(\partial_\theta, \partial_{\bar{\theta}}, \partial_t)^\dagger = (-\partial_{\bar{\theta}}, -\partial_\theta, -\partial_t).$$

Then † extends to an involution of the algebra (it alread acts on fields), and preserves the relation

$$[Q,\bar{Q}] = 2\partial_t.$$

Problem 13. *Check this.*

So: can get a representation of the 1d $\mathcal{N}=2$ SUSY algebra by restricting to the fixed points of †, i.e. to *real* superfields.

What do real superfields look like? Let's only consider the bosonic ones:

$$X(t,\theta,\bar{\theta}) = \underbrace{x(t)}_{\in\mathbb{R}} + \theta\psi - \bar{\theta}\bar{\psi} + \theta\bar{\theta}\underbrace{f(x)}_{\in\mathbb{R}}.$$

This leads to the theory from Matt Bullimore's lectures that resulted in de Rham cohomology.

OR: restrict to Φ such that

$$\bar{D}\Phi = 0;$$

these are called *chiral superfields.* In components:

$$\Phi = \underbrace{\phi}_{\in\mathbb{C}} + \theta\chi + \theta\bar{\theta}\dot{\phi}$$

The adjoint $\Phi^\dagger$ satisfies

$$D\Phi^\dagger = 0,$$

called *antichiral superfields.*

Remark 3.23. To construct actions one requires both chiral and antichiral superfields.

There is also a fermionic version: *fermi superfield* Γ satisfying

$$\bar{D}\Gamma = 0$$

Doing SUSY QM with these sorts of superfields yields the Hermitian models from Matt Bullimore's lectures (i.e. leads to Dolbeault cohomology).

3.4 Lecture 4 (Davide Gaiotto)

3.4.1 2d (N, N) SQFT

Recall the relevant SUSY algebra

$$\begin{aligned}
\{Q_+^i, Q_+^j\} &= \delta^{ij} P_{++} \\
\{Q_-^i, Q_-^j\} &= \delta^{ij} P_{--} \\
\{Q_+^i, Q_-^j\} &= 0
\end{aligned}$$

where, e.g. $P_{++} = \partial_1 + i\partial_2$.

If we are going to insert a defect at a particular point, then since translations arise as a result of the SUSY algebra it *must* be the case that some amount of SUSY is broken in order to preserve the location of the defect.

$\frac{1}{2}$*-BPS 1d defects* preserve

$$Q^i = Q_+^i + Q_-^i,$$

So that we still have

$$\{Q^i, Q^j\} = \delta^{ij}\partial_1$$

i.e. translations in one of the directions. The resulting SUSY on the defect is $N = N$ SQM:

- $(2, 2)$ gives $N = 2$ SQM
- $(4, 4)$ gives $N = 4$ SQM
- We can also preserve fewer supersymmetries: $(4, 4)$ with $\frac{1}{4}$ BPS defects lead to $N = 2$ SQM

If we are studying a theory T on a manifold with boundary, i.e. a half-space, then choosing $\frac{1}{2}$-BPS boundary conditiosn and passing to Q-cohomology leads to a *dg-category of* $\frac{1}{2}$*-BPS boundary conditions*, $\mathcal{B}_2(T)$.

Question: There is a dg-category of $\frac{1}{4}$-BPS boundary conditions in a $(4,4)$ theory, $\mathcal{B}_2(T)$; moreover, a $\frac{1}{2}$-BPS boundary conditions are in particular also $\frac{1}{4}$-BPS boundary conditions, so we have

$$\mathcal{B}^{1/2}(T) \subset \mathcal{B}_2(T).$$

How can we recognise which $\frac{1}{4}$-BPS boundary conditions are in fact $\frac{1}{2}$-BPS boundary conditions? Davide doesn't know how to give a mathematically precise answer to this question.

3.4.2 Scale invariance

Suppose we have a scale-invariant bulk $(2,2)$ SQFT – in particular this is a SCFT. Then there is a scaling action on the $\frac{1}{2}$-BPS boundary conditions $\mathcal{B}_2(T)$, and we can try to understand subcategory of $\frac{1}{2}$-BPS scale invariant boundary conditions $\mathcal{B}_2^{CFT}(T)$.

Remark 3.24. There are a lot of interesting operations that one can perform on the category of boundary conditions. These operations generally don't preserve supersymmetry or scale-invariance – understanding the interplay between these operations and such conditions is interesting.

If we have two theories which are "dual" to eachother in some precise sense (e.g. some sort of equivalence of categories), then certain subcollections should be dual to eachother. E.g. some class of boundary conditions on the one side should correspond to some *other* class of boundary conditions in the dual theory.

Remark 3.25. Understanding boundary conditions which are preserved by some structure or operation leads naturally to the notion of stability conditions. Do stable objects in one theory get mapped to stable objects in the dual theory?[41] Suspect so, but can't prove it.

3.4.3 Theory on a strip

Given a 2d theory T and appropriate classes of left and right boundary conditions, we can now study the theory on a strip $\mathbb{R} \times [0,1]$ (Figure 22).

By the state-operator correspondence we can (up to questions of rotation by $-\pi$) turn this into a question about a 2d theory on the half-plane with two boundary conditions meeting at a junction (Figure 23).

3.4.4 Boundary conditions for B-model on X

Let's start with **Dirichlet BCs**: we require that our fields satisfy

$$\varphi|_\partial = p$$

for a choice of $p \in X$. Call this BC D_p.

Now, suppose that we consider the theory on a strip. On the LHS of the strip place *any* boundary condition B.[42] On the RHS of the strip place the boundary condition D_p. Then for every length scale L of the interval factor of the strip, every boundary condition B and every point $p \in X$ we obtain a theory of super quantum mechanics, which we call $SQM_B[P,L]$ (Figure 24).

Different length scales will lead to quasi-isomorphic theories, so we will often ignore this. So we have a family of theories parametrised by X. Moreover, one can show that

$$\bar{\partial}_p Q = 0,$$

[41] For some compatible dual notions of stability conditions? Unclear of the precise setup here.

[42] We make *no* assumption on this boundary condition. Even for a free theory, we could take a strongly coupled boundary condition

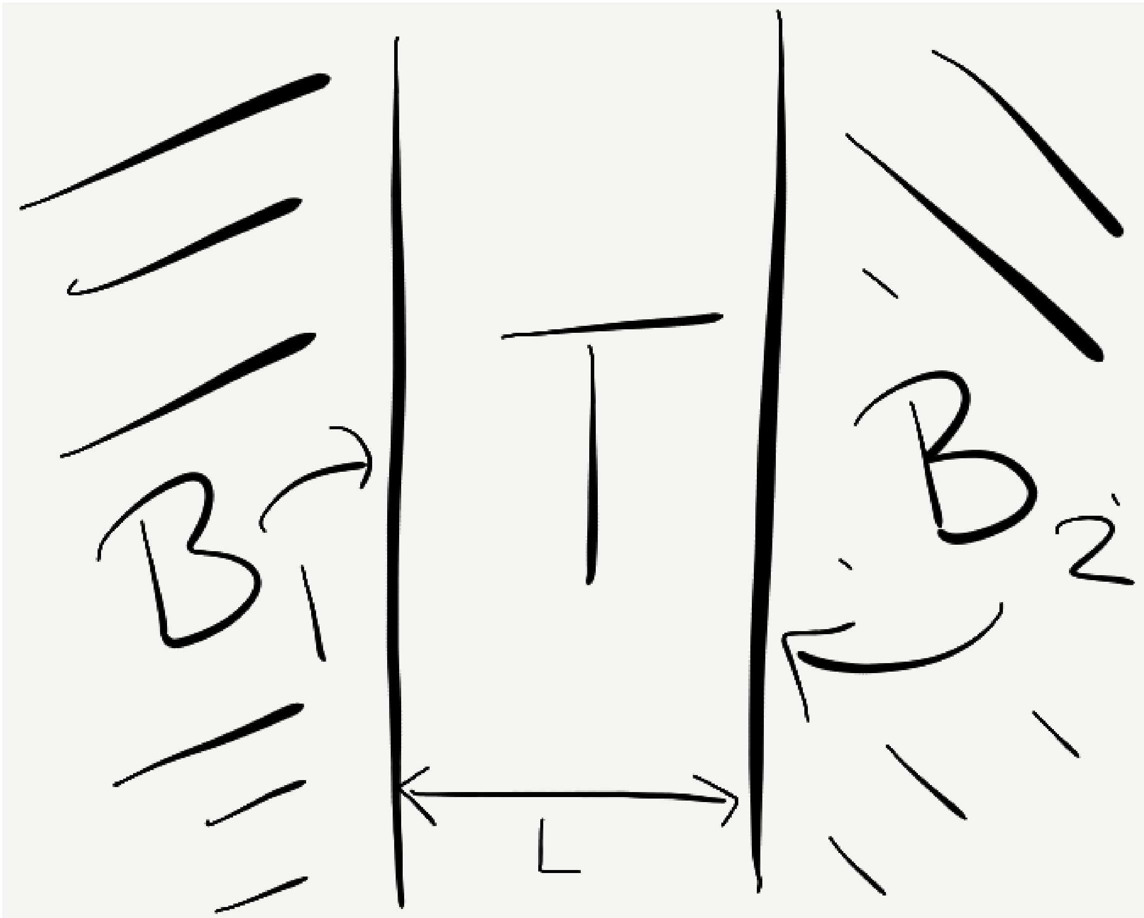


Figure 22: Placing a theory on a strip of width L.

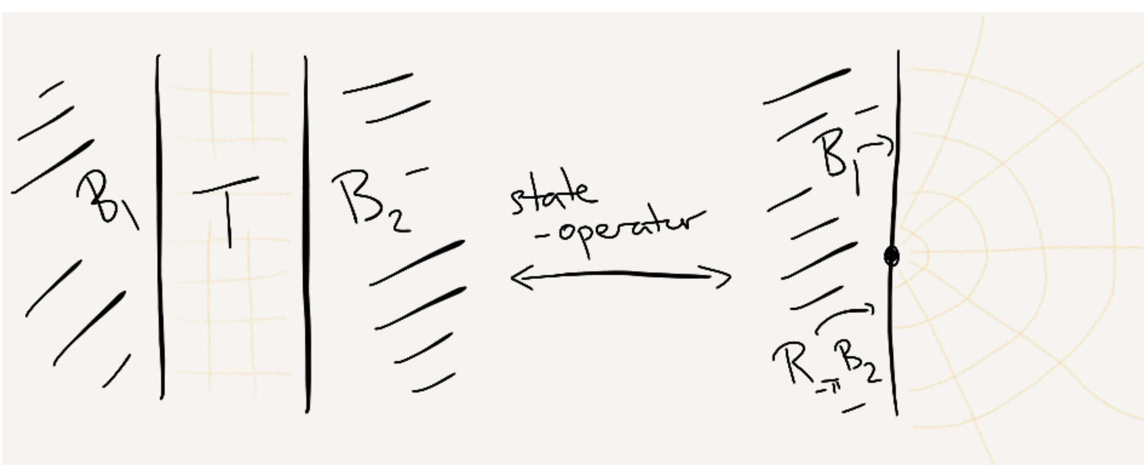


Figure 23: The state-operator map in 2d exchanges theories on a strip and on a half-plane.

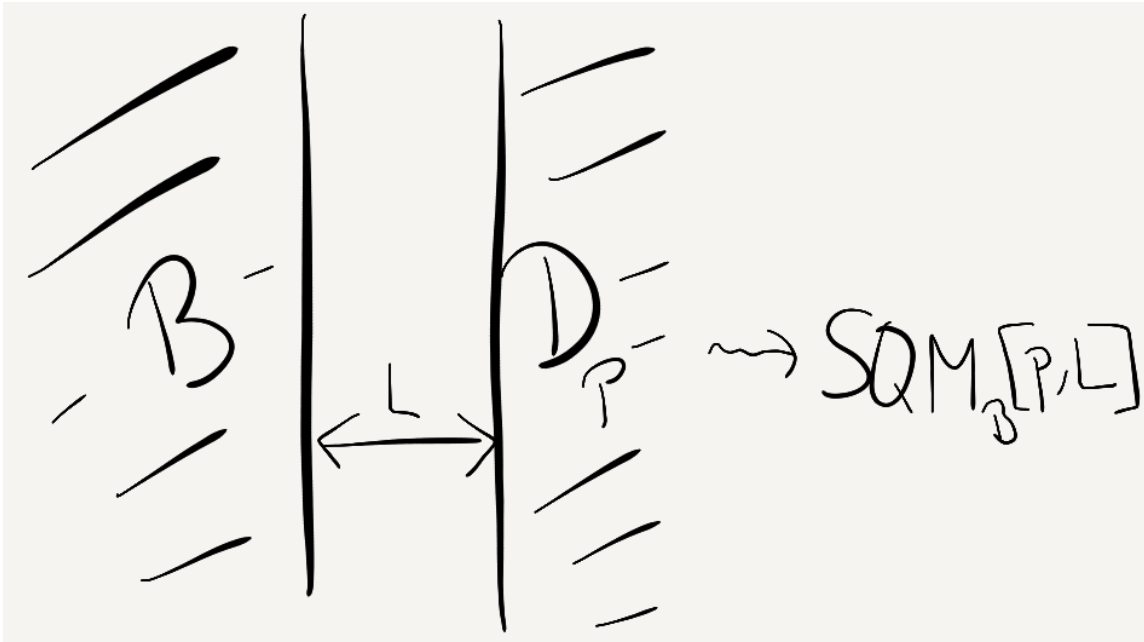


Figure 24: Family of SQM theories from Dirichlet BCs.

so we have holomorphic dependence on $p \in X$.

Upshot: We have a holomorphic family of SQM theories parametrized by X (and L).

Now, let's move on the **Neumann BCs:** we require that our fields satisfy

$$\varphi|_\partial \text{ free}, \quad \partial_\perp \varphi|_\partial = \cdots ,$$

and also other conditions that we can derive by applying the technology of Dylan's talk.

So we get a family of SQM theories parametrized holomophically by X, by taking

$$S^{Neu}_{Bulk} + S^{SQM}[p = \varphi|_\partial].$$

In fact this gives us a map in the *opposite* direction – given a theory of SQM containing a parameter depending holomorphically on X, we can deform our boundary condition in the B-model according to the prescription in the above equation.

Remark 3.26. We now have a pair of (roughly) inverse functors between "boundary conditions in the B-model on X" and "families of SQM theories parametrised by X", denote such theories as "$X \to SQM$".

3.4.5 The boundary condition/quasicoherent sheaf function

Consider $N = n$ SQM: algebra $\{Q^i, Q^j\} = \delta^{ij} H$. Let $\mathcal{H}_E$ be the space of states of energy $H = E$.

If $E > 0$, then the Q^i acting on $\mathcal{H}_E$ generate a Cliff(n) module.

On the other hand, if $E = 0$ then $Q^i|0\rangle = 0$. You don't get the interesting extra structure that arises in the $E \neq 0$ case.

Example 3.12. For $\mathcal{N} = 2$, the non-zero energy states always appear in pairs, mapped to each other by Q and $Q^\dagger$. As you deform your Hamiltonian, you don't change the Q-cohomology provided that you don't allow non-zero energy states to devolve into zero-energy states.

Let's return to the B-model situation. Given a quantum mechanical theory, one can consider the projection $\pi_{E<\Lambda}\mathcal{H}$ to states below a given energy Λ (Figure 25).

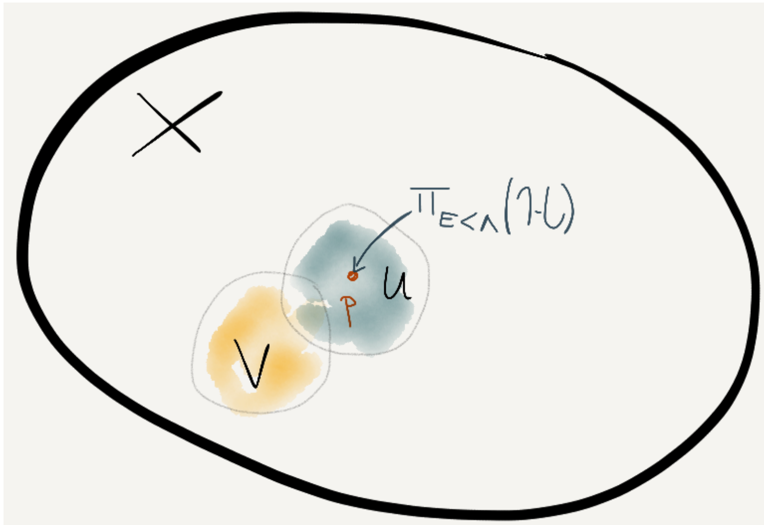


Figure 25: Building a sheaf on X from energy-cutoff Hilbert spaces.

Considering the theory given by a point $p \in X$, we can perform such a cutoff – it won't for instance, change the SUSY ground states. Moreover, moving in a small neighbourhood U of the point p won't move any states over the cutoff $E = \Lambda$, so that Hilbert space may be used to describe all of the theories parametrized by points in the set U.

Given two intersecting open sets U and V, either $\Lambda(U)$ or $\Lambda(V)$ is larger – embed the Hilbert spaces in the Hilbert space with the larger cutoff.

Continuing this procedure over the entire space X, one obtains a quasicoherent sheaf on X. So we have a "functor"

$$\mathcal{B}_2(X) \leftrightarrow \mathrm{QCoh}(X)$$

To prove that this is really a functor one has to do more work, e.g. show how to compatibly map morphisms. This will involve calculations of the following type: given the theory on the strip, we can consider local operators on each of the boundaries; these induce operators in the corresponding SQM theory

$$\begin{tikzcd}
\mathrm{Ops}(B) \arrow[dr] & & \mathrm{Ops}(D_p) \arrow[dl] \\
& \mathrm{Ops}(SQM_B[p, L]) &
\end{tikzcd}$$

and there are compatibilities which must be satisfied, etc.

3.4.6 $\mathcal{N} = 4$ SQM

Consider a matrix describing the supercharges,

$$Q^{A\dot{A}}, \qquad A, \dot{A} = 1, 2$$

concretely this should be the matrix

$$\begin{pmatrix} Q_3 + iQ_4 & Q_1 + iQ_2 \\ -Q_1 + iQ_2 & Q_3 - iQ_4 \end{pmatrix}$$

Then the SUSY algebra is

$$\{Q^{A\dot{A}}, Q^{B\dot{B}}\} = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}H$$

For a general deformation of $\mathcal{N}=4$ SQM described by x,

$$\{Q^{A\dot{A}}, x^{B\dot{B}}\} = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\delta H + \epsilon^{AB}\hat{W}^{(\dot{A}\dot{B})} + \epsilon^{\dot{A}\dot{B}}W^{(AB)}.$$

But there are also two types of special deformation:

- $\hat{W}=0$, and
- $W=0$.

E.g. suppose that we have

$$\{Q^{A\dot{A}}, x^{B\dot{B}}\} = \epsilon^{\dot{A}\dot{B}}(\epsilon^{AB}\delta H + W^{AB});$$

then this is unchanged under linear combinations of deformation parameters[43]

$$x^{B\dot{B}} \to \sum_{C=1}^{2} g^B_C x^{C\dot{B}} = y^{B\dot{B}}$$

Explicitly write

$$\begin{aligned}
y^{+\dot{+}} &= ux^{+\dot{+}} + vx^{-\dot{+}} \\
y^{+\dot{-}} &= ux^{+\dot{-}} + vx^{-\dot{-}} \\
y^{-\dot{+}} &= -\bar{v}x^{+\dot{+}} + \bar{u}x^{-\dot{+}} \\
y^{-\dot{-}} &= -\bar{v}x^{+\dot{-}} + \bar{u}x^{-\dot{-}}
\end{aligned}$$

where the matrix g is

$$g^B_C = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

Upshot: Given a single real deformation of the SQM theory, we automatically get a family of further deformations.

In particular, if we think of u, v as being complex coordinates on the space of deformations, then we obtain a $\mathbb{CP}^1$ of complex structures on deformation space by taking

$$\begin{aligned}
u^\zeta &= u - \zeta\bar{v} \\
v^\zeta &= v + \zeta\bar{u}
\end{aligned}$$

and there is a corresponding $\mathbb{CP}^1$ family of holomorphic deformations[44]

$$\begin{aligned}
\partial_{\bar{u}^\zeta} Q^{\zeta\dot{+}} &= 0 \\
\partial_{\bar{v}^\zeta} Q^{\zeta\dot{+}} &= 0
\end{aligned}$$

where

$$Q^{\zeta\dot{A}} = Q^{+\dot{A}} + \zeta Q^{-\dot{A}}.$$

Further upshot: The deformation space is hyperholomorphic.

[43] Not sure I have this statement quite correct.
[44] Holomorphic in a varying complex structure.

Example 3.13 ($\mathcal{N} = 2$ de Rham and $\mathcal{N} = 4$ Hodge SQM)**.** Studying maps $\mathbb{R} \to M$, $\mathcal{H} =$ forms on M, $Q = d$, $Q^\dagger = d^\dagger$.

Suppose further that M is Kähler. Then we obtain $\mathcal{N} = 4$ Hodge SQM with

$$Q^{A\dot{A}} = \begin{pmatrix} \partial & \bar{\partial}^\dagger \\ \bar{\partial} & \partial^\dagger \end{pmatrix}$$

How much of the structure we have been discussing can we see here?

We have a $\mathbb{CP}^1$ worth of supercharges given by

$$Q^{\zeta\dot{+}} = \bar{\partial} + \zeta\partial.$$

So:

- At $\zeta = 0$ we obtain Dolbeault cohomology. ($\bar{\partial}$)
- At $\zeta = \infty$ we obtain the conjugate to Dolbeault cohomology. (∂)
- At $\zeta = 1$ we obtain de Rham cohomology. ($\partial + \bar{\partial} = d_{\mathrm{dR}}$)
- At generic ζ the cohomology is isomorphic to de Rham cohomology.

So we obtain a family over $\mathbb{CP}^1$ which is generically de Rham cohomology, degenerating to Dolbeault cohomology at the poles.

Example 3.14 (Tri-holomorphic Dolbeault SQM)**.** Start with (X, E) a Kähler manifold equipped with a holomorphic bundle E. Then we get an $\mathcal{N} = 2$ thoery with $Q = \bar{\partial}_E$ and $Q^\dagger = \bar{\partial}_E^\dagger$.

Now suppose that instead we have a hyperkähler X equipped with a *hyperholomorphic bundle* – a bundle with connection A such that the curvature $F \in \Omega^{1,1}$ in *all* complex structures. Then one gets a family of $\mathcal{N} = 4$ SQM theories, with

$$\bar{\partial}_A^\zeta = Q^{+\dot{+}} + \zeta Q^{-\dot{+}}.$$

As a sub example: if $X = \mathbb{C}^2$, then one finds that "hyperholomorphic" is equivalent to being an instanton.

Problem 14. *In the exercise session we will study a family SQM^{ADHM} parametrised by $\mathbb{C}^2$.*

3.5 Lecture 5 (Tudor Dimofte)

Orientation: This week, Tudor, Davide and Si are all describing ways of deriving boundary conditions for 2d theories, with the goal of (hopefully) upgrading this to defects in higher dimensional theories later in the week.

A story of the A-model and B-model: The B-model is nice and algebraic! Things can be written down, and it's all lovely...except for the fact that all of the interesting information winds up being hidden in fermions and singularities, necessitating the use of shifted symplectic spaces and Lagrangians, etc. The A-model is geometric, the boundary conditions really are just geometric Lagrangians – but almost nothing is exact. Bummer.

We'll consider A-model boundary conditions in today's talk, and then in the exercises we'll consider the parallel story of B-model boundary conditions.

3.5.1 1d $\mathcal{N} = 2$ SUSY

Recall: $[Q, \bar{Q}] = 2\partial_t$, acts on superspace $\mathbb{R}_t \times \Pi\mathbb{C}_{\theta,\bar{\theta}}$ via left invariant vector fields

$$Q = \partial_\theta - \bar{\theta}\partial_t \qquad\qquad \bar{Q} = -\partial_{\bar{\theta}} + \theta\partial_t$$

There are also right invariant vector fields, D and $\bar{D}$.

Today we will be interested in *real* superfields:

$$\begin{aligned} X &= x(t) + \theta\psi(t) - \bar{\theta}\bar{\psi}(t) + \theta\bar{\theta}f(t), \\ X &= X^\dagger \end{aligned}$$

The action of Q and $\bar{Q}$ on component fields is given by the following table:

	x	ψ	$\bar{\psi}$	f
Q	ψ	0	$\dot{x} - f$	$\dot{\psi}$
$\bar{Q}$	$\bar{\psi}$	$\dot{x} + f$	0	$-\dot{\bar{\psi}}$

To derive this, act by Q on X and then read off the components.

Problem 15. *Check that* $[Q, \bar{Q}] = 2\partial_t$ *on any component.*

The basic action.

$$S = \int dt d\theta d\bar{\theta} \left[\frac{1}{2}\bar{D}XDX + h(X)\right]$$

where h is a *real superpotential*, a smooth function $h : \mathbb{R}_x \to \mathbb{R}$. In components this is

$$S = \int dt \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}f^2 + fh'(x) + \bar{\psi}\dot{\psi} + \underbrace{h''(x)\bar{\psi}\psi}_{\text{mass term}}\right]$$

The equations of motion are

$$\begin{aligned} \ddot{x} &= fh'' + h'''\bar{\psi}\psi \\ \dot{\psi} &= -h''\psi \\ \dot{\bar{\psi}} &= h''\bar{\psi} \\ f &= h'(x) \end{aligned}$$

Physicists would say that f is an "auxilliary field" (no derivatives of f appear in the EOM) and can be "integrated out" of the theory.

In the language we have been considering this week:

- Local operators Ops: we work modulo the EOM, so can just *set* $f = h'(x)$
- Hilbert space: quantisation of phase space P, where P is the solutions to the EOM on $[0, \epsilon)$. P doesn't contain $f(0)$ (since f is fixed in terms of x – no initial conditions needed or allowed).
- So we obtain an equivalent QFT by solvign the EOM for f.

If we do so, then

$$-\frac{1}{2}f^2 + fh'(x) = +\frac{1}{2}h'(x)^2$$

and the action becomes

$$S = \int dt \left[\frac{1}{2}\dot{x}^2 + \underbrace{\frac{1}{2}h'(x)^2}_{V(x)} + \cdots \right]$$

Local operators.

We'll restrict to polynomials, Ops^{polys}: then we can generate local operators from $x, \dot{x}, \psi, \bar{\psi}$. Let's study the commutators given by Figure 26. (From now on: $\hbar = 1$.)

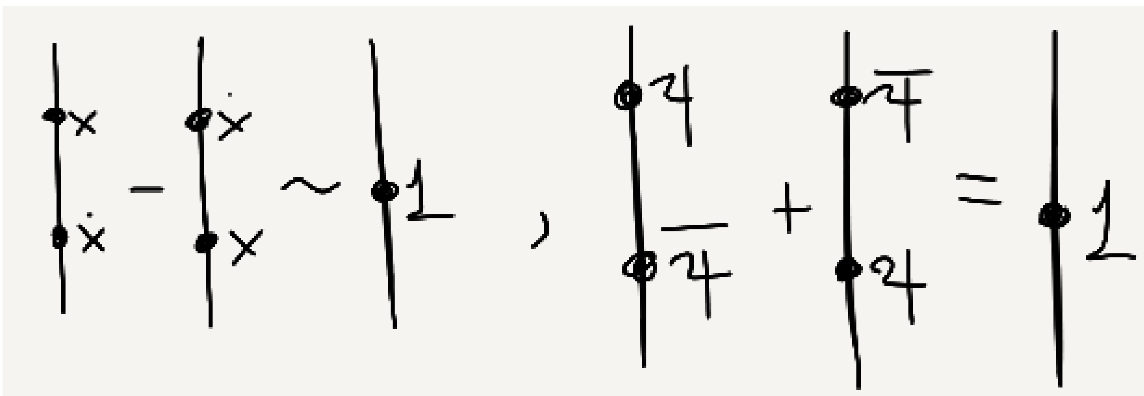


Figure 26: Commutators for a set of generating local operators.

Can we calculate $H^\bullet_Q(\mathrm{Ops}^{polys})$? When we take Q cohomology, we shouldn't have to worry about *precisely* where we insert the operators. We have

$$Q(x) = \psi, \qquad Q(\bar{\psi}) = \dot{x} - h'(x),$$
$$Q(\psi) = 0, \qquad Q(\dot{x}) \neq 0,$$

but

$$Q(\dot{x} - h') = 0.$$

So,

$$H^\bullet_Q(\mathrm{Ops}^{polys}) = \mathbb{C}\langle 1 \rangle.$$

Remark 3.27. If we don't restrict to Ops^{polys} then we get a different answer – there are disorder operators which can be constructed from terms such as $\exp(\dot{x})$. This upset Tudor when he remembered it last night, and we're not talking about it today.

Hilbert space.

The phase space is $P = \{x(0), \dot{x}(0), \psi, \bar{\psi}\}$, so[45]

$$\begin{aligned} \mathcal{H} &\simeq L^2(\mathbb{R}_x) \otimes \mathbb{C}[\psi] \\ &= \{a(x) + b(x)\psi\} \\ &= \Omega^\bullet(\mathbb{R}) \end{aligned}$$

[45] Choosing a polarisation by $x(0)$ and ψ; different polarisations will result in different looking results.

where in the last line we have identified $\psi \leftrightarrow dx$. The correspondence between local operators and operators on $\Omega^\bullet(\mathbb{R})$ is

$$x \leftrightarrow x\cdot, \qquad \psi \leftrightarrow dx\wedge,$$
$$\dot{x} \leftrightarrow -\frac{\partial}{\partial x}, \qquad \bar{\psi} \leftrightarrow \iota_{\frac{\partial}{\partial x}}$$

How does Q act on $\mathcal{H}$? *Via Noether charge!*

$$\begin{aligned}\mathcal{Q} &= (\dot{x} - h'(x))\psi \\ &= \left(-\frac{\partial}{\partial x} - h'(x)\right)\psi \\ &= -d - h'(x)dx\wedge \\ &= -e^{-h}de^h\end{aligned}$$

on $\Omega^\bullet(\mathbb{R})$.

To try and demystify the Noether charge procedure, consider the relation that must hold from Figure 27, i.e. that acting on $\mathcal{H}$

$$Q(\mathcal{O}) = [\mathcal{Q}, \mathcal{O}].$$

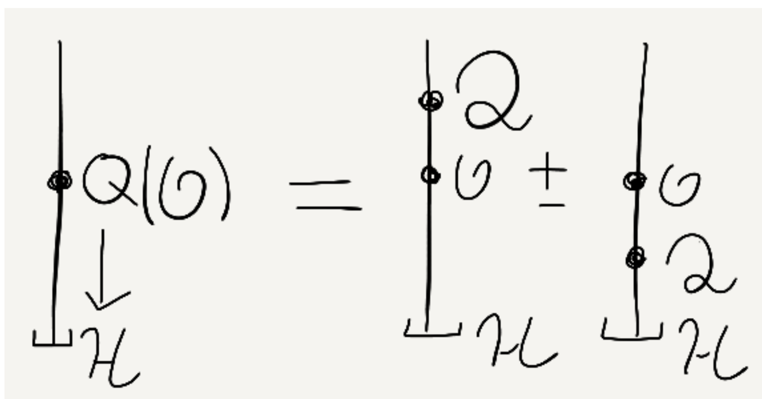


Figure 27: Relation between the vector field Q and the Noether charge local operator $\mathcal{Q}$.

Remark 3.28. Can use this result to find $\mathcal{Q}$.

Note that in the above equation: Q is a vector field and $\mathcal{Q}$ is the Noether charge operator.

Remark 3.29. C.f. higher dimensions: have an action of a symmetry Q on local operators (induced from the action on fields). So given a local operator $\mathcal{O}$ at some point, we can ask what happens when we act on it by Q, $Q(\mathcal{O})$.

There is also a Noether current: a 1-form J satisfying the conservation equation $d \star J = 0$. From this we can construct a surface operator $\mathcal{Q} = \int_\Sigma \star J$. Then the relation between Q and $\mathcal{Q}$ is given by placing the surface operator $\mathcal{Q}$ on a small sphere surrounding the operator $\mathcal{O}$ (Figure 28) – in 1d we are in the special situation where the link of a point is S^0 (two points).

Back to SQM: If the target Z were compact, say $Z = S^1$ (rather than $Z = \mathbb{R}$), then h doesn't matter for computing Q-cohomology,

$$H^\bullet_{e^{-h}de^h}(Z) \simeq H^\bullet_d(Z) = H^\bullet_{\mathrm{dR}}(Z).$$

Can generalize to an n-dimensional Riemannian target Z: locally, we have n superfields X^i and action

$$S = \int dt d^2\theta \left[\frac{1}{2}g_{ij}(X)\bar{D}X^i DX^j + \frac{1}{2}h(X)\right].$$

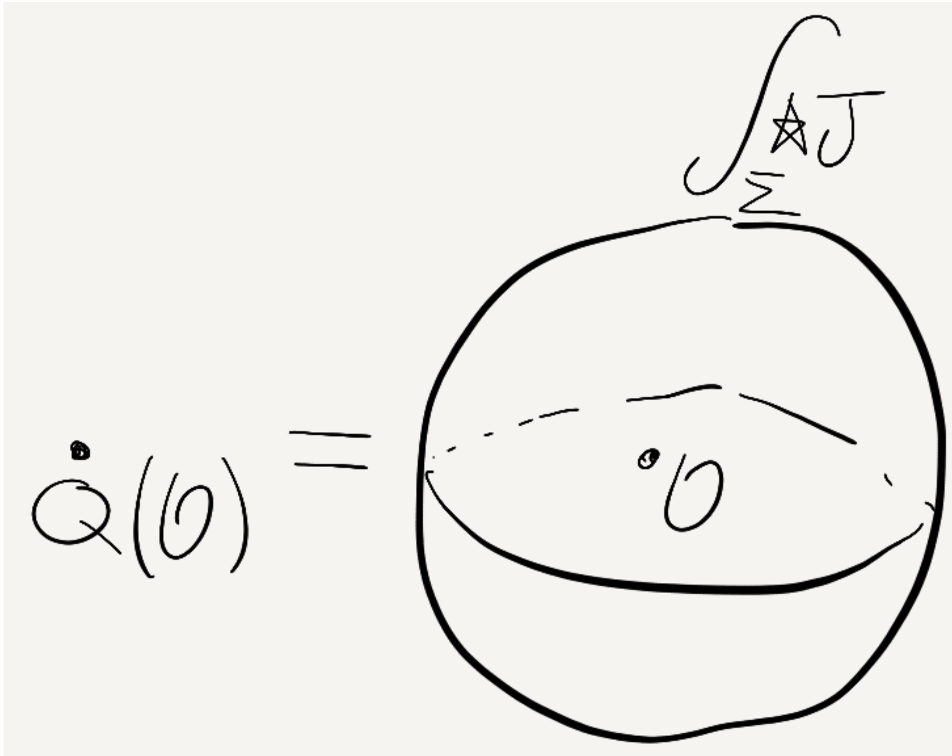


Figure 28: Relation between the vector field Q and the Noether current J in higher dimensions.

If the target is compact, one finds again that the superpotential doesn't matter and

$$H_Q^\bullet(\mathcal{H}) = H_{\mathrm{dR}}^\bullet(Z).$$

On $Z = \mathbb{R}$ noncompact,

$$H_Q^\bullet(\mathcal{H}) \simeq H^\bullet(\text{Morse-Witten complex}).$$

What is the Morse-Witten complex?

- Vector space: $\mathbb{C}\langle dh = 0\rangle$ (generated by critical points of h).
- Differential: from gradient flows.

The Morse-Witten complex arises because we have a potential $V(x)$ whose minima are precisely at the critical points of h. We expect to have a ground state trapped at precisely each local minimum of $V = h'(x)^2$ (Figure 29), and the instantons (gradient flow) arise because the Q-fixed points in the space of fields allow

$$\dot{x} = h'(x).$$

Example 3.15. If $h = x^2$, $H_Q^\bullet(\mathcal{H}) \simeq \mathbb{C}$ (Figure 30).

Example 3.16. If $h \sim x^3 + x$ then there are two critical points and an instanton correction between them (Figure 31),

$$H_Q^\bullet(\mathcal{H}) \simeq \ \mathbb{C} \xrightarrow{d} \mathbb{C} \ \simeq 0$$

Both examples are consistent with $H_Q^\bullet(\mathrm{Ops}) = \mathbb{C}\langle 1\rangle$.

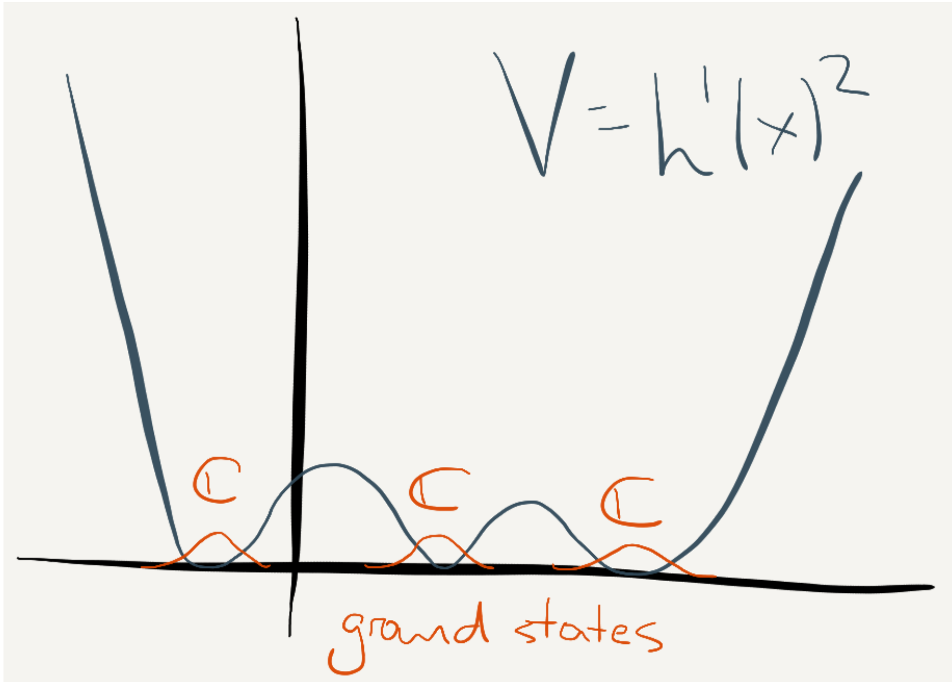


Figure 29: Local ground states generate the Morse-Witten complex.

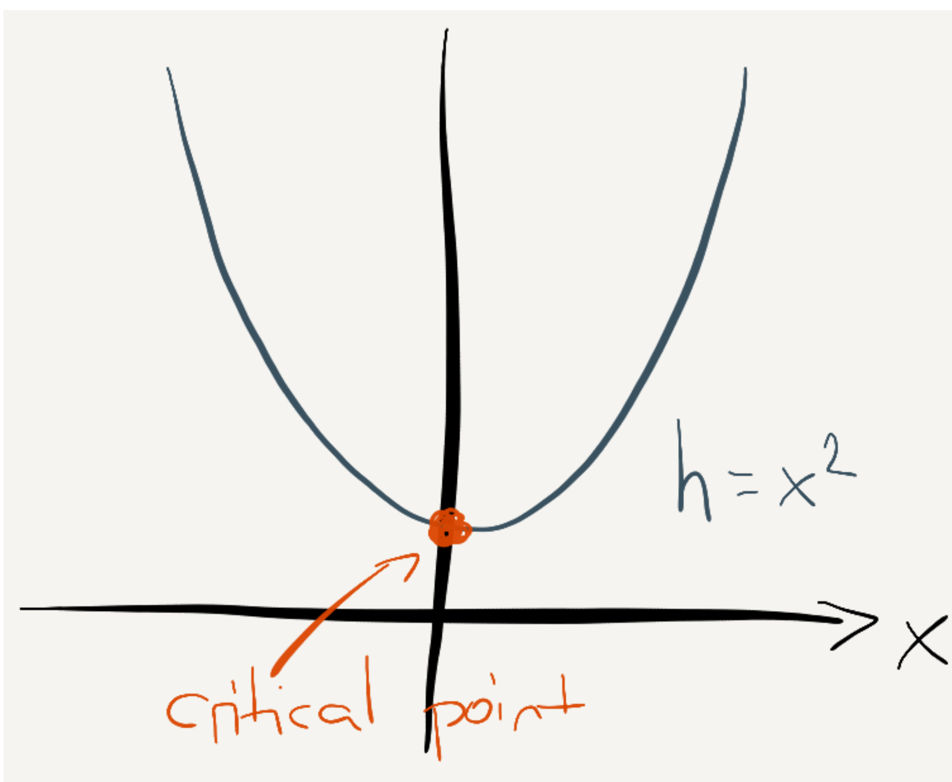


Figure 30: Local ground state associated to $h(x) = x^2$.

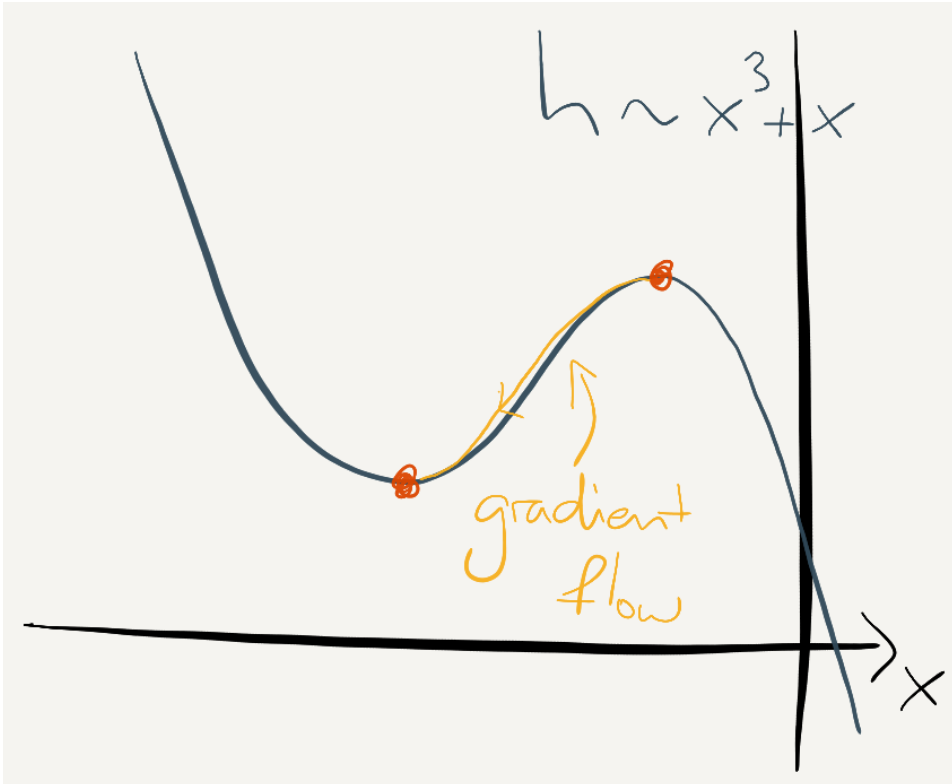


Figure 31: Local ground states associated to $h(x) = x^3$ together with an instanton correction.

3.5.2 A-twist of 2d $\mathcal{N} = (2, 2)$ theory

Recall: we are studying a 2d σ-model with Kähler target $\mathcal{M}$. Have:

- Fields: $\text{Maps}(V^{\mathcal{N}=(2,2)}, \mathcal{M})$
- Local complex coords Φ^i chiral multiplets $i = 1, \ldots, \dim_{\mathbb{C}} \mathcal{M}$; $\Phi^{i\dagger}$
- $\cdots$ – for details see Si's first lecture

The SUSY algebra involves $Q_\pm$, $\bar{Q}_\pm$ satisfying

$$[Q_+, \bar{Q}_+] = 2\partial_z, \qquad\qquad [Q_-, \bar{Q}_-] = 2\partial_{\bar{z}}.$$

There is a nilpotent supercharge

$$Q_A = Q_+ + \bar{Q}_-, \qquad Q_A^2 = 0.$$

Now, what steps to we need to perform to analyse boundary conditions in the theory?

(1) For boundary conditions that preserve Q_A *and* a physical SUSY algebra along the boundary, need to classify 1d SUSY subalgenbas of the 2d $\mathcal{N} = (2, 2)$ algebra containing Q_A.

I.e. where there exists differential operators generating translations parallel to the boundary, but **not** perpendicular to it.

If we fix a boundary as in Figure 32, the only subalgebra is 1d $\mathcal{N} = 2$ generated by Q_A, $\bar{Q}_A = \bar{Q}_+ Q_-$ (this involved a choice depending on the boundary orientation). Check: $[Q_A, \bar{Q}_A] = 2\partial_t$.

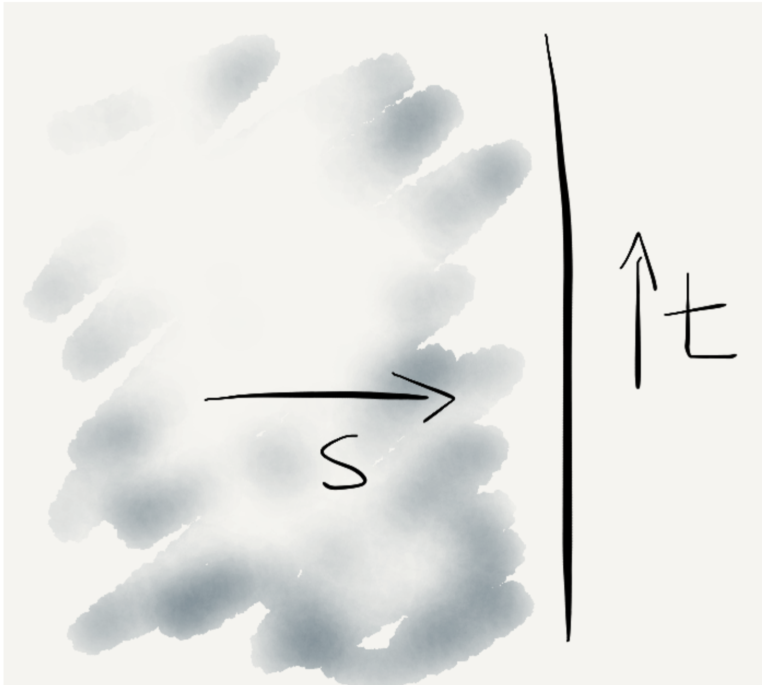


Figure 32: Fixed boundary with boundary coordinate t and normal coordinate s.

(2) Which SQM?

Rewrite entire 2d theory as a 1d $(Q_A, \bar{Q}_A)$ SQM with 1d target Maps$(\mathbb{R}_s, \text{2d target})$.

Main example: 2d target $\mathcal{M} = \mathbb{C}$,

$$\Phi = \underbrace{\phi}_{\mathbb{C}} + \cdots$$

via a linear transformation decomposes into two *real* superfields for $(Q_A, \bar{Q}_A)$,

$$\begin{aligned} X &= x + \theta\psi_x - \bar{\theta}\bar{\psi}_x + f_x \\ Y &= y + \theta\psi_y - \bar{\theta}\bar{\psi}_y + f_y \\ \phi &= x + iy \end{aligned}$$

Note that x, ψ_x etc. now depend on two parameters: $x = x(s,t)$, $\psi_x = \psi_x(s,t)$, etc. The action becomes

$$S_{2d} = \int dt d\theta d\bar{\theta} \left[\frac{1}{2} \int_{\mathbb{R}_s} ds (\bar{D}XDX + \bar{D}YDY) + \underbrace{\frac{1}{2} \int_{\mathbb{R}_s} ds X \partial_s Y}_{h} \right]$$

Since an integral over 1d superspace can only produce t-derivatives, the s-derivatives are accounted for by a potential h-term:

$$h = \int_{\mathbb{R}_s} ds X \partial_s Y = -\int_{\mathbb{R}_s} ds Y \partial_s X.$$

Now:

- $|dh|^2 = \int((\partial_s x)^2 + (\partial_s y)^2)$
- $f_x = \frac{\partial h}{\partial x} = \partial_s y$
- $f_y = \frac{\partial h}{\partial y} = -\partial_s x$

So we are mixing functional derivatives with normal[46] derivatives.

For a general symplectic $(\mathcal{M}, \omega)$ with Λ the Liouville 1-form ($d\Lambda = \omega$) we have

$$h = \int_{\mathbb{R}_s} x^*(\Lambda).$$

In e.g., $\mathcal{M} = \mathbb{C} = T^*\mathbb{R}$, $\Lambda = xdy$.

(3) The (classical) analysis of SUSY boundary conditions boils down to making sure that δh has no boundary terms, i.e.

$$\delta h = \int (EOM) + (\text{no terms on the boundary of space}).$$

Any constraints should be imposed on entire superfields (to ensure they preserve supersymmetry)!

E.g. consider a half-space $\mathbb{R}_{s\leq 0} \times \mathbb{R}_t$ with

$$\begin{aligned} h &= -\int_{\mathbb{R}_{\leq 0}} dsY\partial_s X \\ \delta h &= -\int_{\mathbb{R}_{\leq 0}} ds(\delta Y\partial_s X + Y\partial_s \delta X) \\ (IBP) &= -\int_{\mathbb{R}_{\leq 0}} ds(\delta Y\partial_s X - \delta X\partial_s Y) - Y\delta X|_{s=0} \end{aligned}$$

Then

- $\partial_s X$ and $\partial_s Y$ are the SUSY equations of motion.
- Still need to deal with the boundary term $Y\delta X|_{s=0}$. We *could* impose this constraint by hand[47] – but instead we're going to do something different (and per Tudor, preferable).

So, *don't* impost boundary conditions by hand. Let $\delta h = 0$ do it for us. In our example above, this becomes

$$Y|_{s=0} = 0,$$

and considering the component fields we find that

$$\underbrace{y|_0 = 0}_{\text{Dirichlet on } y} \qquad\qquad f_y|_0 = \underbrace{\partial_s x|_0 = 0}_{\text{Neumann on } x}\ .$$

It will also give us a collection of consistent SUSY boundary conditions on the fermions, which we won't write down. Note that this cuts out (Figure 33)

$$\{y = 0\} \subset \mathcal{M}, \text{ a Lagrangian in } \mathcal{M}.$$

More generally, we could add a boundary action,[48]

$$S_{2d} = \int dtd^2\theta \left[(\cdots) - \int dsY\partial_s X + g(X)|_{s=0}\right]$$

with the variation

$$\delta(h + g(X)|_{s=0}) = 0$$

implying that

$$-Y\delta X + g'(X)\delta X = 0$$

[46] In the sense of "orthogonal".

[47] I.e. we could have constrained our space of fields that we integrate over to have specified behaviour at the boundary.

[48] So $g(X)$ is a function supported only on the boundary.

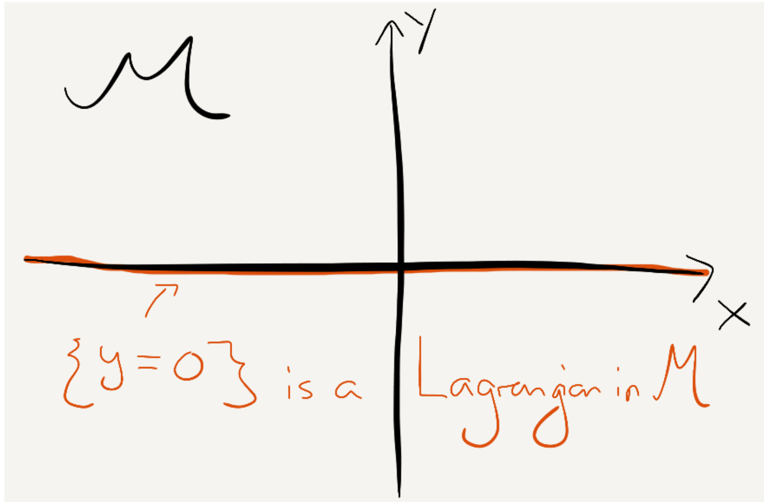


Figure 33: Boundary conditions cutting out the zero section Lagrangian.

i.e. $Y = g'(X)$. Then the Lagrangian we produce is

$$\mathcal{L} = \text{Graph}(dg) \subset \mathcal{M}.$$

Even *more* generally, we can introduce *any* 1d (dR) QM at $s = 0$ If we let Z denote the boundary superfields, then the action becomes

$$S = \int dtd^2\theta \left[(\text{kinetic}) - \int_{\mathbb{R}_{\leq 0}} dsY\partial_s X + g(X, Z) \right]$$

Then classically,

$$\mathcal{L} = \left\{ y = \frac{\partial g}{\partial x}, \frac{\partial g}{\partial z} = 0 \right\}$$

E.g. if $\mathcal{Z} = \mathbb{R}$ and $g = XZ$ then the corresponding Lagrangian is the *vertical* line through 0 in $\mathcal{M}$ (i.e. a cotangent fibre).

One quantum interpretations of this (Davide): the boundary supports SQM on $\mathcal{Z}$ with Morse function $g(X, Z)$, X a parameter (not dynamical), and one obtains

$$H_Q^\bullet(\mathcal{H}_{bdy}) \text{ is fibred over values of } X.$$

So one obtains a constructible sheaf over $\{y = 0\}$. (Simple example: a local system on a Lagrangian in phase space.)

What haven't we done today: shown how to compute Homs. (Ran out of time.)

3.6 Lecture 6 (Davide Gaiotto)

Suppose that we want to construct a d-dimensional TFT, obtained by twisting a SUSY QFT by a supercharge Q – we require at least $d+1$ supercharges in the original theory, since we require supercharges $Q_1, \ldots, Q_d$ satisfying

$$\{Q, Q_i\} = P_i$$

for the Q-cohomology to give a topological theory.

Now: suppose that we want a $(d-1)$-dimensional defect in our theory. It can preserve at most $Q, Q_1, \ldots, Q_{d-1}$ – if it is going to preserve the maximal amount of supersymmetry (which turns out to be at best half the supercharges), we find that we require $2d$ supercharges originally.

We can continue this process – considering defects within defects, etc. – until we consider line operators preserving 2 supercharges Q and $Q^\dagger$. If this is possible, one can show that we require the original theory contain 2^d supercharges.

Now, if we want our original QFT to be physically reasonable (e.g. unitary) then we are allowed to consider *at most* 16 supercharges. Thus, if we want to consider physically reasonable theories with supersymmetric line defects, we are restricted to dimensions $d \leq 4$. In this "best case" we have the following embeddings of SUSY algebras:

$$4\text{d } \mathcal{N} = 4 \supset 3\text{d } \mathcal{N} = 4 \supset 2\text{d } \mathcal{N} = (2,2) \supset 1\text{d } \mathcal{N} = 2$$

With that preamble – let's dive into an interesting 4d theory.

3.6.1 4d $\mathcal{N} = 4$ SYM

To write down the field content and symmetries of this theory we actually only need to know the gauge *Lie algebra*. So, we will consider a more general situation than is often described:

- $4d$ $\mathcal{N} = 4$ SYM
- With gauge Lie algebra $\mathfrak{g}$
- and some topological data.[49]

The field content is

$$\begin{aligned} A_\mu &\in \Omega^1 \otimes \mathfrak{g} \\ \lambda_\alpha^A &\in S^+ \otimes \mathfrak{g} \otimes V_4 \\ \lambda_{\dot\alpha A} &\in S^- \otimes \mathfrak{g} \otimes V_4^* \\ \Phi_{AB} &\in \Omega^0 \otimes \mathfrak{g} \otimes \wedge^2 V_4 \end{aligned}$$

and we will identify $\wedge^2 V_4 \simeq \wedge^2 V_4^*$.

The symmetries of this theory are generated by

$$P_\mu, \quad Q_{\alpha A}, \quad \bar{Q}^A_{\dot\alpha}, \quad M^A_B \in \mathfrak{su}(4)_R \quad (M^A_A = 0).$$

The SUSY algebra is

$$\{Q_{\alpha A}, Q_{\beta B}\} = 0 \qquad\qquad \{\bar{Q}, \bar{Q}\} = 0$$
$$\{Q_{\alpha A}, \bar{Q}^B_{\dot\beta}\} = \delta^B_A P_{\alpha\dot\beta}$$

and the supercharges act on fields schematically as

$$A, \Phi \xrightarrow[\bar{Q}]{Q} \lambda \xrightarrow[\bar{Q}]{Q} F, \partial\Phi.$$

Problem 16. *Write down a reasonable action of $Q, \bar{Q}$ for $\mathfrak{g} = \mathfrak{u}(1)$.*

[49] Which could be, e.g. a choice of gauge group – but could be something more interesting still!

Once a gauge group is specified, the Lagrangian of the theory is unique determined. Schematically it is the sum of terms for the gauge field

$$\frac{1}{g_{YM}^2}\int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \qquad\qquad \underbrace{\theta\int F\wedge F}_{\text{topological term}}$$

Remark 3.30. The topological term affects:

- The quantisation of the theory, but also
- the boundary conditions of the theory (even classically).

Once the gauge field terms are written down the rest of the Lagrangian is uniquely determined by supersymmetry:

$$\frac{1}{g_{YM}^2}\int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}+\lambda\,\not{\partial}\lambda+(\partial_\mu\Phi)^2)+\cdots)+\theta\int F\wedge F.$$

Example 3.17 (Abelian $U(1)$ gauge theory in 4d.)**.** Even in the non-supersymmetric situation, this has an interesting electric-magnetic (EM) duality, $F\leftrightarrow\star F$. For the quantum theory it is *not* enough to simply make this substitution – instead one introduces a Lagrange multiplier field and then integrates out the original field:

$$\frac{1}{g^2}\int|dA|^2\to\int\frac{1}{g^2}|F|^2+dF+B\to g^2\int|dB|^2$$

Thus at $\theta=0$ we find that one must in fact make the exchanges:

$$F\leftrightarrow\star F \qquad g^2\to\frac{1}{g^2}$$

One can show that at $\theta\neq0$ you can still perform the above procedure. Form the complex parameter

$$\tau=\frac{\theta}{2\pi}+i\frac{1}{g^2},$$

then EM duality exchanges

$$\tau\leftrightarrow-\frac{1}{\tau}.$$

Since the θ parameter is periodic, the theory is also invariant under shifts $\tau\to\tau+1$, and so in the end we obtain an $SL(2,\mathbb{Z})$-action

$$\tau\to\frac{a\tau+b}{c\tau+d}.$$

For a nonabelian SUSY theory, there should also be such a duality, which moreover exchanges $\mathfrak{g}\leftrightarrow{}^L\mathfrak{g}$.

There will also be a nontrivial phase multiplier for the supercharges under these dualities. This is due to the different transformation properties of F and $\star F$ (difference in spinor chirality?).

Problem 17. *Check* $\tau\to-\frac{1}{\tau}$.

Research Question: Find the mathematical data required to fully specify a 4d gauge theory. Should involve at least a Lagrangian splitting of the weight lattice times the coweight lattice, but things will be more subtle than that.

3.6.2 3d reduction

Under the $3d \subset 4d$ SUSY embedding, $\mathbb{R}^4 \to \mathbb{R}^3 \oplus \mathbb{R}$, both spinor representations are sent to the same 3d spinor representation,

$$S_+, S_- \to S.$$

In terms of this splitting the SUSY algebra becomes

$$\{Q_{\alpha A}, \bar{Q}^B_{\dot{\beta}}\} = \delta^B_A(\epsilon_{\alpha\dot{\beta}}P^{(4)} + P^{(3)}_{\alpha\dot{\beta}})$$

To preserve 3d SUSY, need to pick an identification $V_4 \simeq V_4^*$,

$$SU(4)_R \to SO(4)_R;$$

we obtain an S^1-family of possible supercharges:

$$Q^{3d}_{\alpha A}[\omega] = e^{\frac{i\omega}{2}} Q_{\alpha A} + e^{\frac{i\omega}{2}} \bar{Q}^A_{\dot{\alpha}}.$$

Then we can consider $\frac{1}{2}$-BPS boundary conditions, which preserve $Q_{\alpha A}[\omega]$.

Problem 18. *Find $\frac{1}{2}$-BPS Dirichlet BCs $A_i = 0$, $i = 1, 2, 3$.*

ω is fixed as a function of τ and the BCs.

Example 3.18. For Dirichlet, $\omega = 0, \pi$.

Remark 3.31. You need to be careful that you don't overconstrain your fields – e.g. can't prescribe vanishing of the scalar field *and* its normal derivative.

Starting with a family of BCs, we can apply EM-dualities to obtain new families of BCs. Denote

$$T : \tau \to \tau + 1, \qquad\qquad S : \tau \to -\frac{1}{\tau}$$

Example 3.19 (U(1))**.** T preserves Dirichlet BCs; but there is a (terminating) chain of BCs:

$$DIR \xrightarrow{S} NEU \xrightarrow{T^k} NEU^{(1,k)} \xrightarrow{S} NEU^{(-k,1)} \dots$$

There are more general boundary conditions $NEU^{(p,q)}$ for coprime p, q which set to zero some particular combination of the field strength at the boundary and the dual field strength at the boundary, with combination determined by τ, p, q.

For $NEU^{(p,q)}$ there two possible values for ω, $\omega^{\pm}(\tau, p, q)$. At $\theta = 0$ one can show that the values of ω for DIR and NEU are $0, \pi$ – things get more complicated at $\theta \neq 0$.

3.6.3 Line operators

In 3d:

There is a dg-category of topological 1d-lines in the bulk – really there is a lot here, it means "all the ways you can fill in a topological cylinder ", i.e. the category that the 3d theory assigns to a circle. So, e.g. you can take a 2d (2, 2) boundary condition B and you can map it to the object that B assigns to the circle to find a map

$$(2,2) - BC \longrightarrow Z^{3d}(S^1)$$

$$B \longmapsto B(S^1)$$

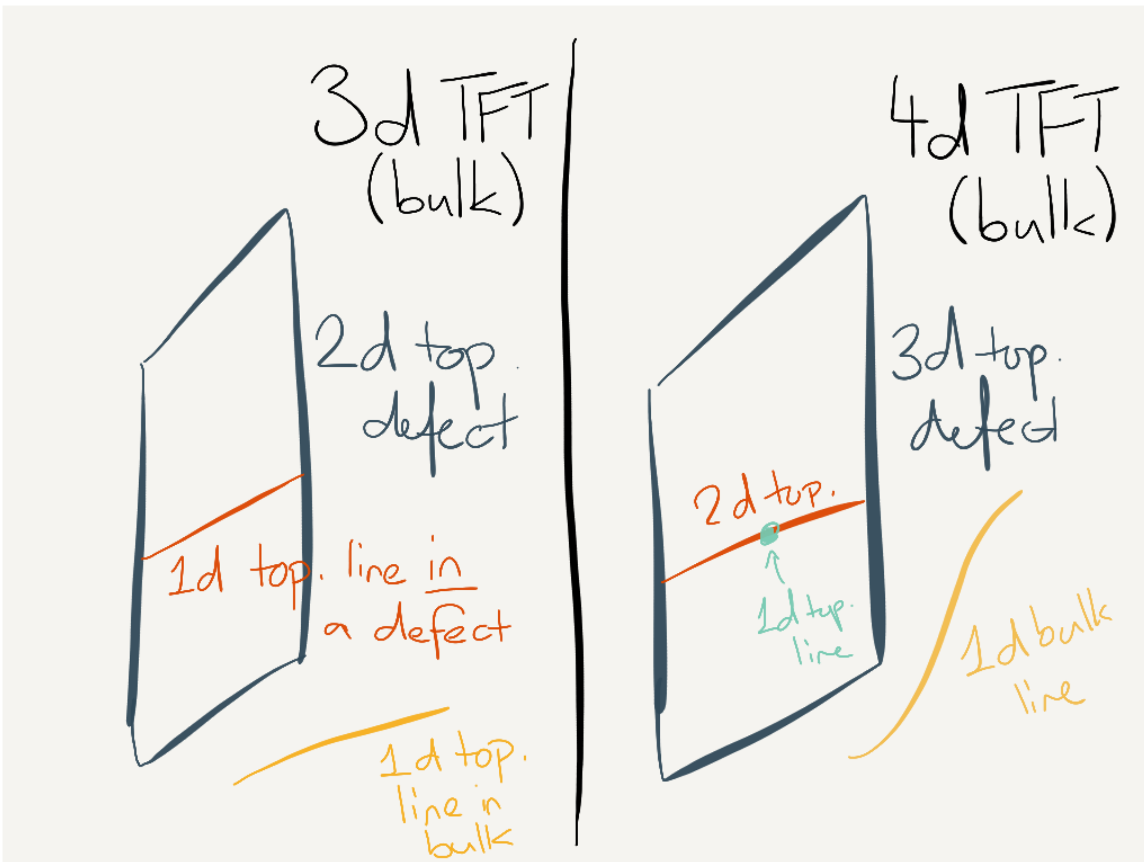


Figure 34: Topological defects in 3d and 4d.

There are also subcategories of e.g. $\frac{1}{2}$-BPS physical lines, possibly living on a defect.

In 4d:

Things are more interesting. E.g. there are Wilson lines (which measure holonomy), and the $\frac{1}{2}$-BPS Wilson lines correspond to $\mathrm{Rep}_{\mathfrak{g}}$.[50] We don't know what the $\frac{1}{4}$-BPS lines are; the $\frac{1}{8}$-BPS lines should be the entire dg-category of "things that look like Wilson lines".

Under S-duality, Wilson lines are sent to 't Hooft lines.[51] Again there is a dg-category of "things that look like 't Hooft lines", which contain $\frac{1}{2}$-BPS lines,[52] $\frac{1}{4}$-BPS lines, etc.

S-duality should exchange Wilson lines and 't Hooft lines (preserving $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS lines, etc.). Mathematically this should result in an equivalence of dg-categories

$$\text{dg-category of Wilson lines} \simeq \text{dg-category of 't Hooft lines}$$

and this should be the *derived Geometric Satake equivalence*.

3.6.4 Junctions between topological lines

We now want to make all of these topological defects live "in the same theory". To do this we will wind up losing some of the purely topological dependence:

- A 3d bulk theory with 2d *holomorphic* surface defects and (topological?) line junctions.
- A 4d bulk theory with 3d topological defects supporting 2d holomorphic junctions (Figure 35).

Example 3.20 (3d $\mathcal{N}=4$)**.** Justin introduced some of the ingredients we need to begin this procedure. Consider the holomorphic twist coming from Q^{+}_{++}. From this we can twist further

- Topological A-twist: $Q^{+}_{++} + \epsilon Q^{-}_{+-}$.
- Topological B-twist: $Q^{+}_{++} + \epsilon Q^{-}_{-+}$

Given the 3d holomorphic theory and a $(2,2)$ holomorphic boundary condition, we can twist further to obtain a 3d topological theory with a 2d topological boundary condition.

But given the 3d holomorphic theory and a $(0,4)$ holomorphic boundary condition, we can't twist to a 2d topological boundary condition in the 3d TFT. Instead, we can *deform* the boundary condition to a *different* 2d $(0,4)$ holomorphic boundary condition.

The same story can be told in 4d with the GL-twist of $\mathcal{N}=4$ SYM; from this one can further apply the machinery of S-duality. This is a very rick (developing) story.

3.6.5 3d Chern-Simons Theory

We've seen a nice axiomatic formulation of 3d Chern-Simons in this workshop, and it is also the first place we found holomorphic boundary conditions for a topological theory – so let's change gears and discuss this theory.

[50] I.e. the $\frac{1}{2}$-BPS Wilson lines are capturing some *underived* information.
[51] C.f. monopole operators.
[52] Something like equivariant D-modules on the affine Grassmannian.

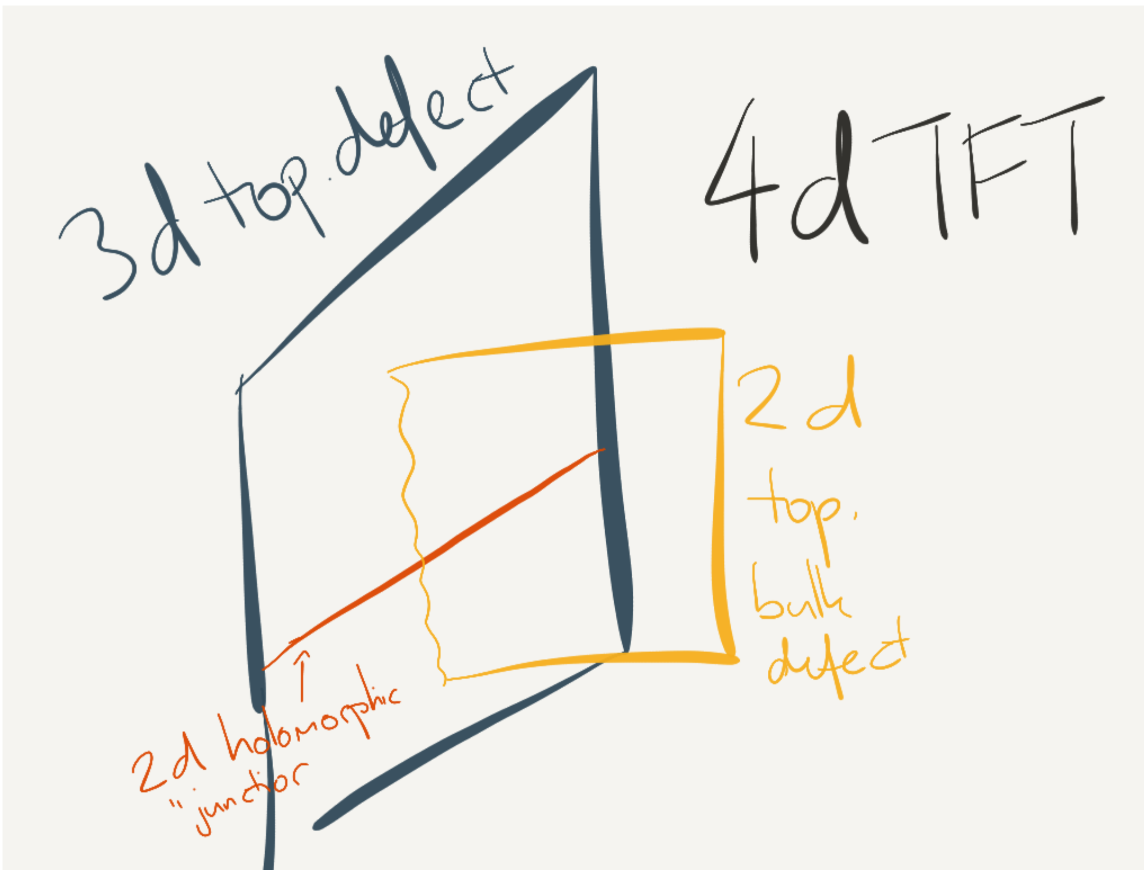


Figure 35: 2d holomorphic junction between 3d defects.

Naively there are not interesting local operators in this theory. But there are interesting line operators: *Wilson lines* W_R, labelled by a representation R of the gauge group and corresponding to an operator insertion

$$P\exp\int A^I_\mu dx^\mu \cdot \sigma^I.$$

Moreover, naively there are no interesting point defects living on a Wilson line – a perturbative argument tells us that

$$(Pert)\,\mathrm{Hom}(W_R, W_{R'}) = \delta_{RR'}\mathbb{C}.$$

Remark 3.32. But – this was too naive! Rather than in perturbation theory, let's consider Chern-Simons at finite coupling k. Then at a junction between W_R and $W_{R'}$ we are led to studying the quantisation of the phase space of flat connections on a sphere with points marked by the representations R and R' (should be related to flags at these points).

Question from Ben G.: What was the mistake in the first calculation?

Answer: A local operator is *anything you can do to field which is local.* A field insertion is one such thing, but "whatever you can do" is a rather unconstrained notion.

Okay, that's great, but we aren't going to use it. Let's step back: consider 3d Chern-Simons with a 2d BC. We can't do things like

- "Set the field to zero on the boundary" or
- "let the field fluctuate freely on the boundary".

These are not good boundary conditions (they do not correspond to Lagrangians, c.f. Dylan's talks).

What can we do? Restrict to gauge transformations which limit to the identity at the boundary. Then we can set

$$A_{\bar{z}} = 0,$$

the $\bar{z}$-part of A to vanish; then the flat connection condition implies $\partial_{\bar{z}} A_z = 0$, so we are studying holomorphic connections. We have discovered a collection of holomorphic boundary operators!

(Caveat lector: I'm not confident the following is correct.)

Example 3.21. In the abelian case,

$$A_z(z)A_z(0) \simeq \frac{1}{k}\frac{1}{z^2},$$

and for $\frac{1}{k}A_z = J_z$ we discover at the boundary the Kac-Moody vertex algebra at level k. (These are the *perturbative observables on the boundary.*)

If we work nonperturbatively, we are led to geometric quantisation of phase space, and we obtain a rational VOA as the (non-perturbative) observables on the boundary: the WZW VOA at level k.

Now: Suppose that we have a Wilson line *in the bulk* ending on the 2d boundary. This gives us a map

$$1d \text{ lines} \to \mathrm{Obs}^\partial - \mathrm{Mod}.$$

Moreover in the bulk there are *extra operators* between the lines – morally these correspond to new higher operators in the boundary theory.

Example 3.22. A line that doesn't have to end on the boundary is a line that can end on a bulk monopole. Bringing this bulk monopole to the boundary yields a new unary operator for the theory.

Universal Bulk Theories.

(C.f. Dylan's talk.) Suppose we want to consider 3d Chern-Simons at *generic* level[53] κ – we can ask if there is a 4d TFT with this theory at the boundary, so that calculations in the 4d theory make sense of the (nonperturbative) Chern-Simons theory.

Indeed there is – it turns out that the $GL[\kappa]$-twisted 4d $\mathcal{N}=4$ SYM TFT with Neumann boundary conditions is such a theory.

We can also consider a 3d topological[54] Dirichlet condition for this theory meeting the 3d Neumann BC at a 2d holomorphic junction (corresponding to the Kac-Moody theory). Interesting questions: what Kac-Moody modules correspond to line defects living in 3d CS ending on the junction, etc.?

3.7 Lecture 7 (Tudor Dimofte)

Stretch Goal: Calculations in the A- and B-models; localisation to 1d theories.

More Important Goal: Discuss some conceptual issues that haven't been covered yet this workshop.

3.7.1 Moduli space of vacua

For a classical field theory on $\mathbb{R}^{d-1,1}$, a *vacuum* v is

$$v \in \mathrm{Fields}(\mathbb{R}^{d-1} \times [0,\epsilon))$$

such that

$$dS(v) = 0$$

and the energy is minimised on v (Figure 36.

In classical field theory, the energy is defined to be

$$E = \int_{\mathbb{R}^{d-1}} h$$

where h is the *Hamiltonian density*, i.e. the Noether current for time translation.

Example 3.23 (φ^4 theory in 4d)**.** Here

$$S = \int d^4\vec{x}\left[\partial_t\varphi^2 - |\partial_{\vec{x}}\varphi|^2 - V(\varphi)\right]$$

where $V(\varphi) = m\varphi^2 + \lambda\varphi^4$. The hamiltonian density is

$$h = (\partial_t\varphi^2) + |\partial_{\vec{x}}\varphi|^2 + V(\varphi),$$

so the vacua are

$$\mathrm{Vacua} = \{\text{constant } \varphi, V(\varphi) = 0\} = \{\varphi \equiv 0\} = \mathrm{pt}\,.$$

In Lorentzian QFT we say the same thing but in a quantum way: vacua v are states

$$v \in \mathrm{Hilb}(\mathbb{R}^{d-1})$$

such that $E = H$, the Hamiltonian, is minimized.

[53] Non-integral.

[54] Really homotopically topological – it has a mild dependence on a choice of complex structure at the boundary.

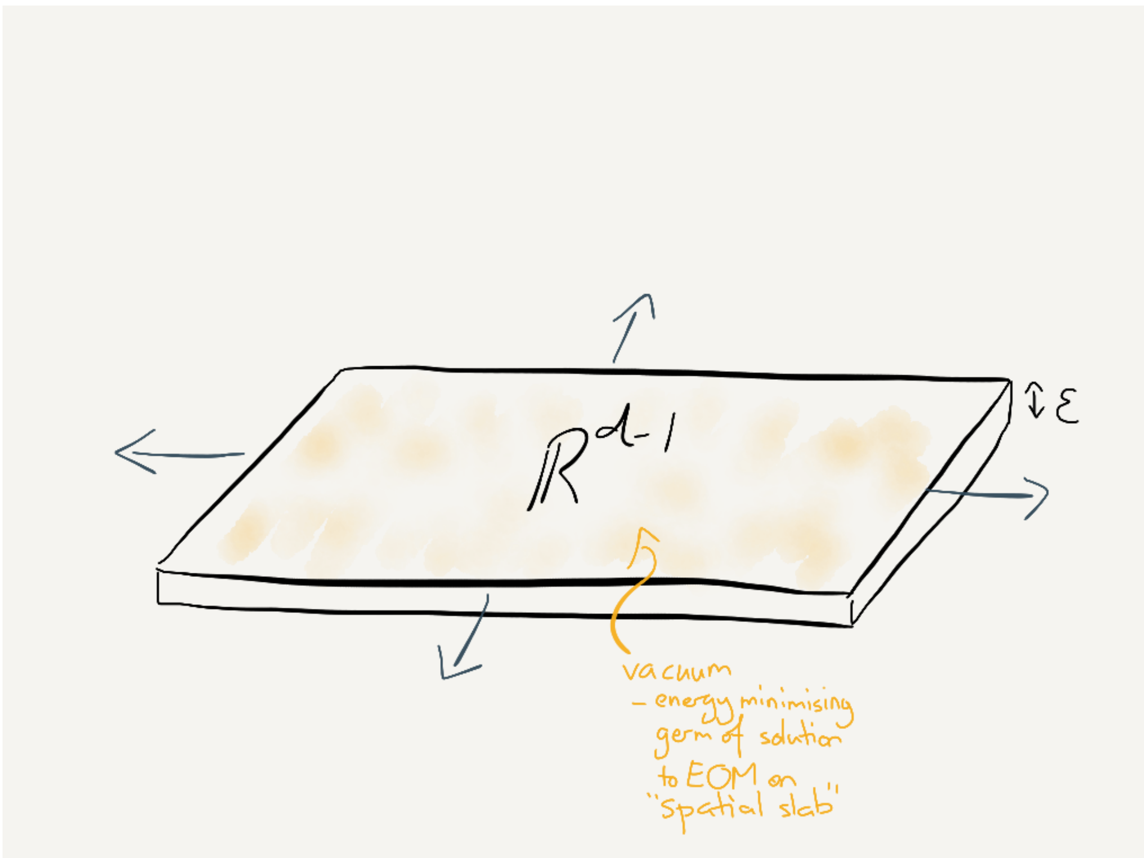


Figure 36: Classical vacuum as germ of minimal energy solution on a spatial slab.

In Euclidean QFT (the case relevant for topological QFT) on $\mathbb{R}^d$ all directions look the same, so we look instead at the asymptotic boundary S^{d-1}_∞. Then a *(generalised) vacuum* is a state

$$v \in \mathrm{Hilb}(S^{d-1}_\infty)$$

such that correlation functions of local operators in this state are finite.

In terms of the path integral, v is an asymptotic boundary condition on the fields such that

$$\langle \mathcal{O}_1(p_1)\cdots\mathcal{O}_n(p_n)\rangle_v = \int_{\text{fields approaching } v} e^{-S}(\cdots)$$

is finite.

We get a pairing

$$\mathrm{Ops}_L \times \mathcal{M}_{vac} \to \mathbb{C}$$

where $\mathrm{Ops}_L = \mathrm{Fact}(B^d_L)$ are the local operators in a ball of radius L, B^d_L, given by

$$\mathcal{O}\cdots\mathcal{O}, v \mapsto \langle \mathcal{O}\cdots\mathcal{O}\rangle_v.$$

We expect (or define[55]) this pairing to be non-degenerate. Then

$$\mathcal{M}_{vac} = \lim_{L\to\infty} \mathrm{Spec}\left(\mathrm{Fact}(B^d_L)\right).$$

Problem 19. *Make sense of this mathematically.*

Remark 3.33. One idea: vacua are asymptotic boundary conditions – the idea of vacua being boundary conditions is a good intuition given this caveat, and of course in the topological situation one can't tell the difference.

Suppose the theory is a cohomological TQFT, e.g. a twist of a SUSY theory,

$$\mathrm{Ops} = \mathrm{Fact}(\mathbb{R}^d)$$
$$\mathcal{A} = H^\bullet_Q(\mathrm{Ops})$$

and $\mathcal{A}$ is commutative if $d \geq 2$. Then the (affinisation of the space of) "topological vacua" is

$$\mathrm{Aff}(\mathcal{M}_{vac}) = \mathrm{Spec}(\mathcal{A}).$$

Incorporating line operators.

Line operators form a 1-category: suppose $\mathcal{L} \in \mathrm{Lines}$ and let 1_L denote the canonical "trivial line". Suppose that we have a line that ends at a point in space – this can be thought of as a homomorphism between the trivial line and $\mathcal{L}$ (Figure 37).

We can see that $\mathrm{Hom}(1_L, \mathcal{L})$ is a module for local operators thought of as endomorphisms of the trivial line, $\mathcal{A} = \mathrm{End}(1_L)$, by sending any $\mathcal{L}$ to the vector space of local operators at its endpoint. This gives a functor

$$\mathrm{Lines} \to \mathcal{A} - \mathrm{Mod} = \mathrm{End}(1_L) - \mathrm{Mod}$$

which may not be faithful (some lines may not end!)

Expect: $\mathrm{Lines} = \mathrm{Sh}(\mathcal{M}_{vac})$.[56]

[55] In quantum theory, vacua are essentially defined such that they distinguish between different operators – if the vacua have the same value on all local operators, we should consider them identical. (Q: Would it be possible to not distinguish local operators but distinguish higher dimensional defects? Ans: You might want to do that – Tudor will consider this in the topological case later in the talk.)

[56] More accurately: we've just noted that in fact this may not be an equivalence, and indeed it is *usually* not an equivalence. But! The discussion above tells us that this should be our "zeroth order guess" for what the category of line operators looks like. This is good for two reasons: we don't need to just blindly flail around guessing what line operators might look like, and if you *do* have a guess for what the full category of line operators looks like you should check that it has a part that looks something like sheaves on the moduli of vacua.

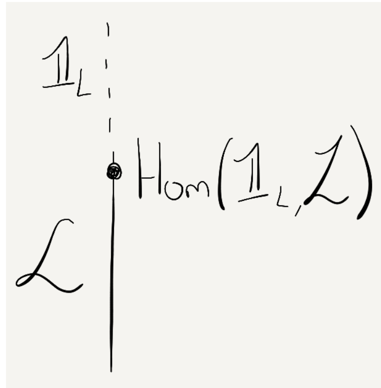


Figure 37: Endpoint of a line as a homomorphism from the trivial line.

Example 3.24 (B-model with target X)**.** Some of the above subtleties appear already in this case:

$$\begin{aligned} \text{Lines} &= \mathrm{Coh}(X \times X) \\ \mathcal{M}_{vac} &= T^*[1]X \\ \mathcal{A} &= \mathrm{PV}(X) \\ \mathrm{Coh}(T^*[1]X) &\overset{KD}{\simeq} \mathrm{Coh}(\underbrace{TX}_{\text{even shift}}) \end{aligned}$$

Keep going to higher dimensional defects.

There is a 2-category of surface operators, Surf; which contains a canonical "trivial surface" S_1. It is not too hard to see that (Figure 38)

$$\mathrm{End}(S_1) = (\text{Lines}, \otimes) = \mathrm{Sh}(\mathcal{M}_{vac}, \otimes).$$

Given any other S, we obtain an action of Lines on $\mathrm{Hom}(S_1, S)$ (Figure 39).

Example 3.25 (3d $\mathcal{N} = 4$ σ-model to $T^*[2]\mathcal{X}$, $\mathcal{X}$ algebraic)**.** Consider the B-twist (Rozansky-Witten (RW) twist). C.f. Kapustin-Rozansky-Sawline. We'll quote some results:

$$\begin{aligned} \mathcal{A} &= H^\bullet_Q(\mathrm{Ops}) \\ &= H^{0,\bullet}(T^*\mathcal{X}) \\ &= H^\bullet(T^*, \mathcal{O}_{T^*\mathcal{X}}) \\ \mathrm{Spec}(\mathcal{A}) &= \mathrm{Aff}(T^*\mathcal{X}) \\ \text{Lines} &= (\mathrm{QCoh}(T^*[2]\mathcal{X}), \tilde{\otimes}) \\ \text{Surf} &= \mathrm{Sh}\,\mathrm{Cat}(T^*[2]\mathcal{X}) \to \text{Lines}-\mathrm{Mod} \end{aligned}$$

The way you would compute this in physics is very similar to the way that Justin performed calculations yesterday. To find Lines, we would compactify the theory on an S^1 that links the line. This gives a 2d B-model on the half-plane with target the (infinite dimensional) loop space $L(T^*[2]\mathcal{X})$.

You might worry – "Oh no, this is gigantic!" But this is where the superpotential comes in. In fact we are thinking of the target $L(T^*[2]\mathcal{X})$ *with* superpotential W, and we localise on $\{dW = 0\} = \{\text{constant loops}\}$. So we conclude Lines $= \mathrm{QCoh}(T^*[2]\mathcal{X})$.

Every time the subscript dR appeared in Justin's talk, that physically had something to do with the appearance of a superpotential which served to localise our theory on constant maps.

Remark 3.34. In the above example the category of lines had an altered tensor product $\tilde{\otimes}$. To say something about where this came from: in a 3d TFT the category of lines should be a *braided* tensor category – or

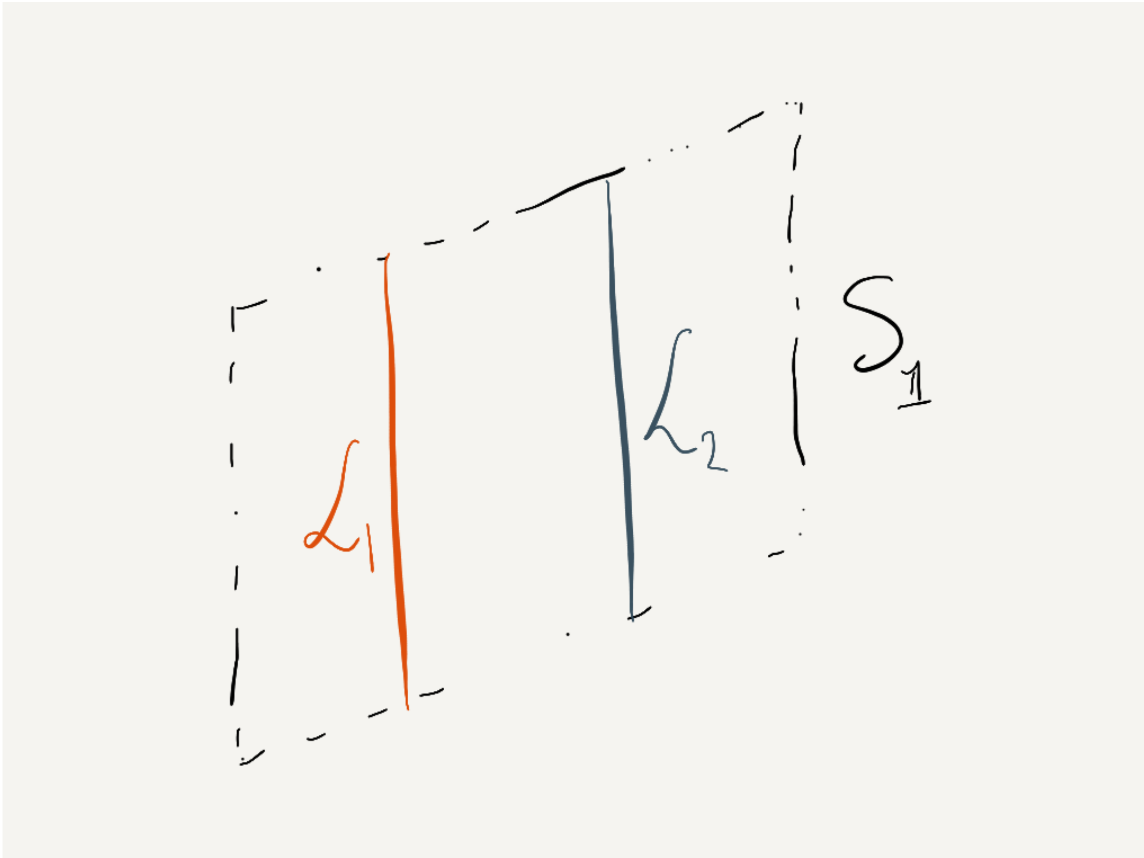


Figure 38: Lines as endomorphisms of trivial surface defect.

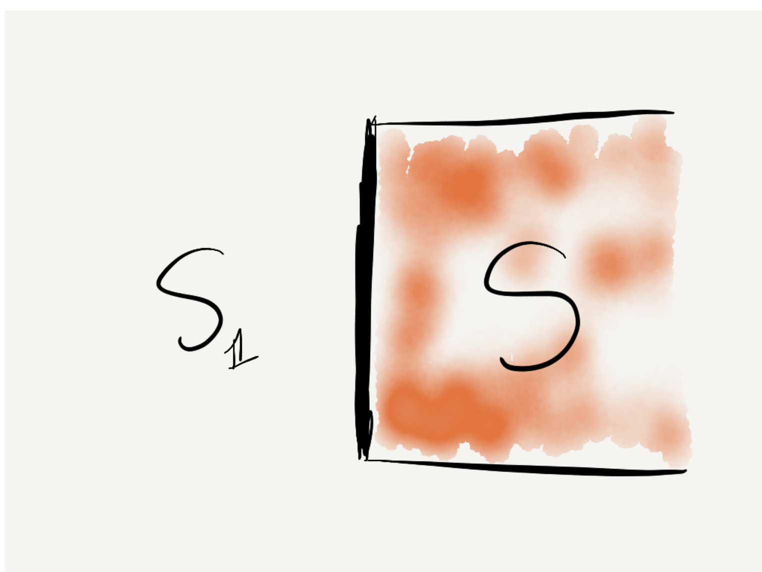


Figure 39: Actions of lines on surface defect with boundary.

perhaps better, Lines in 3d is an E_2-monoidal 1-category. Then the twisted tensor product $\tilde{\otimes}$ is induced from this braided stucture, but it's actually *computed* with the Atiyah bracket. This results in interesting corrections to the usual tensor product.

In 3d (e.g.) boundary conditions are two dimensional "things" – but they are *not* in Surf. Why? (Figure 40)

- A surface operator S is a thing that connects a theory T to itself.
- A boundary condition B is a thing that connects a theory T to the trivial theory $\emptyset$.

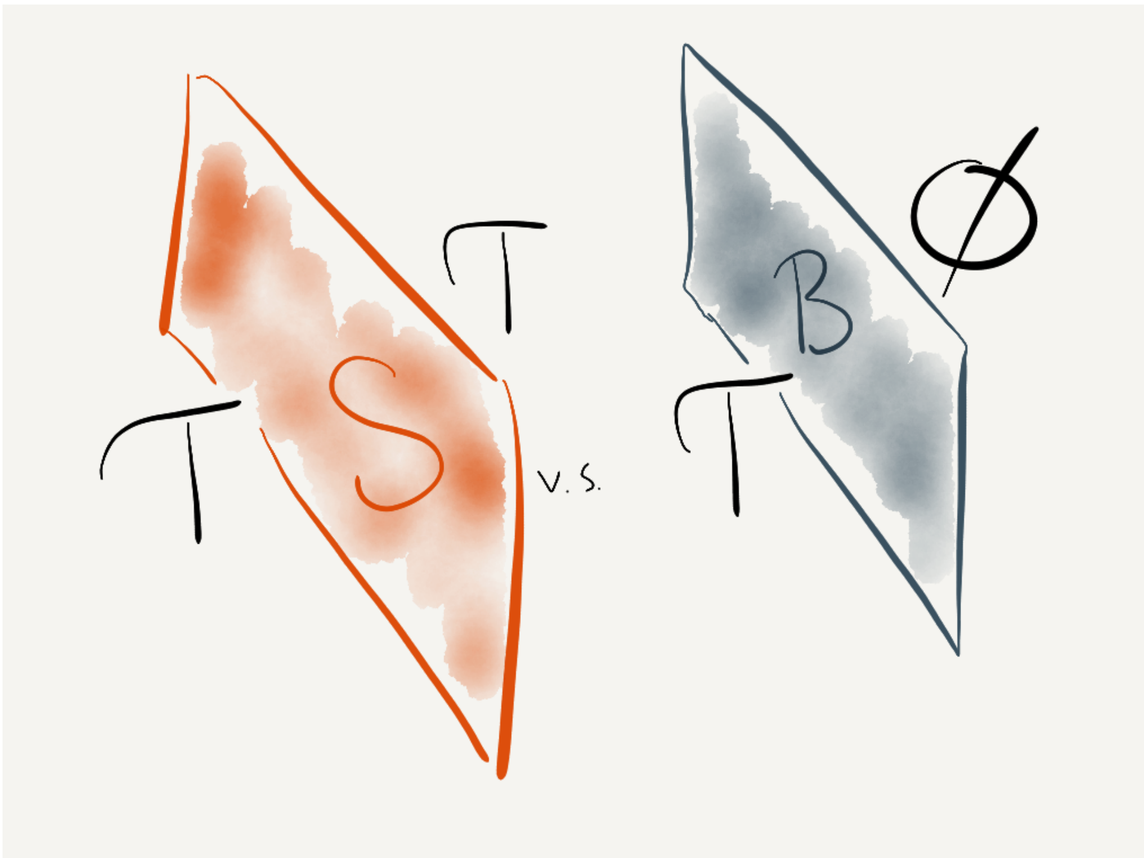


Figure 40: Surface operator versus boundary condition.

Roughly: Bdy $= \sqrt{\text{Surf}}$. What does this mean?

Basic boundary conditions are labelled by *Lagrangians* in $\mathcal{M}_{vac}$.

Example 3.26 (2d B-model)**.** $\mathcal{M}_{vac} \simeq T^*[1]\mathcal{X}$, then e.g. for $\mathcal{Y} \subset \mathcal{X}$, $N^*[1]\mathcal{Y}$ corresponds to $\mathcal{O}_\mathcal{Y}$ in $\mathrm{Coh}(\mathcal{X})$. The passage $N^*[1]\mathcal{Y}$ to $\mathcal{O}_\mathcal{Y}$ is like taking a square-root.

Example 3.27 (3d B-model)**.** In the 3d B-model to $T^*[2]\mathcal{X}$, *one* boundary condition is $B_\mathcal{X}$ supported on $\mathcal{X}$ (thought of as Lagrangian in $T^*[2]\mathcal{X}$). Then

$$\mathrm{End}(B_\mathcal{X}) = \mathrm{Coh}(\mathcal{X}).$$

We can play the same "folding and compactifying" game Justin played yesterday with this canonical BC (Figures 41 and 42) – the result will be the 2d B-model with target $\mathcal{X}$ (instead of $T^*\mathcal{X}$ – Figure 43). Now

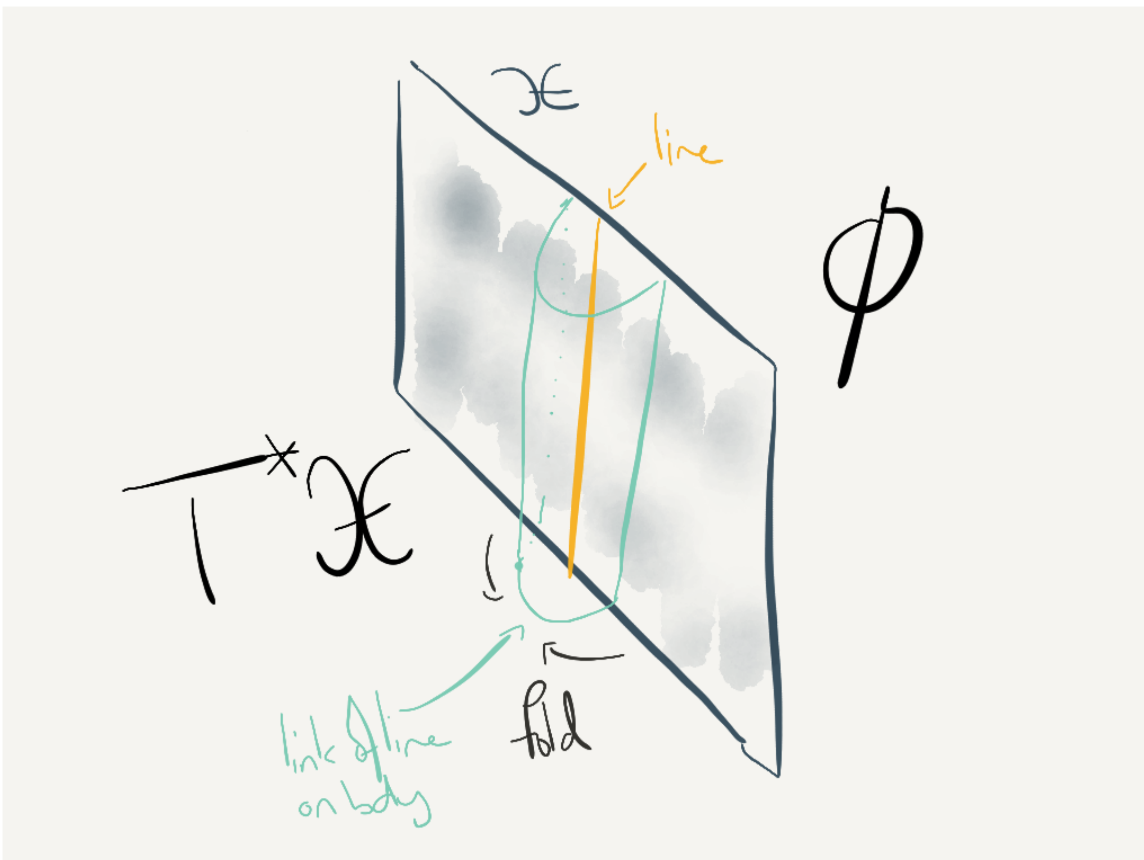


Figure 41: Compactification on the link of a boundary line ("folding").

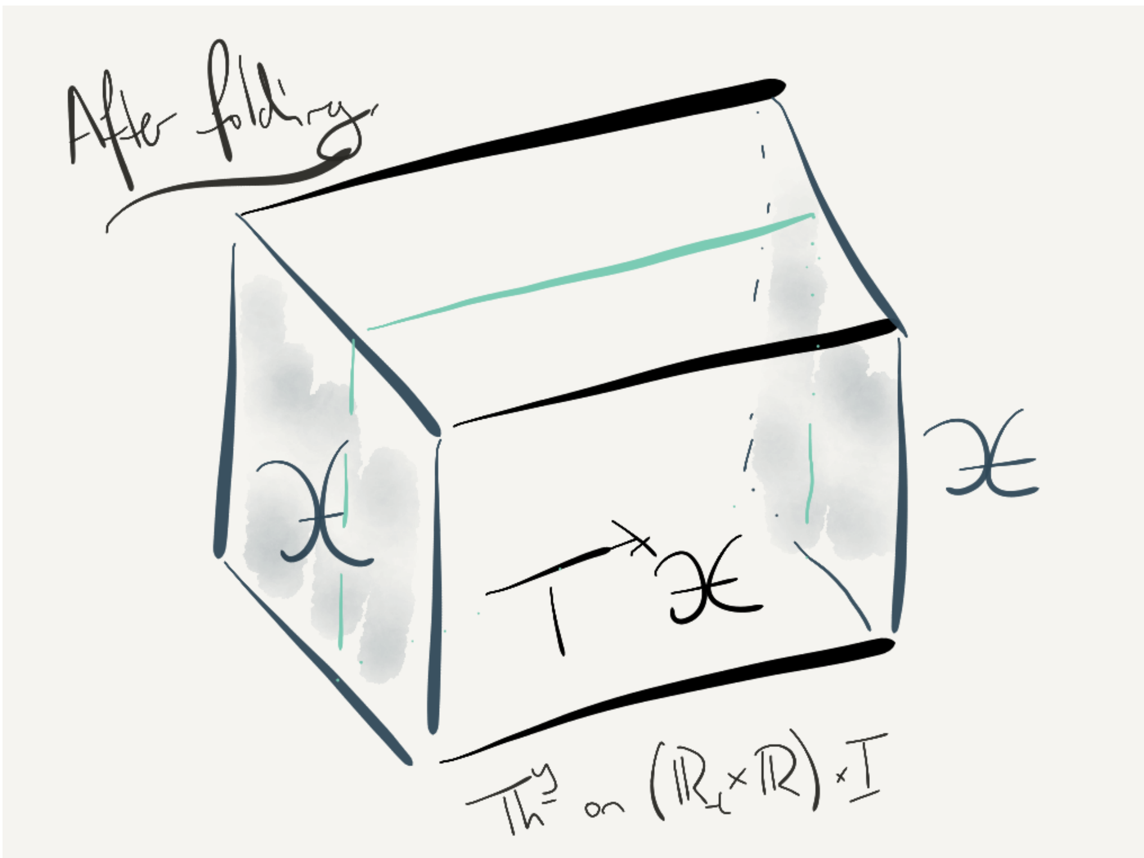


Figure 42: After folding this becomes an interval compactification.

we can find a map (Figure 44)

$$\text{Bdy} \to \text{Coh}(\mathcal{X}) - \text{Mod} = \text{End}(B_{\mathcal{X}}) - \text{Mod}.$$

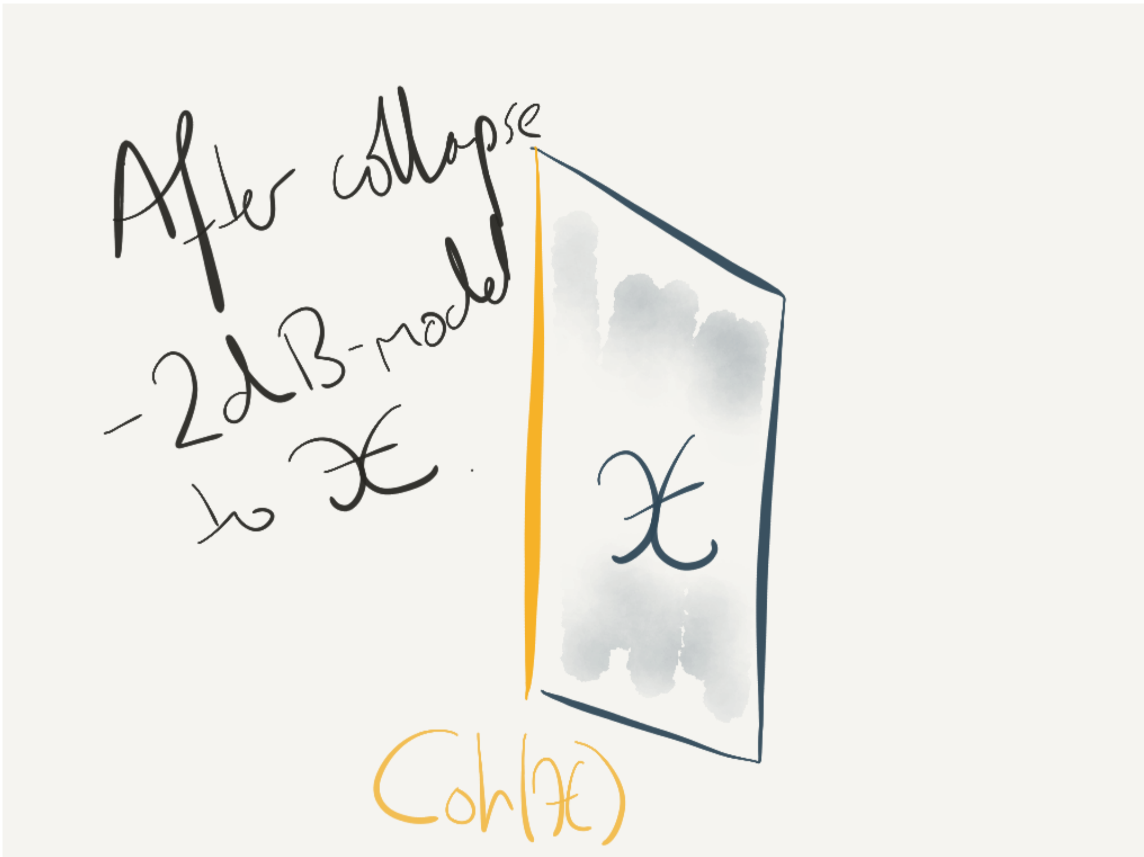


Figure 43: After the interval compactification we find the 2d B-model.

3.7.2 Random remarks

A-twisted analogs: similar, but in computations must always keep loops/paths (i.e. don't localise on constant maps – c.f. B-twisted theory).

Example 3.28 (2d A-model to $\mathcal{X}$)**.** See Figure 45 – after folding cannot collapse the length scale to zero. I.e.

$$\partial_{\bar{z}}(x + iy) = 0$$

where $z = t + is$. So, find A-twisted SQM with target

$$\text{Maps}([0,1], \mathcal{X})$$

and

$$h = \int_0^1 x \partial_s y \, ds.$$

Then

$$\text{Hilb} = H^{\bullet}_{Q}(\text{MW complex})$$

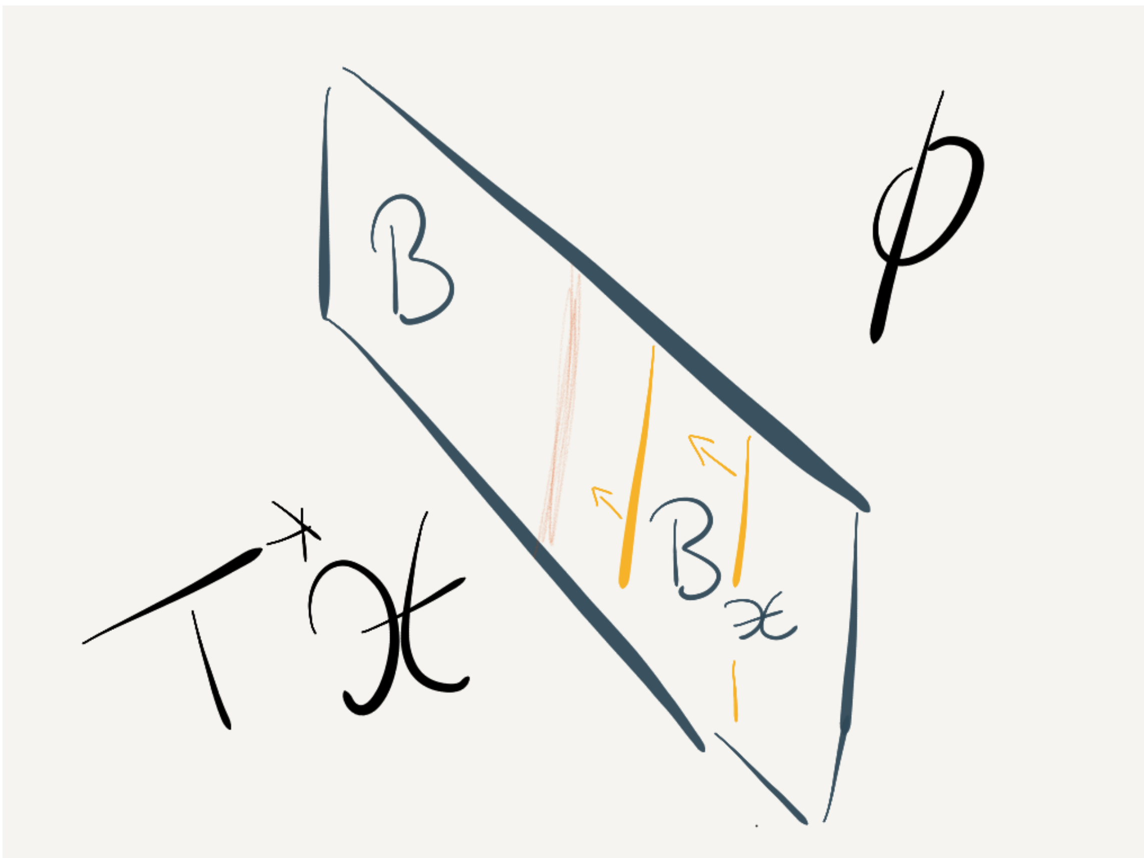


Figure 44: Map from boundary conditions to coherent sheaves in B-model.

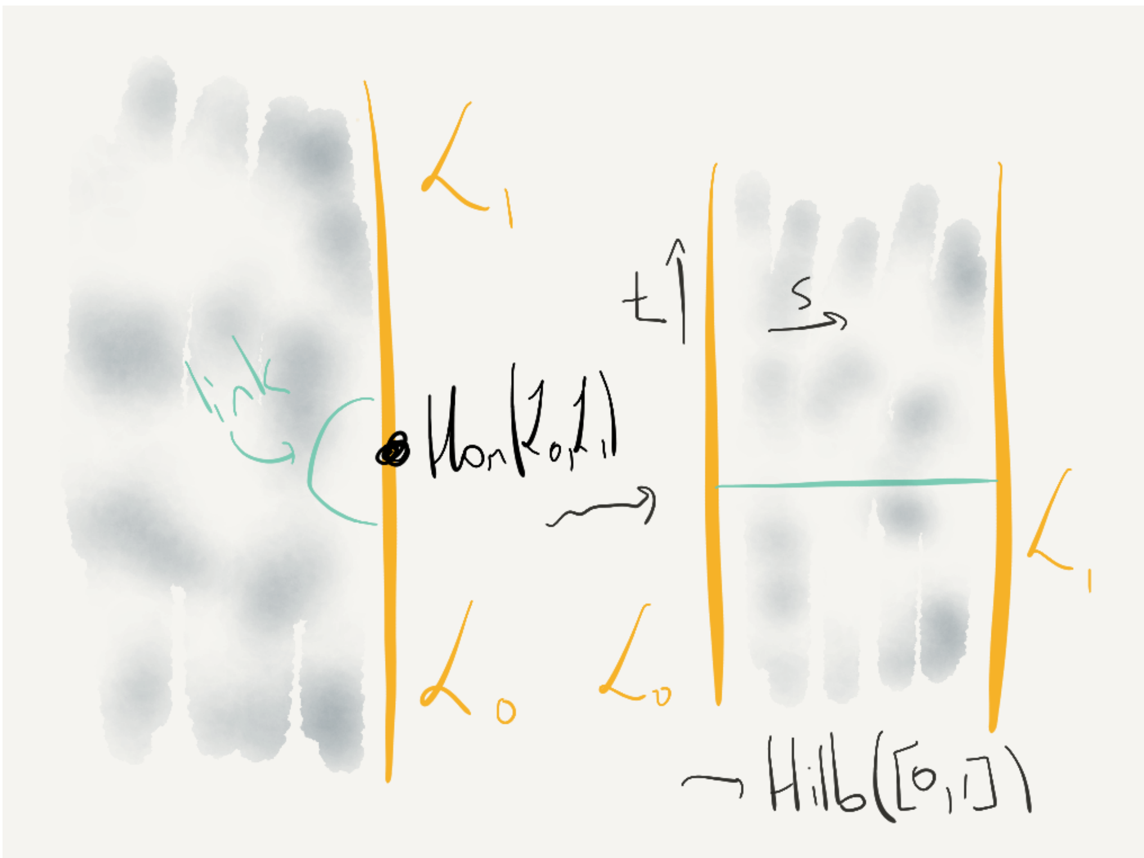


Figure 45: After “folding” in the A-model, cannot collapse the length scale of the interval to zero.

which has generators

$$\{dh = 0\} = \mathcal{L}_0 \cap \mathcal{L}_1$$

and differential arising from *instantons/gradient flows*,

$$\partial_t x = \frac{\partial h}{\partial x} = \partial_s y$$
$$\partial_t y = -\partial_s x$$

The instantons can be pictured as in Figure 46. So the differential on the MW complex comes from counting

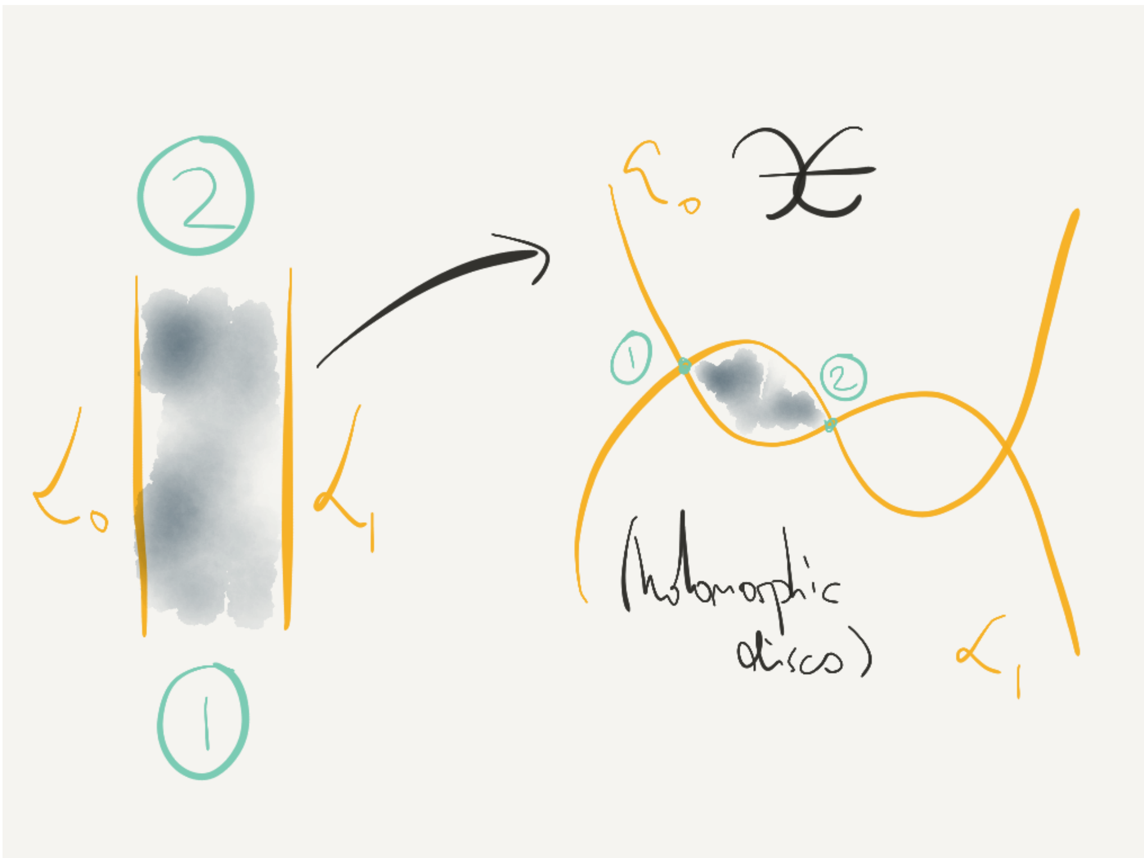


Figure 46: Instantons in the A-model come from holomorphic discs.

holomorphic discs. This is hard.

Also: local operators come from

$$\mathrm{Hilb}(S^1) = \mathrm{Hilb}\ \text{ of SQM with target } L\mathcal{X} \text{ and } h = \oint_{S^1} ds\, x\partial_s y.$$

Example 3.29 (3d A-model to $T^*[2]\mathcal{X}$)**.** [57] What are the line operators? Should find

$$\begin{pmatrix} \text{2d A-model} \\ \text{target } = L(T^*[2]\mathcal{X}) \\ W = \oint dsX\partial_s Y \end{pmatrix} \overset{?}{\simeq} \mathcal{D} - \mathrm{Mod}(L\mathcal{X}).$$

This would agree with calculations of Justin and Phil.

Remark 3.35. Hoping to organise a part II of this workshop in June 2020, focusing on 3d $\mathcal{N} = 4$ theories, with connections to 4d and applications.

[57] This is *way* more interesting for gauge theories.

4 Supersymmetric Field Theory and Topological Twists

4.1 Lecture 1 (Si Li)

4.1.1 Topological QFTs

There are different types of topological QFT:

- **Schwarz type.**
  - No manifest dependence on the metric.
  - E.g. Chern-Simons theory.
- **Witten type.**
  - Variations of the metric are Q-exact..
  - Often arise as topological twists of supersymmetric theories.
  - E.g. twists of 4d $\mathcal{N} = 4$ supersymmetric Yang-Mills.

We'll be focusing on the "Witten type" theories. In these theories we have

- Operator Q, $Q^2 = 0$
- Observables/states arise as Q-cohomology
- If the original theory is supersymmetric, can find an operator G_μ such that

  $$[Q, G_\mu] = \partial_\mu.$$

  I.e. small translations are Q-exact.
- Let $G = dx^\mu G_\mu$; then $[Q, G] = d$ (de Rham differential).
- Implies that if you conjugate by the symmetry generated by G you obtain

  $$e^G Q e^{-G} = Q + d; \tag{4.1}$$

  this relation will be very important.
- For $\mathcal{O}(x)$ a local operator such that

  $$Q\mathcal{O} = 0,$$

  we have

  $$e^G \mathcal{O} = \mathcal{O} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} + \cdots + \mathcal{O}^{(k)} + \cdots$$

  where $\mathcal{O}^{(k)}$ is a k-form. Then

  $$(Q + d)e^G \mathcal{O} = 0 \quad \Rightarrow \quad Q \int_\gamma e^G \mathcal{O} = 0$$

  and so we can construct interesting topological descendant operators.

4.1.2 Super Lie algebra

Definition 4.1 (Super Lie algebra)**.** A *super Lie algebra* is a $\mathbb{Z}/2$-graded vector space

$$\mathfrak{g} = \underbrace{\mathfrak{g}_0}_{\text{(even)}} \oplus \underbrace{\mathfrak{g}_1}_{\text{(odd)}}$$

together with a super Lie bracket $[-,-]$ satisfying:

(1) $[-,-]$ is even:

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \qquad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \qquad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0.$$

(2) Graded skew-symmetry:

$$[x, y] = -(-1)^{|x||y|}[y, x], \qquad |x| = i \text{ if } x \in \mathfrak{g}_i.$$

(3) Graded Jacobi identity:

$$[[x, y], z] = [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]].$$

Example 4.1. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}/2$-graded vector space. Then we have a super Lie algebra

$$\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1,$$

where

$$\mathrm{End}(V)_0 = \mathrm{Hom}(\mathfrak{g}_0, \mathfrak{g}_0) \oplus \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_1), \qquad \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

$$\mathrm{End}(V)_1 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

In this talk we will always assume that the commutator of endomorphisms is graded:

$$[x, y] := x \circ y - (-1)^{|x||y|} y \circ x.$$

Definition 4.2 (Super Hilbert space)**.** A *super Hilbert space* H is a super $\mathbb{C}$-vector space

$$H = H_0 \oplus H_1$$

together with a Hermitian inner product such that H_0 and H_1 are orthogonal.

Let $\alpha : H \to H$ be a bounded linear operator, and denote by α^* the usual adjoint operator. We define the *super adjoint*

$$\alpha^\dagger : H \to H$$

by

$$\alpha^\dagger := \begin{cases} \alpha^*, & \text{if } \alpha \text{ is even,} \\ -\sqrt{-1}\alpha^*, & \text{if } \alpha \text{ is odd.} \end{cases}$$

α is called *super Hermitian* if

$$\alpha = \alpha^\dagger.$$

Properties:

- $(\alpha^\dagger)^\dagger = \alpha$

- $(\alpha\beta)^\dagger = (-1)^{|\alpha||\beta|}\beta^\dagger\alpha^\dagger$ (obeys Koszul sign rule – this comes from our choice of prefactor for the super adjoint of an odd linear operator)
- $[\alpha,\beta]^\dagger = (-1)^{|\alpha||\beta|}[\beta^\dagger,\alpha^\dagger] = -[\alpha^\dagger,\beta^\dagger]$

For an odd operator Q,

$$[Q,Q] = 2Q^2$$

may not be zero; there is a useful positivity condition:

$$\sqrt{-1}(QQ^\dagger + Q^\dagger Q) = (QQ^* + Q^*Q) \geq 0.$$

Define the (infinitesimal) unitary linear operators as

$$\mathfrak{u}(H) = \{\alpha \in \mathrm{End}(H) \,|\, \alpha^\dagger = -\alpha\};$$

the equations above tell us that this is a super Lie subalgebra of $\mathrm{End}(H)$.

Definition 4.3 (Unitary representation)**.** A *unitary representation* of a super Lie algebra $\mathfrak{g}$ on H is a super Lie algebra morphism

$$\mathfrak{g} \to \mathfrak{u}(H).$$

4.1.3 Super Poincaré algebra

Let V be a d-dimensional $\mathbb{R}$-vector space equipped with a quadratic form. In practice we will consider two situations:

- **Minkowski:** $\mathbb{R}^{d-1,1}$, $\mathrm{diag}(-1,+1,\cdots,+1)$.
- **Euclidean:** $\mathbb{R}^d$, $\mathrm{diag}(+1,\cdots,+1)$.

Given such a space we can consider the special orthogonal and spin groups $SO(V)$ and $Spin(V)$

Let S be a $\mathbb{R}$-representation of $Spin(V)$, together with a $Spin(V)$-equivariant symmetric pairing

$$\Gamma : S \otimes S \to V.$$

Then we can define a super Lie algebra from this data as follows.

Define a super Lie algebra

$$\underbrace{V}_{\text{(even)}} \oplus \underbrace{S}_{\text{(odd)}}$$

via the bracket

$$[v_1 \oplus s_1, v_2 \oplus s_2] = -2\Gamma(s_1,s_2).$$

I.e.

$$[V,V] = 0, \qquad [V,S] = 0, \qquad \Gamma : [S,S] \to V.$$

This generates a super Lie group

$$V \times \Pi S,$$

where Π is the parity changing operator (so ΠS is odd) with a $Spin(V)$ equivariant group law

$$(v_1,s_1)\cdot(v_2,s_2) = (v_1 + v_2 + \frac{1}{2}[s_1,s_2], s_1 + s_2).$$

Remark 4.1. This group law will be important – for instance it will generate interesting differential operators.

Definition 4.4 (Super Poincaré Group and Algebra)**.** The *super Poincaré group* is

$$Poin_S(V) = (V \times \Pi S) \rtimes Spin(V),$$

with corresponding *super Poincaré algebra*

$$\mathfrak{poin}_S(V) = (V \oplus \Pi S) \oplus \mathfrak{so}(V).$$

We will also simply call this the *SUSY algebra*.

Remark 4.2. The decomposition of the symmetric square of S looks something like

$$\mathrm{Sym}^2(S) = V \oplus \mathbb{R}^m \oplus \bigoplus_i \wedge^{p_i} V,$$

and including these terms in our algebra yields:

(1) V: this is the usual $\mathfrak{poin}_S(V)$

(2) $\mathbb{R}^m$: central charges

(3) $\wedge^{p_i} V$: central extension by forms (can be coupled to extend objects)

(4) Outer automorphisms: yields the "R-symmetry" group G_R

4.1.4 Superspace

Let's denote

$$V_S := V \times \Pi S,$$

with functions

$$\mathcal{O}(V_S) = C^\infty(V) \otimes \wedge^\bullet(S^\vee).$$

Choose linear coordinates x^μ, θ^α,

$$x^\mu x^\nu = x^\nu x^\mu, \qquad \theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha.$$

Then a function on superspace can be expanded as

$$f(x, \theta) = \sum_I f_I(x) \theta^I, \quad \theta^I = \theta^{i_1} \cdots \theta^{i_k}, \quad I = \{i_1 < \cdots < i_k\}.$$

To think about vector fields on superspace, consider the space of *super derivations*:

$$\mathrm{Der}(V_S) := \{D \in \mathrm{End}(V_S) \,|\, D(f \cdot g) = D(f) \cdot g + (-1)^{|D||f|} f \cdot Dg\}$$

which is generated by

$$\partial_{x^\mu} = \frac{\partial}{\partial x^\mu}, \quad \partial_{\theta^\alpha} = \frac{\partial}{\partial \theta^*}$$

where

$$\partial_{\theta^\alpha} \partial_{\theta^\beta} = -\partial_{\theta^\beta} \partial_{\theta^\alpha}.$$

The action of the super group V_S on itself yields

- right invariant vector fields, and
- left invariant vector fields.

Problem 20. *Right invariant vector fields are generated by*

$$\{\partial_{x^\mu}, D_\alpha = \partial_{\theta^\alpha} + \Gamma^\mu_{\alpha\beta}\theta^\beta\partial_{x^\mu}\}$$

and left invariant vector fields are generated by

$$\{\partial_{x^\mu}, Q_\alpha = \partial_{\theta^\alpha} - \Gamma^\mu_{\alpha\beta}\theta^\beta\partial_{x^\mu}\}$$

satisfying

$$[Q_\alpha, Q_\beta] = -2\Gamma_{\alpha\beta^\mu}\partial_{x^\mu},$$
$$[D_\alpha, D_\beta] = 2\Gamma_{\alpha\beta^\mu}\partial_{x^\mu}.$$

So $\mathcal{O}(V_S)$ forms a SUSY (translation) algebra representation via the operators $\{Q_\alpha, \partial_{x^\mu}\}$.

4.1.5 SUSY in different dimensions

SUSY in $\mathbb{R}^{d-1,1}$:

d	1	2	3	4	5	6	7	8	10
irred.$\mathbb{R}$-rep	$\mathbb{R}$	$\mathbb{R}_\pm$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_\pm$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}_\pm$
$\dim_{\mathbb{R}}$	1	1	2	4	8	8	16	16	16

- If $d \not\equiv 2, 6 \bmod 8$, $S = n\,\mathrm{Irrep}$, say $\mathcal{N} = n$ SUSY
- If $d \equiv 2, 6 \bmod 8$, $S = n_+\,\mathrm{Irrep}_+ \oplus n_-\,\mathrm{Irrep}_-$, say $\mathcal{N} = (n_+, n_-)$ SUSY

Example 4.2. In $d = 2$, $\mathbb{R}^{1,1}$:

$$SO(1,1) = \left\{ e^{\alpha J} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \middle| \, \alpha \in \mathbb{R},\ J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \simeq \mathbb{R}_{>0}$$

$$Spin(1,1) \simeq \mathbb{R}^\times = \mathbb{R} \setminus 0$$

Dual representations $\mathbb{R}_+$ and $\mathbb{R}_-$; $x \in \mathbb{R}^\times$ acts on $\mathbb{R}_\pm$ as $x^{\pm 1}$

$$\mathrm{Sym}^2(\mathbb{R}_\pm) \simeq V_\pm$$

where

$$V = \underbrace{V_+}_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \oplus \underbrace{V_-}_{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Let $Q_\pm$ be a basis of $\mathbb{R}_\pm$ and $\partial_\pm$ be a basis of $V_\pm$. Then the $d = 2$ $\mathcal{N} = (n_+, n_-)$ SUSY algebra is

$$[Q^a_+, Q^b_+] = -2\delta^{ab}\partial_+$$
$$[Q^{\bar a}_-, Q^{\bar b}_-] = -2\delta^{\bar a\bar b}\partial_- [Q^a_+, Q^{\bar b}_-] \qquad = 2Z^{a\bar b} \quad \text{(central charge)}$$

For unitarity,

$$(Q^a_+)^\dagger = Q^a_+, \qquad (Q^{\bar a}_-)^\dagger Q^{\bar a}_-.$$

Example 4.3. $d = 3$, $\mathbb{R}^{2,1}$. Then

$$Spin(2,1) = SL(2,\mathbb{R})$$

and the real irreducible representation is the fundamental representation.

Let

$$V = \mathbb{R}^{2,1} = \{\text{symmetric } 2\times 2 \text{ real matrices}\}.$$

Let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the isomorphism is given by $x = (x^0, x^1, x^2) \in V$ maps to

$$A(x) = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{pmatrix}$$

and

$$|x|^2 = -\det A(x) = -(x^0)^2 + (x^1)^2 + (x^2)^2.$$

Then $Spin(2,1) = SL(2\mathbb{R}) \ni M$ acts on V via

$$M : A(x) \mapsto (M^t)^{-1} A(x) M^{-1}.$$

There is a natural $Spin(2,1)$-equivariant map

$$\begin{aligned} \mathrm{Sym}^2(S) \otimes V &\longrightarrow \mathbb{R} \\ (S, x) &\longmapsto S^t A(x) S \end{aligned}$$

and so we obtain a map

$$\Gamma : \mathrm{Sym}^2(S) \to V^\vee \simeq V.$$

So the $\mathcal{N} = 1$, $d = 3$ SUSY is given by

$$[Q_\alpha, Q_\beta] = -2\sigma^\mu_{\alpha\beta}\partial_\mu$$

where $\sigma^\mu = \eta^{\mu\nu}\sigma_\nu$.

There is a reality condition:

$$Q_\alpha^\dagger = Q_\alpha$$

Explicitly (matrix components of σ's):

$$\begin{aligned} [Q_1, Q_1] &= 2\partial_0 - 2\partial_1 \\ [Q_1, Q_2] &= 2\partial_2 \\ [Q_2, Q_2] &= 2\partial_0 + 2\partial_1 \end{aligned}$$

So: if ∂_2 is represented by a constant, then we may do dimensional reduction in this direction to obtain $d = 2$ $\mathcal{N} = (1,1)$ SUSY.

Upshot:

$$d = 3,\, \mathcal{N} = 1 \xrightarrow[\text{reduction}]{\text{dim}} d = 2,\, \mathcal{N} = (1,1)$$

Example 4.4. $d = 4$, $\mathbb{R}^{3,1}$. Write

$$V = \{\text{hermitian } 2 \times 2 \text{ matrices}\}.$$

Let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and make the identification by

$$x = (x^0, x^1, x^2, x^3) \leftrightarrow A(x) = x^\mu \sigma_\mu,$$

with

$$|x|^2 = -\det A(x).$$

Then $Spin(3,1) \simeq SL(2\mathbb{C}) \ni M$ acts on V via

$$M : A(x) \mapsto (M^t)^{-1} A(x) M^{-1}.$$

$\mathcal{N} = 1$ SUSY in $d = 4$:

$$[Q_\alpha, \widehat{Q}_{\bar{\beta}}] = -2\sigma^\mu_{\alpha\beta}\partial_\mu,$$

i.e. in matrix components

$$\begin{aligned} &[Q_1, \widehat{Q}_1] = -2(\partial_0 + \partial_3), \\ &[Q_2, \widehat{Q}_2] = -2(\partial_0 - \partial_3), [Q_1, \widehat{Q}_2] &= -2(\partial_1 - i\partial_2), [Q_2, \widehat{Q}_1] = -2(\partial_1 + i\partial_2), \end{aligned}$$

Then dimensional reduction along ∂_3,

$$\{\mathrm{Re}(Q_1), \mathrm{Re}(Q_2)\} \quad \text{and} \quad \{\mathrm{Im}(Q_1), \mathrm{Im}(Q_2)\}$$

gives two sets of $d = 3$, $\mathcal{N} = 2$ SUSY.

Upshot:

$$d = 4, \mathcal{N} = 1 \xrightarrow[\text{reduction}]{\text{dim}} d = 3, \mathcal{N} = 2$$

4.2 Lecture 2 (Si Li)

Today we'll discuss $d = 2$, $\mathcal{N} = (2,2)$ SUSY σ-model

- topological twists (A/B model)
- BV formalism

4.2.1 Superspace setup

The $d = 2$ $\mathcal{N} = (2,2)$ superspace is

$$\begin{aligned} V^{\mathcal{N}=(2,2)} &= \mathbb{R}^{1,1} \times \Pi\mathbb{R}_+^{\oplus 2} \times \Pi\mathbb{R}_-^{\oplus 2} \\ &= \mathbb{R}^{1,1} \times \Pi S_+ \times \Pi S_- \end{aligned}$$

where $S_\pm$ are complex spinors. We write the supercharges in complexified notation as

$$\begin{aligned} Q + \pm &= Q^1_\pm + iQ^2_\pm, \\ \bar{Q}_\pm &= Q^1_\pm - iQ^2_\pm \end{aligned}$$

The SUSY algebra is

$$[Q_\pm, \bar{Q}_\pm] = -2i\partial_\pm = H \pm P$$

Write

$$x^{\pm} = x^0 \pm x^1 \qquad\qquad \partial_{\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \mp \frac{\partial}{\partial x^1}\right)$$

Coordinates on $V^{\mathcal{N}=(2,2)}$: even x^0, x^1, odd $\theta^{\pm}, \bar{\theta}^{\pm}$.

SUSY differential operators:

$$\begin{aligned} Q_{\pm} &= \partial_{\theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm} & D_{\pm} &= \partial_{\theta^{\pm}} + i\bar{\theta}^{\pm}\partial_{\pm} \\ \bar{Q}_{\pm} &= \partial_{\bar{\theta}^{\pm}} - i\theta^{\pm}\partial_{\pm} & \bar{D}_{\pm} &= \partial_{\bar{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm} \\ [Q, D] &= 0 \end{aligned}$$

A superfield Φ is of the form

$$\Phi(x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm}) = \phi(x) + \theta^{\pm}\eta_{\pm}(x) + \bar{\theta}^{\pm}\bar{\eta}_{\pm}(x) + \cdots$$

where the $\eta, \bar{\eta}$ are fermionic. This describes a map

$$\Phi : V^{\mathcal{N}=(2,2)} \to \mathbb{C}$$

Definition 4.5. A superfield Φ is called *chiral* if $\bar{D}_{\pm}\Phi = 0$.

Observe: $\bar{D}_{\pm}(\theta^{\pm}) = 0$ and $\bar{D}_{\pm}(y^{\pm}) = 0$ where

$$y^{\pm} = x^{\pm} + i\bar{\theta}^{\pm}\theta^{\pm},$$

so we can write

$$\Phi = \Phi(y^{\pm}, \theta^{\pm}) = \phi(y^{\pm}) + \theta^{\pm}\psi_{\pm}(y) + \theta^{+}\theta^{-}F(y).$$

Definition 4.6. The *chiral multiplet* is $\Phi = (\phi, \psi_{\pm}, F)$.

Remark 4.3. $\bar{\Phi}$ is *antichiral*: $D_{\pm}\bar{\Phi} = 0$.

Also observe: Q maps chiral fields to chiral fields. Consider a chiral superfield as a map

$$\Phi : V_{ch}^{\mathcal{N}=(2,2)} \to \mathbb{C}.$$

4.2.2 SUSY Action

Consider n chiral superfields

$$\Phi : V_{ch}^{\mathcal{N}=(2,2)} \to \mathbb{C}^n.$$

There are two types of SUSY invariant actions.

(1) **D-term:** Let $K(z^i, \bar{z}^i)$ be a smooth function on $\mathbb{C}^n$ (Kähler potential).

$$S_D := \int d^2x \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^i)$$

S_D is SUSY invariant. Recall $Q_{\pm} = \partial_{\theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm}$, then the red terms below are responsible for vanishing:

$$Q_{\pm}S_D := \int d^2x \int d^2\theta d^2\bar{\theta}(\partial_{\theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm})K(\Phi^i, \bar{\Phi}^i)$$

(2) **F-term:** Let $W(z^i)$ be a holomorphic function.

$$S_F = \int d^2x \int d^2\theta W(\Phi^i)\Big|_{\bar{\theta}=0} + c.c.$$

4.2.3 Gluing

Observe that for all holomorphic functions $h(z^i)$ the transformation

$$K \mapsto K + h(z^i) + \bar{h}(\bar{z}^i)$$

leaves the D-term invariant:

$$\int d^2x \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^i) = \int d^2x \int d^2\theta d^2\bar{\theta} \left(K(\Phi^i, \bar{\Phi}^i) + h(\Phi^i) + \bar{h}(\bar{\Phi}^i) \right)$$

Hence the D-term depends only on $g_{i\bar{j}} = \partial_i \bar{\partial}_j K$ (Kähler metric). So we can glue these to a chiral superfield valued in a Kähler manifold

$$\Phi : V_{ch}^{\mathcal{N}=(2,2)} \to X.$$

We can also glue on the worldsheet: $\Phi^i = (\phi^i, \psi^i_I, F^i)$ where

$$\phi^i : \Sigma \to X$$

and

$$\begin{cases} \psi^i_\pm \in \Gamma(\Sigma, S_\pm \otimes \phi^* T^{1,0}_X) \\ \bar{\psi}^{\bar{i}}_\pm \in \Gamma(\Sigma, S_\pm \otimes \phi^* T^{0,1}_X) \end{cases}$$

Hence we get a chiral superfield to a Kähler target

$$\Phi : V_{ch}^{\mathcal{N}=(2,2)}(\Sigma) \to X$$

$$S_D(\Phi) = \int d^2x \left[-\frac{1}{2} g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \bar{\phi}^{\bar{j}} + i g_{i\bar{j}} \bar{\psi}^{\bar{j}}_- D_+ \psi^i_- + i g_{i\bar{j}} \bar{\psi}^{\bar{j}}_+ D_- \psi^i_+ + R_{i\bar{j}k\bar{l}} \psi^i_+ \psi^k_- \bar{\psi}^{\bar{j}}_- \bar{\psi}^{\bar{k}}_+ + g_{i\bar{j}} (F^i - \Gamma^i_{jk} \psi^i_+ \psi^j_-)(\bar{F}^{\bar{j}} - \bar{\Gamma}^{\bar{j}}_{\bar{k}\bar{l}} \bar{\psi}^{\bar{k}}_- \bar{\psi}^{\bar{l}}_+) \right]$$

and we write

$$D_\mu \psi^i_\pm = \partial_\mu \psi^i_\pm + (\partial_\mu \phi^j) \Gamma^i_{jk} \psi^k_\pm.$$

4.2.4 Topological Twist

Given $\Phi : V_{ch}^{\mathcal{N}=(2,2)}(\Sigma) \to X$, want to understand how it transforms under SUSY transformations

$$\delta = \epsilon^\pm Q_\pm + \bar{\epsilon}^\pm \bar{Q}_\pm$$

where $\epsilon^\pm$, $\bar{\epsilon}^\pm$ are sections of the dual spin bundles to the Q or $\bar{Q}$ they contract with (i.e. δ is *not* a section of a spin bundle). Then

$$\delta S = \int_\Sigma (\nabla_\mu \epsilon^\pm) G^\mu_\pm + (\nabla_\mu \bar{\epsilon}^\pm) \bar{G}^\mu_\pm$$

for some expression G (in terms of Noether's theorem, will be a charge for the supersymmetry). Hence *global supersymmetry* implies that for some $\epsilon^\pm$ we have

$$\nabla \epsilon^\pm = 0,$$

a covariantly constant spinor.

Topological twist:

1. Consider a theory with symmetry
$$Spin(V) \times G_R$$
(G_R: R-symmetry). Choose a homomorphism
$$\rho : Spin(V) \to G_R.$$

2. Consider the new Poincaré symmetry

$$Spin(V) \xrightarrow{1\times\rho} Spin(V) \times G_R \longrightarrow GL(S).$$

3. Find $Q \in S$ such that Q lies in a trivial subrepresentation under the new $Spin(V)$ action, and $[Q,Q] = 0$. *Such a Q generates a global SUSY.*

Example 4.5 ($d = 2$, $\mathcal{N} = (2,2)$, Euclidean case)**.** So the spin group is $SO(2)$, and a spin structure is equivalent to choosing a square root of the canonical bundle:

$$S_+ = K^{1/2} \qquad\qquad S_- = \bar{K}^{1/2}$$

We have $[Q_\pm, \bar{Q}_\pm] = H \pm P$. The bosonic symmetry of our theory is

$$SO(2) \times \underbrace{U(1)_L \times U(1)_R}_{R\text{-symmetry}}$$

where the "left" and "right" R-symmetries are

$$\begin{aligned} &S_- \mapsto e^{-i\alpha/2} S_-, && \bar{S}_- \mapsto e^{i\alpha/2} \bar{S}_-, && (U(1)_L) \\ &S_+ \mapsto e^{-i\alpha/2} S_+, && \bar{S}_+ \mapsto e^{i\alpha/2} \bar{S}_+, && (U(1)_R) \end{aligned}$$

We can describe what representation of the symmetry group the various spinor bundles transform under"

	S_+	S_-	$\bar{S}_+$	$\bar{S}_-$
$SO(2) \times U(1)_L \times U(1)_R$	$(1/2, 0, -1/2)$	$(-1/2, -1/2, 0)$	$(1/2, 0, 1/2)$	$(-1/2, 1/2, 0)$

Let's consider two possible twistings.

(I) **A-twist:** Take

$$\rho_A : \begin{array}{ccc} SO(2) & \longrightarrow & U(1)_L \times U(1)_R \\ e^{i\alpha} & \longmapsto & e^{i\alpha} \times e^{i\alpha} \end{array}$$

The bundles now transform in the following representations:

A-twist	S_+	S_-	$\bar{S}_+$	$\bar{S}_-$
$SO(2) \times U(1)_L \times U(1)_R$	$(0, 0, -1/2)$	$(-1, -1/2, 0)$	$(1, 0, 1/2)$	$(0, 1/2, 0)$

The $SO(2)$-weight 0 tells us that the SUSY operators Q_+ and $\bar{Q}_-$ survive globally, and we can take

$$Q_A = Q_+ + \bar{Q}_-.$$

After the twist, the fermions in the chiral multiplet transform under:

$$\begin{cases} \psi^i_+ \in \Gamma(\Sigma, \phi^* T^{1,0}_X) \\ \bar{\psi}^{\bar{i}}_+ \in \Gamma(\Sigma, K \otimes \phi^* T^{0,1}_X) \end{cases}$$

$$\begin{cases} \psi^i_- \in \Gamma(\Sigma, \bar{K} \otimes \phi^* T^{1,0}_X) \\ \bar{\psi}^{\bar{i}}_- \in \Gamma(\Sigma, \phi^* T^{0,1}_X) \end{cases}$$

Observe: The Q_A acts like the de Rham differential.

There is a new action functional

$$S_A = Q_A \int d^2 z V + \int_\Sigma \phi^* W$$

where

$$V = g_{i\bar{j}}(\psi^{\bar{j}}_+ \bar{\partial}_z \phi^i + \psi^i_- \partial_z \bar{\phi}^{\bar{j}}).$$

(II) **B-twist:** Take

$$\rho_B : \begin{array}{ccc} SO(2) & \longrightarrow & U(1)_L \times U(1)_R \\ e^{i\alpha} & \longmapsto & e^{i\alpha} \times e^{-i\alpha} \end{array}$$

The bundles now transform in the following representations:

B-twist	S_+	S_-	$\bar{S}_+$	$\bar{S}_-$
$SO(2) \times U(1)_L \times U(1)_R$	$(1, 0, -1/2)$	$(-1, -1/2, 0)$	$(0, 0, 1/2)$	$(0, 1/2, 0)$

The $SO(2)$-weight 0 tells us that the SUSY operators $\bar{Q}_+$ and $\bar{Q}_-$ survive globally, and we can take

$$Q_B = \bar{Q}_+ + \bar{Q}_-.$$

An analogous discussion to the A-twist can proceed from here – we will now go into this.

4.2.5 BV formalism

Recall the *AKSZ construction*:[58]

Let (X, d_X), (Y, d_Y) be two dg spaces, with dg-algebra $(\mathcal{O}_X, d_X)$, $(\mathcal{O}_Y, d_Y)$. Assume that

- X is equipped with $\int : \mathcal{O}_X \to \mathbb{C}$ of degree $-k$.
- Y is equipped with a symplectic form of degree $k-1$.

Then $\mathrm{Maps}(X, Y)$ is a (-1)-shifted symplectic space, d_X and d_Y correspond to $\{S_X, -\}$ and $\{S_Y, -\}$, and

$$S = S_X + S_Y \text{ satisfies } \{S, S\} = 0 \qquad \text{(CME)}.$$

Example 4.6 (Poisson σ-model)**.** (1) $T[1]\Sigma = (\Sigma, \Omega^\bullet(\Sigma))$ (i.e. $= \Sigma_{\mathrm{dR}}$). The integration map is the standard integration map of degree -2,

$$\mathcal{O}(T[1]\Sigma) = \Omega^\bullet(\Sigma) \xmapsto{\int} \mathbb{C}$$

(here Σ is a surface).

(2) $T^*[1]X = (X, \mathrm{PV}(X))$, where $\mathrm{PV}(X) = \wedge^\bullet T_X$, equipped with the canonical symplectic form of degree 1, which in local coordinates (x^I, θ_I) looks like

$$\omega = dx^I \wedge d\theta_I.$$

Let $P = P^{IJ}(x)\partial_{x^I} \wedge \partial_{x^J} = P^{IJ}(x)\theta_I\theta_J$ be the Poisson tensor. It satisfies $\{P, P\} = 0$, and acts on $\mathcal{O}(T^*[1]X)$ via $\{P, -\}$.

Now, consider

$$\mathrm{Maps}(T[1]\Sigma, T^*[1]X).$$

Locally, for φ^I, η_I forms on Σ, the (-1)-symplectic pairing is given by

$$\langle \varphi^I, \eta_I \rangle = \int_\Sigma \varphi^I \wedge \eta_I.$$

The classical action is

$$S = \int_\Sigma d\varphi^I \wedge \eta_I + P^{IJ}(\varphi)\eta_I\eta_J$$

and satisfies $\{S, S\} = 0$.

[58] This was discussed in Dylan's talk (§5.6).

In the BV formalism:

- $\mathcal{E}$ fields, (-1)-symplectic
- S action solving $\Delta(e^{S/\hbar}) = 0$ (QME)
- Choose $\mathcal{L} \subset \mathcal{E}$ a super Lagrangian submanifold, then

$$\int_{\mathcal{L}} e^{S/\hbar}$$

  is independent of continuous deformations of $\mathcal{L}$. $\mathcal{L}$ is a gauge fixing condition.

Example 4.7 (A-model)**.** Let (X, ω) be Kähler, and consider

$$\mathcal{E} = \mathrm{Maps}(T[1]\Sigma, T^*[1]X).$$

ω yields a Poisson tensor ω^{IJ}, and we have an action functional[59]

$$S_A = \int \omega^{IJ}(\varphi)\eta_I\eta_J = \int^{i\bar{j}} (\varphi)\eta_i\bar{\eta}_{\bar{j}}.$$

Let's see how a complex structure on Σ yields a Lagrangian submanifold of $\mathcal{E}$:

- Fields φ^i, η_i and their complex conjugates are forms on Σ
- Represent the four components of the field as

$$\varphi^i = (\varphi^i_0, \varphi^i_z, \varphi^i_{\bar{z}}, \varphi^i_{z\bar{z}}).$$

- Under the symplectic pairing, the symplectic dual fields are (presented vertically):

$$\begin{array}{c|c|c|c} \color{orange}{\varphi^i_0} & \color{orange}{\varphi^i_z} & \varphi^i_{\bar{z}} & \varphi^i_{z\bar{z}} \\ \eta_{i,z\bar{z}} & \eta_{i,z} & \color{orange}{\eta_{i,\bar{z}}} & \color{orange}{\eta_{i,0}} \end{array}$$

  (There are also complex conjugate fields.)

- Set $\mathcal{M}$ to be the space spanned by the orange fields and their complex conjugates. Then

$$\mathrm{Map}(T[1]\Sigma, T^*[1]X) = T^*[-1]\mathcal{M}.$$

- Consider

$$\Psi = \int_\Sigma d^2z \left(\bar{\varphi}^{\bar{j}}_z \bar{\partial}_z \phi^i_0 + \varphi^i_z \partial_z \phi^{\bar{j}}_0 \right) g_{i\bar{j}} \in \mathcal{O}(\mathcal{M}).$$

- Consider $\mathcal{L}_A = \mathrm{Graph}(d\Psi) \subset T^*[-1]\mathcal{M}$.

Exercise: Check that $S_A|_{\mathcal{L}_A}$ is the A-model action.

Example 4.8 (B-model)**.** Let (X, ω) be Kähler, and consider

$$\mathrm{Maps}\left(\underbrace{T[1]\Sigma}_{d}, \underbrace{T^*[1]T^{0,1}[1]X}_{\bar{\partial}} \right)$$

[59]Comparing with above – we only have the second term of the Poisson σ-model action, not the first term.

Locally we have fields and antifields (antifields vertically below fields):

$$\left\{ \begin{array}{ccc} \phi^i & \bar{\phi}^{\bar{i}} & \bar{\eta}^{\bar{i}} \\ \xi_i & \bar{\xi}_{\bar{i}} & \bar{u}_{\bar{i}} \end{array} \right.$$

We have an action functional

$$S_B = \int_\Sigma d\phi^i \wedge \xi_i + d\bar{\phi}^{\bar{i}} \wedge \xi_{\bar{i}} + du_i \wedge \eta_{\bar{i}} + \xi_{\bar{i}} \wedge \eta^{\bar{i}}$$

To figure out a polarisation, write down fields and their symplectic duals as before:

$$\begin{array}{c|c|c|c|c|c|c|c|c} \phi_0^i & \phi_1^i & \phi_2^i & \bar{\phi}_0^{\bar{i}} & \bar{\phi}_1^{\bar{i}} & \bar{\phi}_2^{\bar{i}} & \bar{\eta}_0^{\bar{i}} & \bar{\eta}_1^{\bar{i}} & \bar{\eta}_2^{\bar{i}} \\ \xi_{i,2} & \xi_{i,1} & \xi_{i,0} & \bar{\xi}_{\bar{i},2} & \bar{\xi}_{\bar{i},1} & \bar{\xi}_{\bar{i},0} & \bar{u}_{\bar{i},2} & \bar{u}_{\bar{i},1} & \bar{u}_{\bar{i},0} \end{array}$$

Again take $\mathcal{M}$ to be determined by the orange fields, so that

$$\mathrm{Maps}\left(T[1]\Sigma, T^*[1]T^{0,1}[1]X\right) = T^*[-1]\mathcal{M}.$$

Consider

$$\Psi_B = \int g_{i\bar{j}} \phi_1^i \star d\bar{\phi}_0^{\bar{j}} - \frac{1}{2}\Gamma^i_{jk}\phi_1^j\phi_1^k\xi_{i,0} \in \mathcal{O}(\mathcal{M})$$

and let $\mathcal{L}_B = \mathrm{Graph}(d\Phi_B)$.

Exercise: Check that $S_B|_{\mathcal{L}_B}$ is the B-model action.

4.3 Lecture 3 (Si Li)

4.3.1 SUSY Localisation

Example 4.9 ($\mathcal{N} = 1$, $D = 10$ SYM)**.** Set up:

- $V = \mathbb{R}^{9,1}$
- S_+ chiral spinor ($\mathbb{R}^{16}$)
- Fields:
$$\mathcal{E} = \Omega^1(V, \mathfrak{g})_A \oplus \Omega^0(V, S_+ \otimes \mathfrak{g})_\psi$$
- Action:
$$\mathrm{SYM}^{\mathcal{N}=1} = \int \frac{1}{4}\langle F_A, F_A\rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi\rangle$$

Berkovits Construction:

$$\Gamma : \mathrm{Sym}^2(S_+) \to V$$

Let $Q_\Gamma = \Gamma^{-1}(0) \subset S_+$. Consider

$$V \times Q_\Gamma \times \Pi S_+.$$

Let

$$B = O(V \times Q_\Gamma \times \Pi S_+)$$

and choose coordinates

$$\left\{ \begin{array}{ccc} x^\mu, & \lambda^\alpha, & \theta^\alpha \\ \text{on } V & \text{on } S_+ & \text{on } \Pi S_+ \end{array} \right\}$$

$$B = \mathbb{R}\{x^\mu, \lambda^\alpha, \theta^\alpha\}/(\Gamma^\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta.)$$

Define

$$Q = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + \Gamma^\mu_{\alpha\beta}\theta^\alpha\lambda^\beta \frac{\partial}{\partial x^\mu},$$

which satisfies

$$Q^2 = \left(\Gamma^\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta\right)\frac{\partial}{\partial x^\mu} = 0 \text{ on } B.$$

So (B, Q) is a dga, and it is a result of Berkovits that

$$\mathrm{Crit}(\mathrm{SYM}^{\mathcal{N}=1})/\sim = \mathrm{MC}(B, Q)/\sim .$$

Example 4.10 ($\mathcal{N} = 4$, $D = 4$ SYM)**.** Easiest to obtain by dimensional reduction from $\mathcal{N} = 1$ $D = 10$. Split

$$\mathbb{R}^{9,1} = \mathbb{R}^{3,1} \times \mathbb{R}^6$$

and declare A, ψ to only vary along $\mathbb{R}^{3,1}$ (constant along $\mathbb{R}^6$). In 4d the spin group reduces to

$$\begin{array}{ccccc} Spin(3,1) & \times & Spin(6) & \hookrightarrow & Spin(9,1) \\ \downarrow\simeq & & \downarrow\simeq & & \\ SL(2,\mathbb{C}) & & SU(4)_R & & \end{array}$$

The chiral spinor decomposes as

$$S_+ \otimes_{\mathbb{R}} \mathbb{C} = (S \otimes \bar{\mathbf{4}}) \oplus (\bar{S} \otimes \mathbf{4})$$

where $S = \mathbb{C}^2$ and $\mathbf{4} = \mathbb{C}^4$ for $SU(4)$.

Topological twist: Euclidean.

$$\begin{array}{ccc} Spin(4) & \times & Spin(6) \\ \| & & \| \\ SU(2)_l \times SU(2)_r & & SU(4)_R \end{array}$$

Look for $\rho : SU(2)_l \times SU(2)_r \to SU(4)_R$ – describe the homomorphism by what it does to the representation $\mathbf{4}$:

(1) $\mathbf{4} \mapsto (2,1) \oplus (2,1)$ ("Vafa-Witten twist")

(2) $\mathbf{4} \mapsto (2,1) \oplus (1,1) \oplus (1,1)$

(3) $\mathbf{4} \mapsto (2,1) \oplus (1,2)$ ("GL-twist")

Now – given a SUSY twist determined by Q, consider the equation $\int Q(-) = 0$.

Example 4.11 (Proto-example: Equivariant localisation)**.** Suppose S^1 acts on a spaec X with generating vector field V. Consider $\Omega^\bullet(X)[u]$ equipped with

$$Q = d + u\iota_V$$
$$Q^2 = u\mathcal{L}_V$$

Define $\Omega^\bullet_{S^1}(X) := (\Omega^\bullet(X)[u])^{S^1}$, on which Q is a differential, and consider $(\Omega^\bullet_{S^1}(X), Q)$. There is an integration map

$$\int_X : \Omega^\bullet_{S^1}(X) \to \mathbb{R}[u]$$

and under this map Q-exact terms vanish

$$\int_X Q(-) = 0.$$

Let $Q\alpha = 0$ and consider $\int_X \alpha$. Let g be an S^1-invariant metric on X and consider the 1-form

$$\Psi = \frac{1}{u} g(V, -).$$

Then

$$\int_X \alpha = \int_X \alpha e^{-\frac{1}{\hbar}Q(\Psi)}.$$

Now,

$$Q\Psi = g(V,V) + \underbrace{d\Psi}_{\text{2-form}}$$

so

$$\int_X \alpha = \int_X \alpha e^{-\frac{1}{\hbar}\|V\|^2 + \text{ 2-form}}.$$

But the integral doesn't depend on the value of $\hbar$ – so we can send $\hbar \to 0$ and localise the integral to the zeros of V (i.e. the S^1-fixed points).

Example 4.12 (A-model)**.** Recall that the action for the A-model is of the form

$$S_A = Q_A \underbrace{\int (\cdots)}_{\int |\bar{\partial}\phi|^2 + \text{ fermions}} + \text{ topological term}$$

where $\phi : \Sigma \to X$. So the theory localises to $\bar{\partial}\phi = 0$, i.e. holomorphic maps.

Example 4.13 (B-model)**.** Recall that

$$S_B = \int |d\phi|^2 + \cdots$$

so the theory localises to constant maps.

4.3.2 B-model

Let's focus on the B-model situation for a moment. The fields of the original theory were maps to a target X. Under localisation, the theory can be described by a local QFT on X.

Example 4.14.

- **Closed string:** Localisation gives Kodaira-Spencer theory. (Related to intersection theory.)
- **Open string:** Localisation gives holomorphic Chern-Simons theory:
  - Fields: $\Omega^{0,\bullet} * X, \mathfrak{g})[1]$
  - Action: $HCS[\mathcal{A}] = \int \left(\frac{1}{2}\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{1}{6}\mathcal{A}^3\right) \wedge \Omega_X$

  Computing correlation functions in Poisson σ-model gives 1st order deformations[60] of HCS – so Kontsevich formality controls deformatiosn of HCS.

  For instance, inserting a closed string field operator $mu \in \mathrm{PV}(X) = \Omega^{0,\bullet}(X, \wedge^\bullet T_X)$ deforms HCS by

$$\int_X (\mu \neg \mathcal{A} \wedge \partial A \wedge \cdots \wedge \partial A).$$

[60] I.e. a deformation of the HCS solution to the CME.

4.3.3 Kuranishi model

Begin with the data

$$\begin{tikzcd} & E \arrow[d] \\ \{s=0\} = \mathcal{M} \arrow[r, hook] & X \arrow[u, bend right, "s"'] \end{tikzcd}$$

Think: $\mathcal{M}$ is the classical moduli space, $s = 0$ are the EOM.

Given this data, can construct the *localised Euler class of* $E \subset \mathcal{M}$, $[\mathcal{M}]^{vir}$.

Kuranishi model: consider the two-term complex over $\mathcal{M}$

$$\begin{array}{c} \left[\; TX|_{\mathcal{M}} \xrightarrow{ds} E|_{\mathcal{M}} \;\right] \\ \downarrow \\ \mathcal{M} \end{array}$$

This has

- ker = tangent space of $\mathcal{M}$
- $(\text{coker})^\vee$ = obstruction space

The zero locus of s may not have expected dimension, and may be singualr. Consider taking $s \to \lambda s$. As $\lambda \to \infty$ we localise on the zeros and obtain $[\mathcal{M}^{vir}] \subset \mathcal{M}$.[61]

4.3.4 Mathai-Quillen formalism

Start with

$$\begin{tikzcd} E \arrow[d] \\ X \arrow[u, bend right, "s"'] \end{tikzcd}$$

and choose

- metric $\langle -, - \rangle$ on E
- connection ∇

Consider the supermanifold

$$\begin{array}{c} T[1](E)[-1]) = T[1]X \oplus E[-1] \oplus E \\ \downarrow \\ X \end{array}$$

Choose a local basis $\{e+_\alpha\}$ for E and local coordinates

	x^μ	ψ^μ	χ^α	B^α
deg	0	1	-1	0
	X	$T[1]X$	$E[1]$	E

[61] Need to look this up later. Have a headache now.

Let δ be the de Rham differential,

$$\delta x^\mu = \psi^\mu, \qquad \delta\chi^\alpha = B^\alpha,$$

and A_α^β be the connection 1-form

$$\nabla e_\alpha = A_\alpha^\beta e_\beta.$$

Construct a function S

$$S = \delta\Psi$$

where

$$\Psi = i\langle\chi, s\rangle + \frac{1}{2}\langle\chi, B\rangle + \frac{1}{2}\underbrace{\langle\chi, A\cdot\chi\rangle}_{A_{\alpha\beta,\mu}\psi^\mu\chi^\alpha\chi^\beta}$$

Note that $e^{-\delta\Psi}$ is δ-closed.

MQ Construction:

$$e_{s,\nabla}(\mathcal{E}) = \int d\chi dB e^{-\delta\chi}$$

is a closed differential form on X representing the Euler class, whose de Rham class is independent of the choice of s and ∇. More explicit expression:

$$e_{s,\nabla}(\mathcal{E}) = \frac{1}{(2\pi)^{\dim(X)}}\int d\chi e^{-\frac{1}{2}\langle s,s\rangle + \frac{1}{2}\langle\chi, R\chi\rangle + i\langle\chi,\nabla s\rangle}$$

where R is the curvature 2-form.

Since this is independent of the choice of section, we can consider rescaling

$$s \mapsto \lambda s$$

Now:

- $\lambda \to 0$: Obtain
$$\frac{1}{(2\pi)}\cdot\int d\chi e^{\frac{1}{2}\langle\chi,R\chi\rangle} = \mathrm{Pf}(R)$$
and we already knew that the Pfaffian $\mathrm{Pf}(R)$ represented the Euler class.
- $\lambda \to \infty$: Localise to $\{s = 0\}$.

Example 4.15 (A-model)**.** Begin by looking at all maps[62] $\varphi : \Sigma \to X$. Consider the bundle

$$\begin{tikzcd} E \arrow[d] \\ \mathrm{Maps}(\Sigma, X) \end{tikzcd}$$

whose fibre over a point φ is

$$E|_\varphi = \Gamma(\Sigma, \mathrm{Hom}(T_\Sigma^{0,1}, \varphi^* T_X^{1,0}))$$

This bundle has a canonical section $s = \bar\partial$. The zero section is (up to subtleties of compactifications) holomorphic maps:

$$\begin{tikzcd} & E \arrow[d] \\ \overline{\mathfrak{M}}_g(\Sigma, X) \arrow[r, hook] & \mathrm{Maps}(\Sigma, X) \arrow[u, bend right, "\bar\partial"'] \end{tikzcd}$$

Applying the MQ construction we obtain the A-model action together with the virtual fundamental class $[\overline{\mathfrak{M}}_g(\Sigma, X)]^{vir}$.

[62] All reasonable maps – e.g. could restrict to smooth maps.

Example 4.16 (SUSY QM)**.** Consider fields $\varphi : S^1 \to X$. The space of these maps is precisely the loop space, and we have a bundle

$$\begin{tikzcd} E \arrow[d] \\ LX \end{tikzcd}$$

whose fibre over a point φ is

$$E|_\varphi = \Gamma(S^1, \varphi^* T_X).$$

This describes infinitesimal motions of a loop in X – i.e. E is precisely the tangent bundle to the loop space, TLX.

Let t be the periodic coordinate on S^1 – then there is a section $s = \partial_t$ given by

$$s(\varphi) = \frac{d}{dt}\varphi,$$

and the zero section is constant maps – i.e. the zero section is precisely X itself. So we have the setup

$$\begin{tikzcd} & TLX \arrow[d] \\ X \arrow[r, hook] & LX \arrow[u, bend right, "\partial_t"'] \end{tikzcd}$$

Applying the MQ-construction, one finds SQM.

Consider fluctuations around the locus of constant maps $S^1 \to X$. If we work with the BV formalism and think of X as a phase space (in particular symplectic) (around the effective neighbourhood of constant maps), we find that the QME is equivalent to the data of a flat connection on the bundle of Weyl algebras $\mathcal{W}(TX) \to X$. Fedosov uses this to constuct the deformation quantisation of a symplectic connection on a symplectic manifold, together with some sort of algebraic index theorem [Fed94].

5 TA Sessions

5.1 Session 1 (Theo)

5.2 Session 2 (Chris): Supersymmetry Algebras

Motivation: When we study QFT on $(\mathbb{R}^n, g)$ with g a pseudo-Riemannian metric of signature (p, q), often we'd like invariance under symmetries of $\mathbb{R}^n$.

Definition 5.1. The *Poincaré group* is

$$ISO(p, q) = SO(p, q) \ltimes \mathbb{R}^n.$$

It has Lie algebra

$$\mathfrak{iso}(p, q) = \mathfrak{so}(p, q) \ltimes \mathbb{R}^n$$

with complexification

$$\mathfrak{iso}(n; \mathbb{C}) = \mathfrak{so}(n; \mathbb{C}) \ltimes \mathbb{C}^n.$$

We have the following no-go theorem:

Theorem 5.1 (Coleman-Mandula)**.** *If $G \supseteq ISO(n-1, 1)$, $n \geq 4$, acts on a "nice" QFT, then $G = ISO(n-1, 1) \times G'$. G' are the "internal symmetries" of the theory.*

One can get around this by studying $\mathbb{Z}/2$-graded extensions of the Poincaré group.

Remark 5.1. We're going to omit discussion of the different real structures on the complexified group/algebra.

Definition 5.2. An n-dimensional *super Poincaré algebra* is a super Lie algebra[63]

$$\mathfrak{A} = \mathfrak{iso}(n; \mathbb{C}) \ltimes \Pi\Sigma$$

with further bracket

$$\Gamma : \Sigma \otimes \Sigma \to \mathbb{C}^n,$$

and where Σ is a spinor representation of $\mathfrak{so}(n, \mathbb{C})$.

Classifying super Poincaré algebras means classifying two pieces of data:

1) Representation Σ

2) Pairing $\Gamma : \Sigma \otimes \Sigma \to \mathbb{C}^n$

We can classify spinor representations of $\mathfrak{so}(n; \mathbb{C})$. There is either

- one irreducible spinor representation S if n is odd, or
- two non-isomorphic spinor representations if n is even.

In terms of the Dynkin diagrams:

- For type B_n we are taking the fundamental representation corresponding to the rightmost node (the unique long simple root).

[63] Π denotes the parity shift functor.

- For type D_n we are taking the fundamental representations associated to the two "branched" simple roots in the diagram.

Remark 5.2. There is a SUSY version of Coleman-Mandula by Haag-Lopusanski-Sohnius for super Poincaré algebras – we won't state it, but it is the reason that we restrict to these fundamental representations.

So, the possibilities for Σ are:

- Odd case: $\Sigma = S \otimes W$, where S is the *Dirac spin representation* and W is a finite dimensional auxilliary vector space.
- Even case: $\Sigma = S_+ \otimes W_+ \oplus S_- \otimes W_-$ where $S_\pm$ are the *Weyl spinor representations* (with Dirac spinor $S = S_+ \oplus S_-$), and $W_\pm$ are again auxilliary spaces.

CLassification of pairings:

These are equivariant symmetric maps

$$\Sigma^{\otimes 2} \to V = \mathbb{C}^n;$$

in other words, we're looking for irreducible summands of $\mathrm{Sym}^2(\Sigma)$ which are isomorphic to V.

- Odd case: $S \otimes S \cong C^+(V)$, the even Clifford algebra. Using that $\wedge^k \cong \wedge^{n-k}$,

$$S \otimes S \cong \bigoplus_{k \text{ even}} \wedge^k V \cong \bigoplus_{k=0}^{\frac{n-1}{2}} \wedge^k V$$

  and in particular, one summand is isomorphic to $\wedge^1 V = V$.

- Even case: We have

$$(S_+ \oplus S_-)^{\otimes 2} \cong C(V) \cong \bigoplus_{k=0}^{n} \wedge^k V = 2\left(\bigoplus_{k=0}^{\frac{n}{2}-1} \wedge^k V\right) \oplus \wedge^{n/2} V,$$

  so there are two copies of V.

Lemma 5.2. *With the exception of $n = 2$:*

- *If n is odd, there exists a unique irreducible summand of $S^{\otimes 2}$ isomorphic to V.*
  - *$n \equiv 1, 3$ mod 8, contained in* $\mathrm{Sym}^2(S)$
  - *$n \equiv 5, 7$ mod 8, contained in* $\wedge^2 S$
- *If $n \equiv 0, 4$ mod 8, there exists a unique irreducible summand of $S_+ \otimes S_-$ isomorphic to V.*
- *If $n \equiv 2, 6$ mod 8, there exists a unique irrducible summand of $S_\pm^{\otimes 2}$ isomorphic to V. There is no such summand in $S_+ \otimes S_-$.*
  - *If $n \equiv 2$ mod 8, inside* $\mathrm{Sym}^2(S_\pm)$
  - *If $n \equiv 6$ mod 8, inside* $\wedge^2 S_\pm$

We can rephrase this in terms of the classification of Γ-pairings. A choice of super Poincaré algebra is a choice of:

- An orthogonal vector space W if $n \equiv 1, 3$ mod 8.

- A pair of orthogonal vector spaces W_+ and W_- if $n \equiv 2 \bmod 8$.
- A single vector space $W = W_+$ with dual $W^* = W_-$ if $n \equiv 0, 4 \bmod 8$.
- A single symplectic vector space W if $n \equiv 5, 7 \bmod 8$.
- A pair of symplectic vector spaces W_+ and W_- if $n \equiv 6 \bmod 8$.

The above is precisely the data required to give a *symmetric* pairing on the representation Σ.

Terminology.

Usually indicate a choice of super Poincaré algebra by giving

$$\dim W = \mathcal{N} \qquad \text{or} \qquad (\dim W_+, \dim W_-) = (\mathcal{N}_+, \mathcal{N}_-)$$

Example 5.1. 3d $\mathcal{N} = 4$ means

$$(\mathfrak{so}(3;\mathbb{C}) \ltimes (\mathbb{C}^3 \oplus \Pi(S \otimes W))$$

with W of dimension 4.[64]

Remark 5.3 (Exception!). If $n \equiv 5, 6, 7 \bmod 8$ (symplectic cases),

$$\mathcal{N} = \frac{\dim W}{2} \qquad \mathcal{N}_\pm = \frac{\dim W_\pm}{2},$$

so that $\mathcal{N} = 1$ is always the least possible.

5.2.1 R-symmetry

Consider G_R, the group of outer automorphisms of $\mathfrak{A}$ super Poincaré fixing the even part. These are given by the automorphisms of W. I.e. by our classification above:

- $O(W)$ if $n \equiv 1, 3 \bmod 8$.
- $O(W_+) \times O(W_-)$ if $n \equiv 2 \bmod 8$.
- $GL(W)$ if $n \equiv 0, 4 \bmod 8$.
- $Sp(W)$ if $n \equiv 5, 7 \bmod 8$.
- $Sp(W_+) \times Sp(W_-)$ if $n \equiv 6 \bmod 8$.

Note that these are the symmetries *of the algebra* – they may not all be present in any particular SQFT. Generally, one chooses $\mathfrak{g} \subseteq \mathrm{Lie}(G_R)$ and considers the SUSY algebra

$$\mathfrak{g} \oplus \mathfrak{so}(n;\mathbb{C}) \ltimes (\mathbb{C}^n \oplus \Pi\Sigma),$$

remembering that $\mathfrak{g}$ acts on Σ (this determines the bracket structure of $\mathfrak{g}$ with odd elements).

5.2.2 Square-zero elements

One can consider $Q \in \Sigma$ such that $\Gamma(Q, Q) = 0$. These determine cohomological structures on SUSY QFTs. From this we can consider *twisting*: add Q to the BV-BRST differential[65] of a SUSY QFT.

Example 5.2. If $\Gamma(Q, -) : \Sigma \to \mathbb{C}^n$ is surjective, we say that Q is *topological.*

[64] Theo has notational qualms about this, but hey, we do the best we can for the moment.
[65] I.e. cook up a double complex, and then take the total complex.

5.2.3 Dimension 1

$\mathfrak{so}(1;\mathbb{C})$ is trivial, so the SUSY algebra is just

$$\mathbb{C} \oplus \Pi W$$

with bracket given by a bilinear pairing on W. Square zero in this case means *null*: $\langle w, w\rangle = 0$.

Example 5.3 ($\mathcal{N} = 2$)**.** Null vectors are of the form $(1, i)$ or $(1, -i)$. In $\mathcal{N} = 2$ SQM these give rise to the supercharges Q and $Q^\dagger$ from Matt Bullimore's first lecture.

5.2.4 Dimension 2

$SO(2;\mathbb{C}) \cong \mathbb{C}^\times$. $S_\pm$ have weight $\pm\frac{1}{2}$, and $V = \mathbb{C}^2$ has weights $(1, -1)$. So,

$$\Sigma = W_+^{(1/2)} \oplus W_-^{(-1/2)}$$

and square zero elements are (w_+, w_-) both null. The first topological twisting (square zero elements) happen for $\mathcal{N} = (2, 2)$.

5.2.5 Dimension 4

There is an exceptional isomorphism

$$\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})_-$$

with $S_\pm$ the corresponding defining representations. $V \cong S_+ \otimes S_-$. Then

$$\Sigma = S_+ \otimes W \oplus S_- \otimes W^*$$

and square zero supercharges correspond to pairs of subspaces $W_{Q_+} \subseteq W$ and $W_{Q_-} \subseteq W^*$ which pair to zero. (E.g. W_{Q_+} is the image of the induced map $S_+^* \to W$.)

5.3 Session 3 (Natalie): 2d Yang-Mills

We're going to discuss 2d Yang-Mills in Euclidean signature – there's a lot of literature on this, but the primary reference for this talk is the review by Cordes-Moore-Ramgaalam.

5.3.1 Review: Yang-Mills

We'll focus on gauge groups $G = U(N)$ and $SU(N)$. The basic action functional is

$$S = \frac{1}{4e^2}\int_\Sigma \mathrm{tr}(F \wedge \star F)$$

where $F \in \Omega^2(\Sigma, \mathfrak{g}_{P_N})$ is the curvature of a connection A on a principal G-bundle P_N.

This simplifies in dimension 2, since we can take:

$$\begin{aligned} \star F &= f, & f &\in \Omega^0(\Sigma, \mathfrak{g}_{P_N}) \\ F &= f\mu, & \mu &\text{ an area form on } \Sigma \end{aligned}$$

Remark 5.4. The above works for oriented surfaces. This can be generalised to the unoriented case by working with densities where appropriate.

So in 2d we have

$$S = \frac{1}{4e^2}\int_\Sigma \mu\,\mathrm{tr}(f^2);$$

this is in fact invariant under a very large symmetry group: $S\,\mathrm{Diff}(\Sigma)$, the orientation preserving diffeomorphisms of Σ.

Classically: (Atiyah-Bott)

$$d_A(\star F) = 0,$$

so we can study covariantly constant sections X of $\mathfrak{g}_{P_N}$. Using this, there is a way to study Yang-Mills solutions by studying "holonomies" – we won't say too much about this.

Hilbert space: There are various derivations of the Hilbert space, with solution

$$L^2(\mathcal{A}/\hat{G}) = \{\text{square-normalisable class functions on } G\}.$$

If we were working in d-dimensions, then on a $(d-1)$-manifold Y the theory would assign the above space with the interpretation:

- $\mathcal{A}$: space of connections on P_N
- $\hat{G}$: $\mathrm{Map}(Y, G)$

In 2d, by the Peter-Weyl theorem,

$$L^2(\mathcal{A}/\hat{G}) = \left(\bigoplus_{R:\text{irreps of } G} R\otimes\overline{R}\right)^G .$$

The pairing on this space is

$$\langle f_1|f_2\rangle = \int_G dU\,\overline{f_1(U)}f_2(U)$$

where dU is the Haar measure on G normalised so that $\mathrm{vol}(G) = 1$.

Consider the theory on a cylinder $S^1_L \times \mathbb{R}$ where S^1 has circumference L. Then

$$\psi[A^a(x)] = \psi\left[\underbrace{P\exp\int_0^L dx\,A(x)}_{\equiv U}\right]$$

is a class function, and the wavefunctions are given by

$$\chi_R(U) = \langle U|R\rangle$$

(χ_R the character of the representation R).

The Hamiltonian is

$$H = \frac{e^2}{2}\int_0^L \frac{\delta}{\delta A(x)}\frac{\delta}{\delta A(x)}$$

and so

$$H|R\rangle = \frac{\lambda L}{2}C_2(R)|R\rangle$$

where $\lambda = e^2 N$ is the *'t Hooft coupling* and $C_2(R)$ is the value of the quadratic Casimir on R.

Recall from Davide's lectures: To define a QFT, can try to define a discretized theory and then take a continuum limit. For 2d Yang-Mills, this discretization is an exact procedure – the answers in the theory discretized on a mesh agree with the continuum limit (in particular we will see that partition functions are invariant under subdivisions of a mesh).

5.3.2 Discretisation

Cover Σ by a polygonal mesh. Let's consider a lattice gauge theory:

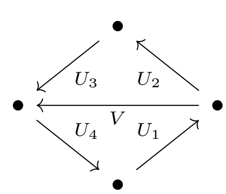
- For $\mathcal{V}$ the vertex set we assign
$$\begin{aligned} \mathcal{V} &\longrightarrow G \\ x &\longmapsto g_x \end{aligned}$$
- To a directed edge $\gamma : x \to y$, assign holonomy variables $U_\gamma \in G$. Moving from x to y is the parallel transport
$$g_y U_\gamma g_x^{-1}$$
- Discretized action:
$$e^{-\int \mathcal{L}} \to \prod_i e^{-\int_{W_i} \mathcal{L}_i}$$
where Σ is divided into *plaquettes* W_i, $i = 1, \ldots$

If $\mathcal{U} = U_1 U_2 \cdots$ is a loop of edges surrounding a plaquette W with area a_W, then the local contribution to the action is
$$\left(e^{-\int_W \mathcal{L}} =\right) \qquad \Gamma(\mathcal{U}, a_W) := \sum_{\alpha \in \mathrm{Irrep}(G)} \dim(\alpha) \chi_\alpha(\mathcal{U}) \exp\left(-a_w \frac{c_2(\alpha)}{2}\right)$$
As $a \to 0$ this limits to the δ-function $\delta(U-1)$; i.e. there is no holonomy around a "constant" loop.[66]

Let the subdivision on Σ be denoted by X, assign Σ area a (choice of area form); then the partition function is
$$\mathcal{Z}_{\Sigma,X}(a) = \int \prod_{\gamma:\ \text{edges}} dU_\gamma \prod_{i:\text{plaquettes}} \Gamma(\mathcal{U}_i, a_i)$$

Claim: If X' is a subdivision of X,
$$\mathcal{Z}_{\Sigma,X}(a) = \mathcal{Z}_{\Sigma,X'}(a).$$

Consider a subdivision of a square plaquette:



where the areas of the top and bottom triangle are a' and a'', and the original area of the square plaquette is $a_0 = a' + a''$.

[66] There is presumably also some argument regarding flatness of connections?

Now,

$$\Gamma'\Gamma'' = \sum_{\alpha,\beta} \dim(\alpha)\dim(\beta)\chi_\alpha(U_1U_2V)\chi_\beta(V^{-1}U_3U_4)\exp(\cdots),$$

and then

$$\int dV\, \Gamma'\Gamma''$$

by the orthogonality relation

$$\int dV\, \chi_\alpha(AV)\chi_\beta(V^{-1}B) = \delta_{\alpha\beta}\frac{1}{\dim(\alpha)}\chi_\alpha(AB).$$

Now: the fundamental group of Σ is generated by elements $\{a_1,\ldots,a_g,b_1,\ldots,b_g\}$ satisfying the relation $a_1b_1a_1^{-1}b_1^{-1}\cdots a_g^{-1}b_g^{-1}$; so take the corresponding "single plaquette" discretisation of Σ with holonomy variables U_i, V_i corresponding to the a_i and b_i generators. Then:

$$\mathcal{Z}_\Sigma(a) = \sum_{\alpha\in\mathrm{Irrep}(G)} \dim(\alpha)e^{-a\frac{C_2(\alpha)}{2}}\int dU_i dV_j \chi_\alpha(U_1V_1U_1^{-1}V_1^{-1}\cdots) = \sum_\alpha \frac{e^{-a\frac{C_2(\alpha)}{2}}}{\dim(\alpha)^{2g-2}}$$

There is a corresponding analysis for a surface with b boundaries. Here one needs to give boundary conditions – prescribed holonomies U_i around each of the boundary circles; the corresponding answer is

$$\mathcal{Z}_\Sigma(a,U_1,\ldots,U_b) = \sum_\alpha \frac{e^{-a\frac{C_2(\alpha)}{2}}}{\dim(\alpha)^{2g+b-2}}\prod_{i=1}^b \chi_\alpha(U_i).$$

Remark 5.5. One can also arrive at this answer by starting with the TFT axioms of Atiyah and Segal – one needs to know what to assign to a pair of pants, disk, etc. all labeled by an area parameter.

5.3.3 First order formalism

For the purposes of today's talk, we'll treat this as just a way to rewrite the action. The 2d Yang-Mills action becomes

$$S = -\frac{1}{2}\int i\,\mathrm{tr}(BF) + \frac{1}{2}e^2\int \mathrm{tr}(B^2)\mu.$$

Various aspects of the theory become more transparent in this formalism – e.g. taking the area of the surface to zero kills the second term and one is left with BF-theory, an honest topological field theory.

Reference: Mnev-Irasu from 2018.

5.3.4 Wilson Lines

Choose a collection of simple closed curves Γ, and write the complementary components of Σ as

$$\Sigma \setminus \coprod \Gamma = \coprod_C \Sigma^C$$

so that C labels the connected components. Then we can take the expectation value of a product of Wilson line observables labeled by representations R_Γ:

$$\begin{aligned}\langle\prod_\Gamma W(R_\Gamma,\Gamma)\rangle &= \int \prod_\Gamma dU_\Gamma \prod_C \Big(\mathcal{Z}_{\Sigma^C}(a_C, U_{C^l_\Gamma=C}, U^{-1}_{C^r_\Gamma=C})\Big)\prod_\Gamma W(R_\Gamma,\Gamma)\\ &= \sum_{R(C)}\prod_C (\dim R(C))^{\chi(\Sigma^C)}\times(\cdots)\times(\text{factors of Clebsch-Gordon coefficients})\end{aligned}$$

(Recall that the CG coefficients are the multiplicities of irreps appearing in the tensor product of irreps.)

Similarly, if one takes *intersecting* Wilson lines, there is a similar story where instead of the Clebsch-Gordon coefficients, the 6j-symbols appear. Related to work of Kevin, Witten, Yamazaki(?), others...

5.3.5 Another interesting limit

Recall the 't Hooft coupling $\lambda = e^2 N$. One can also consider taking large N limits, looking at expansions in powers of $\frac{1}{N}$ – no time to talk about this today, however.

5.3.6 Questions

Question 4. *What about gauge theories with matter?*

Answer 5.1. Introducing matter breaks metric invariance and messes up solubility of the theory.

Question 5. *Any interesting subtleties by considering groups outside of type A?*

Answer 5.2. Yes. Answers written down in the cited review, at least for the classical groups.

Question 6. *Do higher Casimirs appear at all?*

Answer 5.3. Possibly in a non-polynomial deformation of the theory? Who knows.

5.4 Session 4 (Chris): 4d Yang-Mills and Asymptotic Freedom

Rough plan:

(1) Classical Yang-Mills in BV formalism

(2) The β-function and asymptotic freedom

(3) Quantization of Yang-Mills

5.4.1 Yang-Mills on $\mathbb{R}^4$

We're going to talk about 4d Yang-Mills theory with matter. Choose:

- G simple compact Lie group
- V a representation of G

The fields of the theory are:

- A: gauge field in $\Omega^1(\mathbb{R}^4; \mathfrak{g})$ – boson
- ψ: spinor section in $\Omega^0(\mathbb{R}^4; S \otimes V)$ where $S = S_+ \oplus S_-$ (Dirac spinor bundle) – fermions

Transform under the gauge group by conjugation – for $c \in \Omega^0(\mathbb{R}^4; \mathfrak{g})$,

$$A \mapsto A + d_A(c)$$

To define the action, choose:

- G-invariant pairing $\mu : V \otimes V \to \mathbb{R}$
- positive operator $m : V \to V$ (mass matrix)

Then the action is

$$S(A,\psi) = \int_{\mathbb{R}^4} \frac{1}{2}\|F_A\|^2 + \mu(\psi, (\not{d}_A + m)\psi)$$

where $\not{d}_A$ is the Dirac operator, coming from Clifford multiplication $\rho : \Omega^1 \otimes S \to S$ as

$$\not{d}_A \psi = \rho(d_A\psi).$$

The BV complex for pure Yang-Mills is

$$(\mathcal{E}_{pure}, Q) = \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d\star d} \Omega^3 \xrightarrow{d} \Omega^4$$

but there is no choice for Q^{GF}. To deal with this, we'll rewrite this in *first order formalism.*

Introduce $B \in \Omega^2_+(\mathbb{R}^4;\mathfrak{g})$ a self-dual field, $\star B = B$. The first order formalism action is

$$S_{FO}(A,B,\psi) = \int_{\mathbb{R}^4} \langle F_A, B\rangle_{L^2} - \frac{1}{2}\|B\|^2 + \mu(\psi, (\not{d}_A + m)\psi).$$

The classical BV complex in first order formalism with matter is[67]

$$\begin{tikzcd}
0 & 1 & 2 & 3 \\
\Omega^0(\mathbb{R}^4,\mathfrak{g})_C \arrow[r, "d"] & \Omega^1(\mathbb{R}^4;\mathfrak{g})_A \arrow[r, "d\star"] & \Omega^2_+(\mathbb{R}^4;\mathfrak{g})_{B^\vee} & \\
& \Omega^2_+(\mathbb{R}^4,\mathfrak{g})_B \arrow[r, "d"] \arrow[ur, "-\mathrm{id}"] & \Omega^3(\mathbb{R}^4,\mathfrak{g})_{A^\vee} \arrow[r, "d"] & \Omega^4(\mathbb{R}^4;\mathfrak{g})_{C^\vee} \\
& \Omega^0(\mathbb{R}^4;S\otimes V)_\psi \arrow[r, "m+\not{d}"] & \Omega^0(\mathbb{R}^4;S\otimes V)_{\psi^\vee} &
\end{tikzcd}$$

with

$$I = \langle B, [A\wedge A]\rangle + \mu(\psi, \not{A}\psi) + (A^\vee, [c,A]) + ([c,\psi],\psi^\vee) + ([c,c],c^\vee)$$

This is homotopy equivalent to 2nd order YM coupled to trivial B.

Steps for BV quantization:

1) The purely free part:
   a) Choose a gauge fixing Q^{GF}, $[Q,Q^{GF}]$ a generalised Laplacian
   b) Calculate the kernel K_L mollifying the kernel for $[Q,Q^{GF}] \in \mathcal{E}\otimes\mathcal{E}$; splits into a sum over "particle species" i.e. pairs $\alpha\otimes\alpha^\vee$ paired by symplectic form.
   c) Calculate the propagator

   $$P(\epsilon, L) = \int_\epsilon^L dt\,(Q^{GF}\otimes 1)K_t,$$

   splits into a sum as above over $\alpha\otimes\beta$ where $|\alpha| + \beta| + 1 = 3$.
2) Calculate $I[L]$, first step is to try

   $$\lim_{\epsilon\to 0} W(P(\epsilon,L),I);$$

   this will be divergent. Choose $I^{CT}(\epsilon)$ so that

   $$\lim_{\epsilon\to 0}(W(P(\epsilon,L),I) - I^{CT}(\epsilon))$$

   exists; many ways of doing this. (Result is a free theory – i.e. free theories are unobstructed.)

[67]Subscripts indicate names of corresponding fields.

3) Try to solve the QME, try adding some J to

$$\tilde{I}[L] = \lim_{\epsilon \to 0} (W(P(\epsilon, L), I) - I^{CT}(\epsilon));$$

J must be a potential for failure of $\tilde{I}$ to solve QME. I.e. no guarantee that a solution to QME will exist.[68]

5.4.2 Local RG flow

Kevin explained today that

$$R_\lambda I[L] = I[L] + \log \lambda + \cdots + \text{ higher order terms} \in \mathcal{O}_{loc}[\lambda^{\pm 1}, \log \lambda].$$

Definition 5.3. The *β-functional at scale L* is the observable

$$\mathcal{O}_\beta[L] = \left. \frac{d}{d \log \lambda} R_\lambda I[L] \right|_{\lambda = 1} \in \mathcal{O}_{loc}(\mathbb{R}^n)[[\hbar]].$$

We'd *like* to have just a nice function of the coupling constant. Write

$$\mathcal{O}_\beta[L] = \sum_{i=0}^{\infty} \mathcal{O}_\beta^{(i)}[L] \hbar^i.$$

Fact: For scale invariant theories, $\lim_{L \to 0} \mathcal{O}_\beta^{(1)}[L]$ exists and is BV-closed. Moreover the cohomology class

$$\left[\lim_{L \to 0} \mathcal{O}_\beta^{(1)}[L] \right]$$

is independent of choices of $I^{CT}(\epsilon)$, etc.

Definition 5.4. The 1-loop β-function is this class $\beta^{(1)} := \left[\lim_{L \to 0} \mathcal{O}_\beta^{(1)}[L] \right]$.

Theorem 5.3 (Physics Theorem: Gross-Wilczek-Politzerm 1973)**.** *For Yang-Mills,*

$$\beta^{(1)}(g) = \frac{-g^3}{16\pi^2} \left(\frac{11}{3} C(\mathfrak{g}) - \frac{4}{3} C(V) \right).$$

A theory is *asymptotically free* if $\beta^{(1)}$ is negative.

Example 5.4. $SU(N)$ with f fundamental flavours: $\beta^{(1)}$ is negative if $f < \frac{11}{2} N$.

Lemma 5.4. *For scale and translation invariant theories which are strictly renormalisable at 1-loop, $\mathcal{O}_\beta^{(1)}[L]$ is cohomologous to the log part of the 1-loop counterterm $I^{CT}_{\log \epsilon}(\epsilon)$.*

5.4.3 BV Quantisation of Yang-Mills

Choose gauge fixing as below:

$$\begin{tikzcd}
0 & 1 & 2 & 3 \\
\Omega^0(\mathbb{R}^4, \mathfrak{g})_C \arrow[r, shift left, "d"] & \Omega^1(\mathbb{R}^4; \mathfrak{g})_A \arrow[l, shift left, "d^*"] \arrow[r, shift left, "d\star"] & \Omega^2_+(\mathbb{R}^4; \mathfrak{g})_{B^\vee} \arrow[l, shift left, "2d^*"] & \\
 & \Omega^2_+(\mathbb{R}^4, \mathfrak{g})_B \arrow[u, "-2d^*"] \arrow[ur, "-\mathrm{id}"] \arrow[r, shift left, "d"] & \Omega^3(\mathbb{R}^4, \mathfrak{g})_{A^\vee} \arrow[u, "-2d^*"'] \arrow[l, shift left, "-2d^*_+"] \arrow[r, shift left, "d"] & \Omega^4(\mathbb{R}^4; \mathfrak{g})_{C^\vee} \arrow[l, shift left, "d^*"]
\end{tikzcd}$$

$$\Omega^0(\mathbb{R}^4; S \otimes V)_\psi \underset{\not{d} - m}{\overset{m + \not{d}}{\rightleftarrows}} \Omega^0(\mathbb{R}^4; S \otimes V)_{\psi^\vee}$$

[68]The calculation for 4d Yang-Mills is the done by Kevin in his book; the obstruction there vanishes.

Here the vertical differentials are the compositions of the relevant diagonal and horizontal maps.

The heat kernel K_t splits into a sum proportional to scalar

$$k_t(x,y) = \frac{1}{(4\pi t)^2} e^{\frac{-|x-y|^2}{4t}}$$

$$\mathcal{E} = C^\infty(\mathbb{R}^4) \otimes (\underbrace{Y \otimes \mathfrak{g}}_{\text{pure}} \oplus \underbrace{\mathcal{S} \otimes V}_{\text{matter}})$$

and end up with

$$K_t = K_{AA^\vee} + K_{BB^\vee} + K_{CC^\vee} + K_{\psi\psi^\vee}.$$

Propagator.

Apply $Q^{GF} \otimes 1$ to K_t.

Lemma 5.5. *The propagator has the form*

$$P(\epsilon, L) = \int_\epsilon^L dt \left(\frac{\partial k_t}{\partial x^i}(x,y)(P^i_{AB} + P^i_{A^\vee C}) + \frac{\partial^2 k_t}{\partial x^i \partial x^j} P^{ij}_{AA} + \frac{\partial k_t}{\partial x^i} P^i_{\psi\psi} \right)$$

where the subscripts on the propagator denote which factor of the tensor square of $\mathcal{E}$ it belongs to.

The 1-loop Feynman diagrams all have the forms of "wheels" – Γ_k is a single loop with k vertices, each with a single external edge ("spoke") protruding.

Lemma 5.6. *The weight associated to Γ_k has no $\log(\epsilon)$ divergence unless $k = 2$.*

So the only diagrams that contribute have the shape (ignore arrow directions):

$$\begin{tikzcd} {} \arrow[r] & \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[r] & {} \end{tikzcd}$$

There are six possibilities for the external edges and internal propagators:

$$\begin{tikzcd} {} \arrow[r, "A"] & \bullet \arrow[r, bend left, "AB"] \arrow[r, bend right, "BA"'] & \bullet \arrow[r, "A"] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "A"] & \bullet \arrow[r, bend left, "A^\vee C"] \arrow[r, bend right, "CA^\vee"'] & \bullet \arrow[r, "A"] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "B"] & \bullet \arrow[r, bend left, "AB"] \arrow[r, bend right, "AA"'] & \bullet \arrow[r, "A"] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "B"] & \bullet \arrow[r, bend left, "AA"] \arrow[r, bend right, "AA"'] & \bullet \arrow[r, "B"] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "A"] & \bullet \arrow[r, bend left, "\psi\psi"] \arrow[r, bend right, "\psi\psi"'] & \bullet \arrow[r, "A"] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "\psi"] & \bullet \arrow[r, bend left, "AA"] \arrow[r, bend right, "\psi\psi"'] & \bullet \arrow[r, "\psi"] & {} \end{tikzcd}$$

It turns out that we don't need to actually calculate the grey diagram, as its weight is BV-exact.

5.5 Session 5 (Du Pei): Verlinde Algebras and 2d TQFTs

Think of this as part two of a lecture series started by Natalie in 5.3.

5.5.1 2d Yang-Mills Theory

Recall the following facts about 2d Yang-Mills theory:

(1) Almost topological, dependence on $e^2 \times$ (area).

(2) As $e^2 \to 0$ on obtains a "2d TQFT" called "BF theory". It doesn't satisfy the Atiyah-Segal axioms however – e.g. $Z(T^2)$ should compute the dimension of the space of states on a circle – but this in infinite dimensional, so $Z(T^2) = \infty$. Similarly $Z(S^2) = \infty$.

(3) In situations where it is well-defined however, it can be used to compute the volume of $\mathcal{M}_{flat}(\Sigma, G)$.

For today,

- G is a compact, simple, simply-connected Lie group
- Σ is a closed Riemann surface

Today: Construct a family of 2d TQFTs parametrised by (G, k), $k \in \mathbb{Z}_{\geq 0}$ the "level".

Fact:

$$\text{2d TQFT} \leftrightarrow \text{commutative Frobenius algebra}$$

For the TQFT we will study today, the corresponding Frobenius algebra is the *Verlinde algebra for G_k*.

(1) In the world of 2d CFTs/Vertex operators algebras/affine Lie algebras, the Verlinde algebra computes the "Fusion rule of the Wess-Zumino-Witten model".

(2) In the world of 3d TQFTs/quantum topology, the Verlinde algebra gives you the "algebra of line defects in G_k Chern-Simons theory".

(3) Count "non-abelian theta functions" on $\mathcal{M}_{flat}$.

(4) Quantum cohomology/K-theory on Grassmannians.

So: how do we go from the data (G, k) to a commutative Frobenius algebra $(V, \star, (-,-), 1_V)$:

- V:finite dimensional vector space over $\mathbb{C}$
- $\star : V \times V \to V$ multiplication ("fusion rule")
- $(-,-) : V \times V \to \mathbb{C}$ symmetric bilinear pairing
- $1_V : \mathbb{C} \to V$ unit

Conditions:

- $(V, \star, 1_V)$ is an associative and commutative unital algebra
- $(-,-)$ is nondegenerate

- Compatibility of pairing and product: $(a \star b, c) = (a, b \star c)$

Example 5.5 ($G = SU(2)$)**.** For $SU(2)$ at level k, $\dim(V) = k+1$. There exists a nice basis $e_0, \ldots, e_k$, where one thinks of e_λ as the integrable representation of $LSU(2)_k$ associated with highest weight $\lambda \in \Lambda_{wt}(SU(2)) \simeq \mathbb{Z}$.

We use the following notation for the fusion product structure coefficients

$$e_{\lambda_1} \star e_{\lambda_2} = \sum_{\lambda_3} f^{\lambda^3}_{\lambda_1\lambda_2} e_{\lambda_3}$$

and the pairing

$$(e_{\lambda_1}, e_{\lambda_2}) = \eta_{\lambda_1\lambda_2}.$$

Use η to lower indices:

$$f_{\lambda_1\lambda_2\lambda_3} := \sum_{\mu} f^{\mu}_{\lambda_1\lambda_2} \eta_{\mu\lambda_3}.$$

We can write

$$\begin{aligned} f_{\lambda_1\lambda_2\lambda_3} &= \begin{cases} 1, & \text{if } \lambda_1+\lambda_2+\lambda_3 \text{ is even, } \Delta(\lambda_1,\lambda_2,\lambda_3) \le 0, \\ 0, & \text{otherwise,} \end{cases} \\ \Delta(\lambda_1,\lambda_2,\lambda_3) &= \max\{\lambda_1-\lambda_2-\lambda_3, \lambda_2-\lambda_1-\lambda_3, \lambda_3-\lambda_1-\lambda_2, \lambda_1+\lambda_2+\lambda_3-2k\}, \\ \eta_{\lambda_1\lambda_2} &= \delta_{\lambda_1\lambda_2}. \end{aligned}$$

Finally, e_0 is the unit.

The relation between the 2d TQFT Z and the Frobenius algebra is as follows:

- $Z(S^1) = V$
- $\star$ corresponds to the pair-of-pants
- $(-,-)$ corresponds to the macaroni
- 1_V corresponds to the cup

Example 5.6. For $SU(2)$ and Σ genus 2, we can decompose into two joker hats, and so compute[69]

$$Z(\Sigma) = \sum_{\lambda_1\lambda_2\lambda_3} f^2_{\lambda_1\lambda_2\lambda_3} = \sum_{\lambda_1\lambda_2\lambda_3} f_{\lambda_1\lambda_2\lambda_3} = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1$$

which is the "$(k+1)^{st}$ tetrahedral number".

The leading coefficient, $\frac{1}{6}$ can be rewritten in at least 2 interesting ways:[70]

(1) $\frac{1}{6} = \mathrm{vol})\mathbb{CP}^3$

(2) $\frac{1}{6} = \frac{1}{\pi^2}\zeta(2)$

To explain the first expression, note that

$$\mathbb{CP}^3 = \mathcal{M}_{flat}(SU(2), \Sigma)$$

[69]Using that $f^2_{\lambda_1\lambda_2\lambda_3} = f_{\lambda_1\lambda_2\lambda_3}$ since the only possible values are 0 and 1.
[70]Interesting $\neq$ meaningful, although presumably the following is meaningful.

equipped with the Atiyah-Bott symplectic form $\omega \in H^2(\mathbb{CP}^3;\mathbb{Z})$, and that

$$\int_{\mathbb{CP}^3} e^\omega = \int_{\mathbb{CP}^3} \frac{\omega^3}{6} = \frac{1}{6}.$$

To relate this to the 2d Yang-Mills partition function, note that

$$\frac{1}{\pi^2} Z_{BF}(G,\Sigma) = \frac{1}{\pi^2} \sum_{R\in \mathrm{Irrep}(SU(2))} \frac{1}{(\dim R)^2} = \frac{\zeta(2)}{\pi^2}$$

which explains the second expression.

In general, if one considers the expression

$$\lim_{k\to\infty} \left(Z_{Ver}(\Sigma_g) k^{-(3g-3)} \right) = \mathrm{vol}(\mathcal{M}_{flat}) = \frac{2}{(2\pi^2)^{g-1}} \underbrace{Z_{BF}(\Sigma_g)}_{\zeta(2g-1)}.$$

Remark 5.6. Recall that Z_{BF} is the zero area limit of Z_{YM}.

One can ask what the Verlinde partition function is counting in general. In totality (i.e. not coefficient by coefficient),

$$Z_{Ver}(\Sigma) = \dim H^0(\mathcal{M}_{flat}, \mathcal{L}^k).$$

To understand this, note that the moduli space of flat connections can be identified with the moduli space of holomorphic $G_{\mathbb{C}}$-bundles over Σ, and this moduli space is equipped with a natural "determinant line bundle". The sections are called "non-abelian theta functions" by analogy with the following example:

Example 5.7 ($G = U(1)$)**.** Then $G_{\mathbb{C}} = \mathbb{C}^\times$, $\mathcal{M}_{flat} \simeq \mathrm{Jac}(\Sigma)$, and the sections of $H^0(\mathrm{Jac}(\Sigma), \mathcal{L}^k)$ are the Riemann theta functions of order k.

We have a nice explicit description in the following familiar example:

Example 5.8 ($G = SU(2)$)**.** Take Σ to have genus 2, so that we are looking at $\mathbb{CP}^3$. Then $\mathcal{L} = \mathcal{O}(1)$ and

$$H^0(\mathbb{CP}^3;\mathcal{O}^k) = \left\{ \begin{array}{c} \text{homogeneous polynomials in} \\ \text{4 variables of degree } k \end{array} \right\} = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1.$$

Further, this is equal to

$$\chi(\mathbb{CP}^3, \mathcal{O}(k)) = \int_{\mathbb{CP}^3} \mathrm{Td}(\mathbb{CP}^3) e^{k\omega}.$$

For more general G, choose a maximal torus $T \subset G$ with $\mathrm{Lie}(T) = \mathfrak{t}$. Then

$$(k+h)\langle -,-\rangle : \mathfrak{t} \to \mathfrak{t}^*$$

where $\langle -,-\rangle$ is the Killing form and h is the dual Coxeter number. This can be exponentiated to a map

$$\chi : T \to T^*$$

and we set

$$F := \chi^{-1}(1).$$

Then the Verlinde algebra is

$$\mathbb{C}[F^{reg}/W]$$

with the algebra structure given by multiplication of functions. A complete basis is given by

$$\Theta_\lambda := \sum_{w\in W} \frac{e^{w(\lambda)}}{\prod_{\alpha<0}(1-e^{w(\alpha)})} \in V$$

– which is just the Weyl character formula – where λ is a dominant weight such that $\langle\lambda, \theta\rangle \leq k$ (θ is the highest root). The pairing is

$$(\Theta_{\lambda_1}, \Theta_{\lambda_2}) = \delta_{\lambda_1 \lambda_2^*}$$

and the unit is Θ_0, the constant function.

How can we diagonalize the fusion rule? I.e. find

$$w_\lambda \star w_{\lambda'} \sim \delta_{\lambda\lambda'} w_\lambda.$$

One can canonically identify

$$\begin{array}{ccc} F^{reg}/W & \xleftarrow{\sim} & \{\text{integrable representations}\} \\ \exp\left(\frac{(\lambda+\rho)^\vee}{k+h}\right) & \longleftarrow\!\!\shortmid & \lambda \end{array}$$

Let f_μ be the δ-functions non-zero on $\mu \in F^{reg}/W$. Then the change of basis is given by the matrix $\Theta_\lambda(\mu)$. In fact, the S-matrix of the 3d TQFT is given by

$$S_{\lambda\mu} = c\Theta_\lambda(\mu)$$

for some constant c (this is in Verlinde's original paper).

Upshot: The S-matrix diagonalises the fusion rule!

Problem 21. *Do the following calculation:*

$$Z_{Ver}(\Sigma_g, G) = \sum_\lambda (S_{0\lambda})^{2-2g}$$

This equality is called the Verlinde formula*.*

Example 5.9. For $SU(2)$,

$$S_{\lambda\mu} = \left(\frac{2}{k+2}\right)^{\frac{1}{2}} \cdot \sin\left(\frac{\pi(\lambda+1)(\mu+1)}{k+2}\right).$$

Question: Is there a Lagrangian one can start with to derive this?

Answer: G/G gauged WZW model. But this theory is the same as 3d Chern-Simons theory on S^1, and it is easier to use 3d Chern-Simons theory to derive this formula. Moreover, 3d Chern-Simons theory can be boosted up to (non-topological) 3d $\mathcal{N}=2$ super Chern-Simons theory – one can ask whether one finds extra information by doing this (open problem).

5.6 Session 6 (Dylan): AKSZ and Boundaries

Today will be *fast* and about *doing calculations*.

Also: everything is classical today (except maybe right at the end).

5.6.1 Preliminaries

Definition 5.5. An *n-shifted symplectic structure on* X is a non-degenerate, closed[71] 2-form

$$\omega_X : T_X \otimes T_X \to \mathbb{K}[n].$$

[71] In the fancy sense.

Example 5.10. For $S \in \mathcal{O}(Y)$, $d\,\mathrm{Crit}(S) = \mathcal{O}(T^*[-1]Y)$, ι_{dS}. This is (-1)-shifted symplectic.

Very explicit: $Y = C^\infty(M)$,

$$T^*[-1]Y = \begin{array}{ccc} C^\infty(M) & \xrightarrow{d\star d} & \Omega^n(M) \\ 0 & & 1 \end{array}$$

Example 5.11. Let G be a group,

$$\langle -, - \rangle : \mathfrak{g}^{\otimes 2} \to \mathbb{K}$$

a non-degenerate ad-invariant form. Then BG is 2-shifted symplectic;

$$T_0 BG = \mathfrak{g}[1]^{\otimes 2} \to \mathbb{K}[2].$$

The following definition is in square quotes.

Definition 5.6. A *d-orientation* on M is a non-degenerate integration map

$$\int_M : \Gamma(M, \mathcal{O}_M) \to \mathbb{K}[-d].$$

Example 5.12. For M a smooth manifold there is a "space" M_{dR}. This is a locally ringed space over M with structure sheaf defined by

$$\Gamma(U, \mathcal{O}_{M_{\mathrm{dR}}}) = (\Omega^\bullet_U, d_{\mathrm{dR}}).$$

For M^d closed and oriented, M_{dR} is d-oriented.

Example 5.13. For X a Calabi-Yau variety of dimension d,

$$\Gamma(X, \mathcal{O}_X) \to \mathbb{K}[-d]$$

by Serre duality.

5.6.2 AKSZ Theories

These are theories that arise by studying shifted symplectic mapping stacks.

Theorem 5.7. *Let X be n-shifted symplectic and M d-oriented. Then $\mathrm{Maps}(M, X)$ is $n-d$ shifted symplectic.*

"Proof". $T_f \,\mathrm{Maps}(M, X) = \Gamma(M, f^* T_X) = \mathcal{E}$. Take

$$T_f \,\mathrm{Maps}(M, X)^{\otimes 2} \xrightarrow{f^*\omega_X} \Gamma(M, \mathcal{O}_M)[n] \xrightarrow{\int_M} \mathbb{K}[n-d]$$

Then prove that this construction gives us a shifted symplectic structure. □

Example 5.14. Let V be a symplectic vector space,

$$T_0 \,\mathrm{Maps}(\mathbb{R}_{\mathrm{dR}}, V) \simeq \Omega^\bullet_{\mathbb{R}} \otimes V$$

and we recall from Kevin's lectures that quantising this will give us the Weyl algebra on V. This is (-1)-shifted symplectic.

Example 5.15. $T_0 \,\mathrm{Maps}(M^3_{\mathrm{dR}}, BG) = \Omega^\bullet_{M^3} \otimes \mathfrak{g}[1]$, phase space for Chern-Simons theory. This is (-1)-shifted symplectic.

Example 5.16. In Example 5.15, replace BG by a holomorphic symplectic manifold X; then we get "Rozansky-Witten theory", a twist of 3d $\mathcal{N} = 4$ SYM. This gives you a (-3)-shifted symplectic space – so some extra work is required to understand the theory.[72]

[72] The theory is "only $\mathbb{Z}/2$-graded". In particular – there is no $U(1)$ R-charge enhancing the fermion grading to a $\mathbb{Z}$-grading. If X were a cone we should be able to fix this.

Example 5.17. The *B-model to X a complex variety* is $\mathrm{Maps}(\Sigma_{\mathrm{dR}}, T^*[1]X)$.

Example 5.18 (4d Chern-Simons)**.** $\mathrm{Maps}(\mathbb{C} \times \mathbb{R}^2_{\mathrm{dR}}, BG)$

Example 5.19 (5d Chern-Simons)**.** $\mathrm{Maps}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, BG)$

Example 5.20 (6d Chern-Simons)**.** $\mathrm{Maps}(X^3, BG)$, X^3 Calabi-Yau 3-fold.

Example 5.21. For π the 2-shifted symplectic structure on BG we have: $\mathrm{Maps}(M^4_{\mathrm{dR}}, T^*_{(\pi)}[3]BG)$ is the Kapustin-Witten B-twist (generic twist).

Example 5.22. $\mathrm{Maps}(M^3_{\mathrm{dR}} \times \mathbb{C}, T^*[3]BG)$ twist of 5d $\mathcal{N} = 2$.

5.6.3 Classical Field Theory on Manifolds with Boundary

Let M be a compact oriented d-manifold with boundary, with $\partial M = N$ closed and oriented. If we take X a $(d-1)$-shifted symplectic space, then

$$\mathrm{Maps}(M_{\mathrm{dR}}, X)$$

would have defined an AKSZ-theory. Since M has boundary, however, this is no longer symplectic. However,

$$\mathrm{Maps}(N_{\mathrm{dR}}, X) \text{ is 0-shifted symplectic;}$$

think of this as like a phase space (we've restricted to a codimension 1 slice).

But good news! There is a restriction map

$$\mathrm{Maps}(M_{\mathrm{dR}}, X) \xrightarrow{res} \mathrm{Maps}(N_{\mathrm{dR}}, X)$$

and it can be given the structure[73] of a Lagrangian map.

Remark 5.7. Didn't need to study an AKSZ theory, however describing the version of the space $\mathrm{Maps}(N_{\mathrm{dR}}, X)$ one gets in that situation is a bit trickier (will get some space of germs of solutions to differential equations).

Fact: If $L_1, L_2 \to X$ Lagrangian mapping to X k-shifted symplectic, then

$$L_1 \times_X L_2 \text{ is } (k-1)\text{-shifted symplectic.}$$

So in our situation, if we *choose* some Lagrangian $L \to \mathrm{Maps}(N_{\mathrm{dR}}, X)$, then the fibre product is (-1)-shifted.

Remark 5.8. The choice of L will be a choice of boundary condition. We will later try to rule out *non-local* boundary conditions, an example of which is precisely the restriction map from M.

What does the choice of L give us, and why are they boundary conditions? We *define* the classical observables on an open set intersecting the boundary to be (Figure 47)

$$\mathrm{Obs}^{cl}(V) = \mathcal{O}\left(\underbrace{\mathcal{E} \times_{\mathcal{E}(V \cap \partial M)} L}_{=:\mathcal{E}^L}\right),$$

so that L is precisely telling us "which fields on V we are allowed to restrict to the boundary in our theory".

Definition 5.7. A *local boundary condition for $\mathcal{E}$* is a Lagrangian subbundle of $\mathcal{E}^\partial$ "compatible" with the equations of motion.

[73] We're working derived dontcha know.

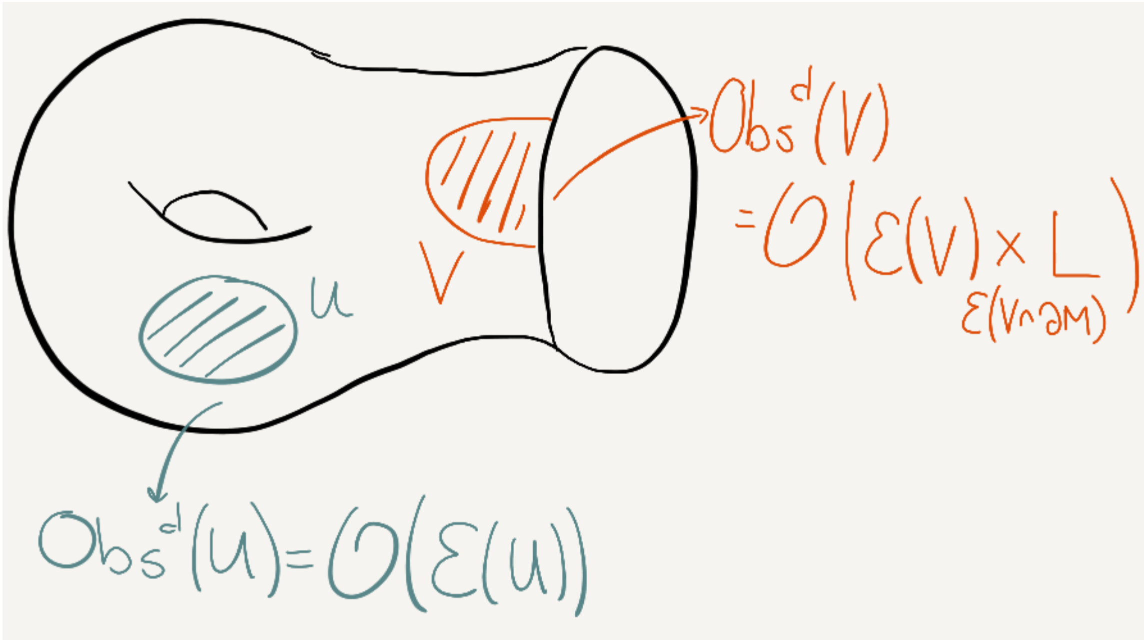


Figure 47: "Lagrangians are boundary conditions": A choice of L yields a choice of boundary observables.

Boundary Theories.

The fields which survive at ∂M have their own dynamics: if the equations of motion are well-posed with respect to the boundary condition L, we have an isomorphism from solving the equations of motion

$$L(V \cap \partial M) \simeq \mathcal{E}^L(V).$$

But now: we had a $\mathbb{P}_0$ factorisation algebra in the bulk M, and performing this extension procedure on open sets in N we conclude that

$$\mathcal{O}(L) \text{ defines a } \mathbb{P}_0\text{-factorisation algebra on } N.$$

Example 5.23 (1-dimensional case)**.** Let X be a symplectic manifold,[74] and let

$$\mathcal{E} = \mathrm{Maps}(\mathbb{R}^{\geq 0}_{\mathrm{dR}}, X).$$

Then $\mathcal{E}^\partial = X$ is 0-shifted symplectic. Let $L \to X$ be a Lagrangian map to X. Consider the formal completion of X around L,

$$\hat{X}_L = (NL, Q)$$

where NL is the normal bundle to L in X and Q is a homological vector field. Via the symplectic form we have an isomorphism $NL \simeq T^*L$, and Q turns out to preserve the symplectic form. Under reasonable assumptions Q is Hamiltonian with Hamiltonian function $S \in \mathcal{O}(T^*L)[1]$,

$$\hat{X}_L \simeq (T^*L, Q = \{S, -\}).$$

Now $\mathcal{O}(T^*L)[1] = \mathrm{PV}^\bullet_{-1}(L)$ (shifted polyvector fields), so,

$$S = Q + \Pi + \Pi^{(3)} + \cdots,$$

and $\vec{\Pi}$ defines a homotopy $\mathbb{P}_0$ structure if and only if $\{S, S\} = 0$.

[74] 0-shifted symplectic.

Example 5.24. Let

$$\mathcal{E} = T_0 \operatorname{Maps}(\Sigma_{\mathrm{dR}}, T^*[1]BG) = \left(\begin{array}{cccc} -1 & 0 & 1 & 2 \\ \Omega^0 \xrightarrow{d} & \Omega^1 \xrightarrow{d} & \Omega^2 & \\ & \Omega^0 \xrightarrow{d} & \Omega^1 \xrightarrow{d} & \Omega^2 \end{array} \right)$$

where the first row is valued in $\mathfrak{g}$ and the second row is valued in $\mathfrak{g}^*$. Call fields in the first row A and fields in the second row B; then we can define BF-theory

$$S(A,B) = \int_\Sigma B(dA + \frac{1}{2}[A \wedge A]).$$

The boundary complex is

$$\begin{array}{rcccc} & -1 & 0 & 1 & \\ \mathcal{E}^\partial = & \Omega^0 \xrightarrow{d} & \Omega^1 & & \mathfrak{g} \\ & & \Omega^0 \xrightarrow{d} & \Omega^1 & \mathfrak{g}^* \end{array}$$

There are two immediately apparent boundary conditions: either the A fields survive ("L_A", Neumann) or the B fields survive ("L_B", Dirichlet). Let's consider L_B. Then

$$\mathcal{E}^\partial = T^* L_B$$

with

$$S = \int B[A,A] \in L_B^* \otimes L_B^{\otimes 2}.$$

The functions on L_B are

$$\mathcal{O}_\hbar(L_B) \cong \mathcal{O}_\hbar(\mathfrak{g}^*) = U_\hbar(\mathfrak{g}) = \operatorname{End}_{U\mathfrak{g}}(U\mathfrak{g}, U\mathfrak{g}),$$

where $U_\hbar(\mathfrak{g})$ is the deformation of the enveloping algebra given by setting

$$xy - yx = \hbar[x,y].$$

We could also have considered Neumann BCs: then

$$\mathcal{O}(L_A) \cong C^\bullet(\mathfrak{g}) = \mathbb{R}\operatorname{Hom}_{U\mathfrak{g}}(\mathbb{C}, \mathbb{C})$$

and there is no Π.

Example 5.25. From Chern-Simons theory in the bulk we can find Kac-Moody PVA and affine $W(\mathfrak{sl}_n)$ on the boundary.

From Kapustin-Witten in the bulk we can find Chern-Simons on the boundary.

From the 5d $\mathcal{N} = 2$ twist we can find 4d holomorphic Chern-Simons on the boundary.

5.7 Session 7 (Justin): Tricks with SUSY algebras

Today: Learn how to read off information and make conjectures just by relating different SUSY algebras.

5.7.1 Review of Si's talk

Recall the 2d $\mathcal{N} = (2,2)$ SUSY algebra (complextified):

$$\underbrace{\mathbb{C}^\times}_{Spin(2)} \times \underbrace{(\mathbb{C}^\times_A \times \mathbb{C}^\times_B)}_{R\text{-symmetry}} \ltimes \left(\underbrace{V_{2d}}_{\mathbb{C}_1 \oplus \mathbb{C}_{-1}} \cdot \Pi \left[\mathbb{C}^2 \otimes \underbrace{S^{2d}_+}_{=\mathbb{C}_{+1/2}} \oplus \mathbb{C}^2 \otimes \underbrace{S^{2d}_-}_{\mathbb{C}_{-1/2}} \right] \right)$$

with inner product on the $\mathbb{C}^2$ factors given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The $\mathbb{C}^\times_{A/B}$ act on the $\mathbb{C}^2$ factors in the odd part as follows: there is a $\mathbb{C}^\times$ action on $\mathbb{C}^2 \oplus \mathbb{C}^2$, then $\mathbb{C}^\times_A$ acts antidiagonally and $\mathbb{C}^\times_B$ acts diagonally. Write supercharges $Q^{\beta\gamma}_\alpha$ with:

- α the weight under spin;
- β and γ the weights under R-symmetry;
- There is a constraint: $\alpha\beta\gamma = +$ (where we think of the indices as just recording the sign of the weight).

Si found two topological twists:

$$Q_A = Q^{++}_- + Q^{-+}_+$$
$$Q_B = Q^{++}_- + Q^{+-}_+$$

- Q^{++}_- is fixed under the combined action of $\mathbb{C}^\times$ and *either* $\mathbb{C}^\times_A$ or $\mathbb{C}^\times_B$.
- Q^{-+}_+ is fixed under the combined action of $\mathbb{C}^\times$ and $\mathbb{C}^\times_A$.
- Q^{+-}_+ is fixed under the combined action of $\mathbb{C}^\times$ and $\mathbb{C}^\times_B$.

Remark 5.9. The "special charge" fixed under both the A and B twists,

$$Q_H := Q^{++}_-$$

is a *holomorphic twist*. See the exercises below.

Problem 22. *We have*

$$Im[Q_A, -] = Im[Q_B, -] = V_{2d}$$

and $Im[Q_H, -]$ *is 1d; one thinks of the operator as* $[Q_H, -] =: \partial_{\bar{z}}$.

Remark 5.10. The holomorphic twist is available even in $\mathcal{N} = (0,2)$ SUSY, gives rise to the chiral de Rham complex (in the infinite volume limit where instanton corrections drop out of the Fukaya category).

After twisting, one gets solutions to EOM for the example of a σ-model with target X a Kähler manifold:

- **H-twist:** $\mathrm{Maps}(T[1]\Sigma, T^*[1]X)$. Notation: $\Sigma_{\mathrm{Dol}} = T[1]\Sigma$.
- **B-twist:** $\mathrm{Maps}(\Sigma_{\mathrm{dR}}, T^*[1]X)$, $\Sigma_{\mathrm{dR}} = (T[1]\Sigma, d_{\mathrm{dR}})$.
- **A-twist:** Roughly,[75] $\mathrm{Maps}(\Sigma, T^*_\pi T[-1]X)$, with differential on the target given by bracketing with π the Poisson bivector, $\{\pi, -\}$.

[75] Quote: "The A-model is bullshit." (It *does not behave well in this formalism*, problem has to do with the existence of instanton corrections – suppose one could again consider a large volume limit, but let's just not.)

5.7.2 3d $\mathcal{N} = 4$ SUSY

The complexified SUSY group is

$$Spin(3) \times G_R \ltimes \left(V_{3d} \oplus \Pi\left(W^{3d} \otimes S_{3d}\right)\right)$$

where W^{3d} is an R-symmetry invariant 4d vector space that we must supply with a symmetric form. Let's understand the components of this:

- $Spin(3) = SL_2$
- $S_{3d} = \mathbb{C}^2$ is the defining representation of SL_2
- $V_{3d} \cong \mathrm{Sym}^2(S_{3d})$
- $G_R = (SL_2)_A \times (SL_2)_B$

What about W_{3d}? Let ω be the standard symplectic form on $\mathbb{C}^2$. Then we'll take

$$W^{3d} := (\mathbb{C}^2_A, \omega) \otimes (\mathbb{C}^2_B, \omega) = (\mathbb{C}^2 \otimes \mathbb{C}^2, \omega \otimes \omega).$$

Remark 5.11. As Theo points out: there is an exceptional isomorphism $Spin(4) \cong SL_2 \times SL_2$, and the above is simply telling us how to write down this isomorphism (by choosing $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$).

We'll write supercharges as

$$Q^{\alpha\beta}_\gamma = e^\alpha \otimes e^\beta \otimes e_\gamma$$

where $e^\alpha \otimes e^\beta \in W^{3d} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and $e_\gamma \in S_{3d} = \mathbb{C}^2$.

Example 5.26. Write elements of the symmetric square as $v_{\gamma\gamma'} = e_\gamma e_{\gamma'} \in \mathrm{Sym}^2(S_{3d})$. Then

$$[Q^{\alpha\dot{\alpha}}_\gamma, Q^{\beta\dot{\beta}}_\mu] = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} e_\gamma e_\mu = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} v_{\gamma\mu}.$$

Claim: 2d $\mathcal{N} = (2, 2)$ embeds into 3d $\mathcal{N} = 4$.

$$\begin{tikzcd}
SL_2 & (SL_2)_A \times (SL_2)_B & V_{3d} = \langle e_+e_+, e_+e_-, e_-e_- \rangle & (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (S^{3d} = S^{2d}_+ \oplus S^{2d}_-) \\
\mathbb{C}^\times \arrow[u, hook, "\text{standard max. torus}"] & \mathbb{C}^\times_A \times \mathbb{C}^\times_B \arrow[u, hook, "\text{standard}"] & V_{2d} = \langle e_+e_+, e_-e_- \rangle \arrow[u, hook] & (\mathbb{C}^2 \otimes S^{2d}_+) \oplus (\mathbb{C}^2 \otimes S^{2d}_-) \arrow[u, hook]
\end{tikzcd}$$

The only map not explicitly given in the diagram is the embedding of spin representations – this is given by:

$$\begin{aligned}
e^- \otimes e^+ \otimes e_+ = Q^{-+}_+ &= e^+ \otimes e_+ \qquad & e^+ \otimes e^+ \otimes e_- = Q^{++}_- &= e^+ \otimes e_- \\
e^+ \otimes e^- \otimes e_+ Q^{+-}_+ &= e^- \otimes e_+ \qquad & e^- \otimes e^- \otimes e_- = Q^{--}_- &= e^- \otimes e_-
\end{aligned}$$

Problem 23. *Check the commutation relations. (This involves pairings from 2d and 3d – the reason that we wrote them in the strange way that we did was to make this computation easier.)*

Corollary 5.8. *The 2d $\mathcal{N} = (2, 2)$ supercharges Q_H, Q_A and Q_B all embed into 3d $\mathcal{N} = 4$.*

Problem 24. *Compute $Im[Q_\star, -] \subseteq V_{3d}$ for $\star = H, A, B$.*[76]

Problem 25. *Show that the 3d $\mathcal{N} = 4$ algebra embeds into the 4d $\mathcal{N} = 4$ algebra. (Hint: Use the exceptional isomorphism $Spin(6) \cong SL_4$.)*

[76] Should find that $\mathrm{Im}[Q_{A/B}, -] = V_{3d}$ and $\dim(\mathrm{Im}[Q_H, -]) = 2$, so that Q_H gives a holomorphic-topological twist.

5.7.3 What does this buy you?

If you want to study boundary conditions for 3d $\mathcal{N} = 4$, one good class is those that preserve 2d $\mathcal{N} = (2, 2)$ at the boundary.

Mathematically: There ought to be relations between 3d $\mathcal{N} = 4$ theories and ordinary mirror symmetry, quantum cohomology, etc.

5.8 Session 8 (Dylan): Sequel to Session 6

5.8.1 A series of slogans

Slogan: *Quantum field theories over M are the same as factorization BD_0-algebras over M.*

The factorisation algebra that we are construct associated to a QFT is

$$U \mapsto \mathcal{O}(\mathcal{E}(U))[[\hbar]], \{S, -\} + \hbar\Delta \tag{5.1}$$

A precursor to this was the slogan: *Classical field theories over M are the same as factorization P_0-algebras over M.* This went via the construction

$$U \mapsto \mathcal{O}(\mathcal{E}(U)), \{S, -\} \tag{5.2}$$

Really though, (5.1) is *more* than just a QFT – it is really a quantum field theory together with a classical field theory that is being quantised. To recover the classical theory, we can set $\hbar = 0$. We can't set $\hbar = 1$, but we can formally invert it and pass to the generic fibre.

So *really* our slogan should be: *Quantum field theories over M are the same as factorization E_0-algebras over M.*

Remark 5.12. The passage from $P_0 \to E_0$ is a bit strange, but should be thought of as analogous to the passage from $P_1 \to E_1 \sim \mathrm{Ass}$ (c.f. Rozenblyum).

One more slogan: *A topological field theory over M is the same as a locally constant factorisation algebra over M.*

When $M = \mathbb{R}^n$ this is equivalent to an $\mathcal{E}_n$-algebra – i.e. we are considering algebras over the operad of little n-discs, and the multiplication induced by disc inclusion should be considered a "topological OPE map".

5.8.2 What about boundary conditions?

On $\mathbb{R}_{\geq 0}$ consider a locally constant factorisation algebra. To an open interval not containing $0 \in \mathbb{R}_{\leq 0}$ we obtain a factoristion algebra A – but to an open interval containing 0 we assign a (potentially different) space H, and the factorisation structure/defect OPE tells us that we must have a map

$$\rho : A \otimes H \to H$$

i.e. we obtain a (factorisation) A-module structure on H.

Similarly: suppose we take a 2d theory on $M = \mathbb{R} \geq 0 \times \mathbb{R}$. Then the local constancy condition allows us to define a factorisation algebra on $\{0\} \times \mathbb{R}$ by extending open intervals in $\mathbb{R}$ to open half-discs in $\mathbb{R} \geq 0 \times \mathbb{R}$ (the space we obtain is insenstive to precisely what open set we use to extend the interval). So part of the data is an associative algebra (the factorisation algebra on $\mathbb{R}$).

More generally: a theory on $M = \mathbb{R}_{\leq 0} \times N$ includes as part of its data a factorisation algebra over the boundary N. There is also an "internal" E_1-algebra with E_0-module associated to each open set in N, corresponding to the theory being "topological in the normal direction to the boundary".

$$\begin{tikzcd} A \in \text{ factorization } E_1\text{-algebras over } N \arrow[r, hook] \arrow[d, dotted] & \text{factorization algebras on } M \\ H \in \text{ factorization } E_0\text{-algebra over } N & \end{tikzcd}$$

Remark 5.13. Classical situation: the notion of a theory which is topological normal to the boundary is a factorisation P_1-algebra on the boundary together with maps

$$\begin{tikzcd} \text{factorization } P_1\text{-algebras on } N \arrow[r, hook] \arrow[d, "\text{Lagrangian map}"'] & \text{factorization } P_0\text{-algebras on } M \\ \text{factorization } P_0\text{-algebra on } N & \end{tikzcd}$$

In practice the P_0 widget will actually have a shifted symplectic structure, and so will know precisely what is meant by a "Lagrangian map".

A theory being topological in the normal direction on $M = \mathbb{R}_{\geq 0} \times N$ is equivalent at the classical level to the existence of a splitting

$$\mathcal{E} = \Omega^\bullet_{\mathbb{R}_{\leq 0}} \boxtimes \mathcal{E}^\partial$$

for some factorisation P_1-algebra on the boundary N, $\mathcal{O}(\mathcal{E}^\partial)$.

Example 5.27 (Chern-Simons on $M = \mathbb{R}_{\leq 0} \times \Sigma$)**.** Here

$$\mathcal{E} = \Omega^\bullet_M \otimes \mathfrak{g} = \Omega^\bullet_{\mathbb{R}_{\leq 0}} \boxtimes (\Omega^\bullet_\Sigma \otimes \mathfrak{g}).$$

Note that to see this has the correct shifts, recall the description in the AKSZ formalism $\mathrm{Maps}(M, BG) = \mathrm{Maps}(\mathbb{R}_{\leq 0}, \mathrm{Maps}(\Sigma, BG))$.

Definition 5.8. A (regular embedded)[77] boundary condition for $\mathcal{E}$ is a Lagrangian subbundle $L \hookrightarrow \mathcal{E}^\partial$ over $N = \partial M$.

Remark 5.14. Naively this looks like a bad definition – choosing a subbundle appears to be only allowing Dirichlet boundary conditions. However, since we're working in the BV formalism our bundle $\mathcal{E}^\partial$ also contains antifields, which broadens the types boundary conditions that this definition encompassing. E.g. by the equations of motion, vanishing of an antifield can correspond to vanishing of normal derivatives for a field and so we can obtain Neumann boundary conditions.

If we weren't working in the first order BV formalism, this would be a pretty silly definition.

Question 7. *Given a boundary condition as per above, how do we construct a map from a P_1-algebra over N to a P_0-algebra over N?*

Answer 5.4. $N \supset U \mapsto \mathcal{O}(L(U))$ is a a factorisation P_0-algebra over N.

We need to find a Poisson structure

$$\pi \in \mathrm{PV}_{-1}(L)[1] = \mathcal{O}(T^*L)[1].$$

Equivalently, we are finding $S \in \mathcal{O}(T^*L)[1]$ such that

- $Q = \{S, -\}$ is square-zero,

[77] This is the assumption that deformation to the normal cone holds. Alternatively, think of it as saying that there are no boundary degrees of freedom.

- $\{S^{original} + S, S^{original} + S\} = 0$ (the classical master equation – here $S^{original}$ is the Hamiltonian for the original theory)[78]

Since L is Lagrangian (and we are working in formal geometry) we can think of

$$\mathcal{E}^\partial(T^*L, Q = \{S, -\}) =: T^*_\pi L.$$

Upshot: This description of $\mathcal{E}^\partial$ gives us the Poisson bivector π.

So:

- Given the data $(\mathcal{E}, L)$ we can produce π.
- But also, given (L, π) we can produce[79] $T^*_\pi L =: \mathcal{E}^\partial$, $\mathcal{E} = \Omega^\bullet_{\mathbb{R}} \boxtimes \mathcal{E}^\partial$. This is the "Universal Bulk Theory" – a universal theory with given boundary theory and a canonical boundary condition.

5.8.3 Interval compactifications

What happens if you have a bulk theory which is topological in the normal direction on $[0,1] \times N$ and two boundary conditions? Since we are topological in the normal direction, we should be able to "squish the theories together to produce a new theory".

What do we have?

- A factorisation E_1-algebra A over N.
- Two factorisation modules M_1 and M_2.
- So we just take $M_1 \otimes_A M_2$ – this is our new factorisation algebra (so our new "squished together" theory).

There is a dual fibre product picture:

$$L \times_{\mathcal{E}^\partial} L = L \times_{\mathcal{E}^\partial} \mathcal{E} \times_{\mathcal{E}^\partial} L.$$

Note that if $\mathcal{E}^\partial$ is 0-shifted, the resulting fibre product is (-1)-shifted.

Example 5.28. We could take

$$\begin{aligned}
\mathcal{E} &= T_0 \operatorname{Maps}([0,1]_{\mathrm{dR}} \times \mathbb{R}_{\mathrm{dR}}, T^*[1]X) \\
\mathcal{E}^\partial &= T_0 \operatorname{Maps}(\mathbb{R}_{\mathbb{R}}, T^*[1]X) \\
L &= T_0 \operatorname{Maps}(\mathbb{R}_{\mathrm{dR}}, X) \subset \mathcal{E}^\partial
\end{aligned}$$

Then

$$L \times_{\mathcal{E}^\partial} L = T_0 \operatorname{Maps}(\mathbb{R}_{\mathrm{dR}}, T^*X).$$

For another boundary condition: take a 1-form, assume of the form dW (exact), and take

$$\tilde{L} = T_0 \operatorname{Maps}(\mathbb{R}_{\mathrm{dR}}, \operatorname{Graph}(dW)).$$

Then

$$L \times_{\mathcal{E}^\partial} \tilde{L} = T_0 \operatorname{Maps}(\mathbb{R}_{\mathrm{dR}}, \operatorname{Crit}(W))$$

is SQM with superpotential W.[80]

[78] This is equivalent to π being a Poisson bivector.

[79] $T^*_\pi L$ is the derived Poisson centre of L – the P_0 version of Hochschild cochains.

[80] Up to (nontrivial!) questions of the state-operator correspondence/existence of a Serre functor on the category of boundary conditions, this is the procedure of taking homs between boundary conditions.

Now: we could *also* imagine the situation where the theory continues to the left of the boundary at 0 – then there would be another algebra B, and M_1 would be a B-A-bimodule.

Dylan finished by drawing the following picture and giving an inspiring speech about how we can get cool maths from string theory (Figure 48).

5.9 Session 9 (Justin): Defects in higher dimensional TFTs

5.9.1 Last time

There are embeddings
$$2d^{\mathcal{N}=(2,2)} \subset 3d^{\mathcal{N}=4} \subset 4d^{\mathcal{N}=4}$$

There are three square-zero supercharges we can use to twist.

Q: How many translations are in the cohomology?

	$2d$	$3d$	$4d$
Q_H	1	2	3
Q_A	2	3	4
Q_B	2	3	4

5.9.2 Warmup: B-model

Consider the B-model on X Calabi-Yau, $\mathcal{O}_X \simeq \omega_X[-d]$.

Classical EOM:
$$EOM(\Sigma) = \mathrm{Maps}(\underbrace{\Sigma_{\mathrm{dR}}}_{(T[1]\Sigma, d_{\mathrm{dR}})}, T^*X)$$

Want to produce a 2d TQFT,
$$Z : 2-\mathrm{Cob} \to \mathrm{Cat}$$

Ansatz: $Z(M) = GQ(EOM(M \times \mathbb{R}^d))$ where $\dim(M) = d = 2$. (Geometric Quantization)

Okay: but how do I get (higher) categories out of geometric quantization?

The AKSZ formalism tells us that
$$\mathrm{Maps}\left(\underbrace{M^d_{\mathrm{dR}}}_{\text{compact}} \times \underbrace{\mathbb{R}^{2-d}_{\mathrm{dR}}}_{\text{irrelevant}}, T^*[1]X\right)$$

has a symplectic form of degree $1-d$. So:

- If I plug in a *point* then I get a symplectic form of degree 1, and I want to produce a category. Hmm.
- If I plug in a *circle* then I get a symplectic form of degree 0, and I want to produce a vector space. I know how to do this – usual geometric quantisation.
- If I plug in a *surface* then I get a symplectic form of degree -1, and I want to produce a number.
- So to an n-shifted symplectic stack I want to associate an n-category.

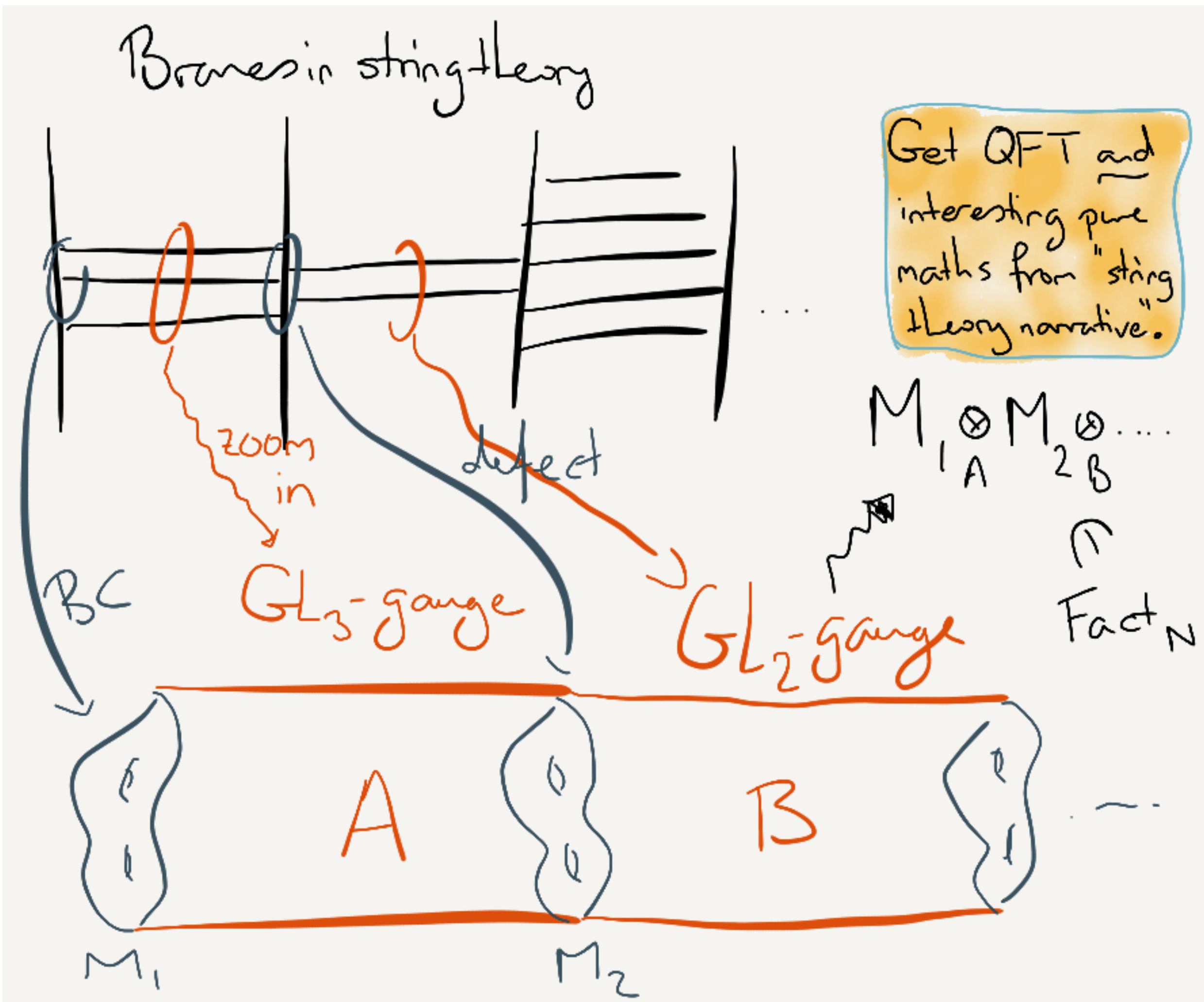


Figure 48: Gauge theories and boundary conditions from branes in string theory.

If I consider manifolds with boundary, with "inward" and "outward" pointing assignments, we obtain a Lagrangian correspondence

$$\begin{tikzcd} & EOM(M) \arrow[dl] \arrow[dr] & \\ EOM(\partial_{in}M) & & EOM(\partial_{out}M) \end{tikzcd}$$

Rather than going into the details for how we geometrically quantize n-shifted symplectic objects, Justin presents the following –

Ansatz:[81]

$$GQ(T^*[n]X) = n\mathrm{QCoh}(X)$$

I.e. want:

$$\begin{array}{rr} T^*[-1]X & 1 \\ T^*X & \mathcal{O}_X \\ T^*[1]X & \mathrm{QCoh}(X) \\ T^*[2]X & \mathrm{QC\,Shv\,Cat}(X) = (\mathrm{QCoh}(X), \otimes)\text{-Mod} \end{array}$$

Remark 5.15. For $T^*[2]X$ we really want the KRS[82] 2-category, but so far this is our best approximation.

So now:

$$\begin{aligned} Z(\mathrm{pt}) &= GQ(EOM(\mathrm{pt})) = GQ(T^*[1]X) = \mathrm{QCoh}(X) \\ Z(S^1) &= GQ(EOM(S^1)) = GQ(\mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[1]X)) = \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) = (\mathrm{PV}(X))^\vee \simeq \mathrm{PV}(X)[d] \end{aligned}$$

Above we have use that

$$\mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[1]X) = T^* \underbrace{\mathrm{Maps}(S^1_{\mathrm{dR}}, X)}_{\mathcal{L}X = T[-1]X}$$

where the equality $\mathcal{L}X = T[-1]X$ holds if X is actually a scheme, and also that X is Calabi-Yau (to identify PV with a shift of its linear dual).

Remark 5.16. In general we need to actually also consider the data of a framing[83] on our spacetime manifold, so we really have $\pi_1(SO(2)) = \mathbb{Z}$ worth of circles in our theory. Label them as $S^1_{(n)}$ – $n = 0$ corresponds to the blackboard framing, $n = 1$ is cylinder framing, etc. Then we assign

$$Z(S^1_{(n)}) = \Gamma(\mathcal{L}X, \pi^! \omega_X^{\otimes n})$$

where $\pi : \mathcal{L}X \to X$ is the "basepoint map". Our Calabi-Yau condition means that we don't need to worry about these subtleties.

Remark 5.17. Justin has also lied about IndCoh versus QCoh subtleties.

5.9.3 Let's talk about line operators in 2d!

The link of a line in 2d is S^0 (i.e. two points). Let's rewrite the theory as maps into maps from S^0. Rewrite the complement of a straight line in $\mathbb{R}^2$ as the product

$$\mathbb{R}_{>0} \times S^0 \times \mathbb{R}$$

[81] Corresponds to polarising by the fibres of the cotangent bundle.
[82] Kapustin-Rosansky-?
[83] In this situation we mean a normal framing of M^d in $M^d \times \mathbb{R}^{d-2}$.

and use the mapping stack adjunction to write the space of fields in the B-model as

$$\mathrm{Maps}(\mathbb{R}_{\geq 0}\times\mathbb{R}_{\mathrm{dR}}, \underbrace{\mathrm{Maps}((S^0)_{\mathrm{dR}}, T^*[1]X)}_{T^*[1]X\times X = EOM(S^0)})$$

So the category of line operators is

$$\mathrm{QCoh}(X\times X) = Z(S^0).$$

What is the trivial line?

$S^0 \hookrightarrow I$, so we get

$$\begin{tikzcd} & EOM(I) \arrow[r] & EOM(S^0) \arrow[d, equal] \\ & T^*[1]X \arrow[r, "\Delta"] \arrow[d, equal] & T^*[1](X\times X) \\ & N^*_X[1](X\times X) & \end{tikzcd}$$

Now we're trying to quantise a *Lagrangian*. We haven't done this before, so again rather than go into detail Justin will make an –

Ansatz: Given $f: Y\to X$, $GQ(N^*_Y[n]X) = f_*(n\mathcal{O}_Y)$, where

$$1\mathcal{O}_Y = \mathcal{O}_Y, \qquad 2\mathcal{O}_Y = \mathrm{QCoh}(Y), \qquad \cdots$$

So by our ansatz,

$$GQ(I) = \Delta_*\mathcal{O}_X,$$

thought of as an integral kernel.

Remark 5.18. Should also note that the trivial line should be the monoidal unit for the tensor structure on lines coming from the "open pair of pants".

What does it mean to compactify BCs on an S^0?

AKSZ tells us that

$$N^*_{\mathrm{Maps}(M,Y)}[1]\,\mathrm{Maps}(M,X) = \mathrm{Maps}(M, N^*_Y[1]X) \to \mathrm{Maps}(M, T^*[1]X)$$

is Lagrangian. The geometric quantisation of this Lagrangian is $f_*(\mathcal{O}_{Y\times Y})$.

Remark 5.19. Classical boundary conditions are (basically) only Lagrangians in the target. (Further discussion in this digression was deferred to later.)

5.9.4 3d $\mathcal{N}=4$ σ-models

3d $\mathcal{N}=4$ σ-models are defined by T^*X thought of as a holomorphic symplectic variety/hyperkähler manifold. In a slight generalisation of the B-model[84]

$$EOM^{3d}_B(M^3) = \mathrm{Maps}(M^3_{\mathrm{dR}}, T^*[2]X).$$

We want to make an ansatz as to what the TQFT $Z: 3-\mathrm{Cob}\to 2\,\mathrm{Cat}$ will be.

Local operators.

[84] Which has a $\mathbb{Z}/2\mathbb{Z}$ grading problem.

A local operator is an insertion at a point. The link of a point in 3d is an S^2. So (leaving implicit the crossing with extra real dimensions to bring us to the correct dimension for the QFT),

$$Z(S^2) = GQ\left(\mathrm{Maps}(S^2_{\mathrm{dR}}, T^*[2]X)\right)$$

whigh if X is a scheme is just $T^*T^*[2]X$, so that $Z(S^2) = \mathcal{O}_{T^*[2]X}$ – not so interesting.

Line operators.

The link of a line in 3d is an S^1; do our rewriting trick for the complement of the line,

$$\mathbb{R} \times \mathbb{R}_{\geq 0} \times S^1$$

and then use the mapping stack adjunction to rewrite

$$Z(S^1) = GQ(\mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[2]X)).$$

We usually have that

$$\mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[2]X) = T^*[1]\,\mathrm{Maps}(S^1_{\mathrm{dR}}, X),$$

and if X is a scheme we have

$$\mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[2]X) = T^*[1]T^*[2]X.$$

So there are two (obvious) possibilities for which polarisation to choose in order to geometrically quantise, and we get[85]

$$Z(S^1) = \begin{cases} \mathrm{QCoh}(T^*[2]X) \\ \mathrm{QCoh}(T[-1]X) \end{cases}$$

However, according to BZ-Nadler these two categories are equivalent (Koszul dual).

Moral: In ordinary quantum mechanics, there is a Fourier transform relating two different polarisations of $\mathbb{R}^2$,

$$L^2(\mathbb{R}_p) \simeq L^2(\mathbb{R}_q).$$

In the categorified story, the Fourier transform implies Koszul duality.[86]

What is the trivial line?

Consider $S^1 \hookrightarrow D^2$. In one choice of polarisation the map

$$T^*[2]X \to \mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[2]X) = T^*[1]T^*[2]X$$

is just the inclusion of the zero section. So in this polarisation the trivial line is just $\mathcal{O}_{T^*[2]X} \in \mathrm{QCoh}(T^*[2]X)$. In the other polarisation,

$$T^*[2]X \to \mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[2]X) = T^*[1]\,\mathrm{Maps}(S^1_{\mathrm{dR}}, X) \leftarrow N^*_{\mathrm{Maps}(D^2_{\mathrm{dR}}, X)}[1]\,\mathrm{Maps}(S^1_{\mathrm{dR}}, X)$$

and the trivial line corresponds to $\mathcal{O}_X \in \mathrm{QCoh}(\underbrace{T[-1]X}_{\mathcal{L}X})$. This is good – these two objects are in fact exchanged by Koszul duality.

Remark 5.20. If you're careful with the functional analysis, you'll find that Justin told a lie! The two polarisations we've chosen don't match, and correcting for this one finds that the formula for the trivial line in the second (loop space) perspective is off by some powers of the canonical bundle.

[85] Up to issues of functional analysis of sheaves.

[86] This will make BZ very happy.

5.9.5 3d A-model

For any algebraic curve C, some version of the A-twist has space of fields

$$T^*[-1]\,\mathrm{Maps}(C^\times M^1_{\mathrm{dR}}, X_{\mathrm{dR}}).$$

To find local operators in this story, we actually need a *holomorphic* version of the sphere: write

$$S^2 = D \cup_{D^\times} D$$

where $D = \mathrm{Spec}\mathbb{C}[[t]]$ and $D^\times = \mathrm{Spec}\mathbb{C}((t))$. BFN tell us that the space of local operators in this theory is $H^\bullet(\mathrm{Maps}(\mathbb{B}, X))$ where $\mathbb{B}$ is a particular space.

References

[Fed94] Boris V. Fedosov. A simple geometrical construction of deformation quantization. *J. Differential Geom.*, 40(2):213–238, 1994.