# FOURIER-MUKAI TRANSFORM OF AN ABELIAN SUBVARIETY.

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Consider an abelian subvariety $A \subset B$, so that there is an exact sequence of abelian varieties

$$0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$$

This gives as exact sequence of dual abelian varieties

$$0 \longrightarrow C^D \xrightarrow{p^D} B^D \xrightarrow{j^D} A^D \longrightarrow 0$$

Denote by $\mathrm{QC}(X)$ the symmetric monoidal dg category of quasicoherent sheaves on $X$ (assume that we are working in characteristic 0). The monoidal structure on QC is assumed to be tensor product of sheaves "$\otimes$" – when I wish to work with the monoidal convolution product "$\star$" I will make this explicit in the notation.

Observe that $j : A \to B$ is an affine map, i.e. it is the relative spectrum of the quasicoherent sheaf of $\mathcal{O}_B$-algebras

$$\mathcal{A}_A := j_*(\mathcal{O}_A) \in \mathrm{Alg}(\mathrm{QC}(B))$$

and so we have an equivalence

$$j_* : \mathrm{QC}(A) \simeq \mathrm{Mod}^{\otimes}_{\mathcal{A}_A}(\mathrm{QC}(B)).$$

Under the Fourier-Mukai transform for $B$ this is sent to

$$FM_B : \mathrm{Mod}^{\otimes}_{\mathcal{A}_A}(\mathrm{QC}(B)) \simeq \mathrm{Mod}^{\star}_{FM_B(\mathcal{A}_A)}(\mathrm{QC}(B^D)),$$

i.e. modules for $FM_B(\mathcal{A}_A)$ under the convolution product $\star$.

**Proposition 0.1.** *The Fourier-Mukai transform of $\mathcal{A}_A$ is $\mathcal{A}_{C^D} = p^D_*(\mathcal{O}_{C^D})$. Furthermore, there is a commuting diagram of equivalences*

$$\begin{array}{ccc} \mathrm{Mod}^{\otimes}_{\mathcal{A}_A}(\mathrm{QC}(B)) & \xrightarrow[\sim]{FM_B} & \mathrm{Mod}^{\star}_{\mathcal{A}_{C^D}}(\mathrm{QC}(B^D)) \\ {\scriptstyle j_*}\big\uparrow{\scriptstyle \sim} & & {\scriptstyle (j^D)^*}\big\uparrow{\scriptstyle \sim} \\ \mathrm{QC}(A) & \xrightarrow[\sim]{FM_A} & \mathrm{QC}(A^D) \end{array}$$

*Proof.* Since $j^D$ is a smooth map between abelian varieties with fibres torsors for another abelian variety, it has trivial relative canonical bundle. Hence the $*$- and !-pullbacks of $j^D$ differ only by a shift in degree. The diagram will follow from the standard canonical equivalences relating Fourier-Mukai transforms

$$FM_B \circ j_* \simeq (j^D)^* \circ FM_A \tag{1}$$

once we have proven that $FM_B(\mathcal{A}_A) \simeq \mathcal{A}_{C^D}$ and shown that the algebra $(\mathcal{A}_{C^D}, \star)$ controls the descent monad for $j^D : B^D \to C^D$.

To prove the first part, use (1) to obtain

$$FM_B(\mathcal{A}_A) = FM_B(j_*\mathcal{O}_A) = (j^D)^*(FM_A(\mathcal{O}_A)) = (j^D)^*(\mathcal{O}_{\hat{0}})$$

where $\mathcal{O}_{\hat{0}}$ is the skyscraper sheaf supported at the identity element $\hat{0} \in A^D$. The pullback is therefore the structure sheaf of the kernel of $j^D$, i.e.

$$(j^D)^*(\mathcal{O}_{\hat{0}}) = (p^D)_*(\mathcal{O}_{C^D}) = \mathcal{A}_{C^D}.$$

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Since $\mathcal{A}_A$ was an algebra with respect to the $\otimes$-monoidal structure, the Fourier-Mukai transform is an algebra with respect to the $\star$-monoidal structre.

To simplify notation,[1] let us prove the corresponding statement for descent along $p : B \to C$. Consider the adjunction

$$p_* : \mathrm{QC}(B) \longleftrightarrow \mathrm{QC}(C) : p^!$$

and the corresponding monad $T = p^! p_*$. I claim that $T(-) \simeq \mathcal{A}_A \star (-)$. To see this, apply a Fourier-Mukai transform to both sides of the equation:

$$\begin{aligned} FM_B(p^! p_*(-)) &= p_*^D(FM_C \circ p_*(-)) = p_*^D(p^D)^*(FM_B(-)) \\ &= p_*^D(\mathcal{O}_{C^D}) \otimes FM_B(-) = FM_B(\mathcal{A}_A) \otimes FM_B(-) = FM_B(\mathcal{A}_A \star -) \end{aligned}$$

Taking the inverse Fourier-Mukai transform then yields the desired result. To finish, let us show that we are in a situation where the $\infty$-categorical Barr-Beck theorem applies. It is sufficient to show that $p^!$ is conservative and cocontinuous:

- Since $p$ is a smooth map (in particular, faithfully flat), $p^*$ – hence $p^!$ – is conservative.
- Since $p^!$ is (up to a shift) equivalent to the left adjoint $p^*$, it is cocontinuous.

We conclude that $p^! : \mathrm{QC}(C) \simeq \mathrm{Mod}^{\star}_{\mathcal{A}_A}(\mathrm{QC}(B))$. □

[1] I.e. to avoid annoying proliferation of superscript "$D$".