15/08/13

Atiyah-Singer Index Theorem Seminar.

Hirzebruch-Riemann-Roch (Lee Cohn).

Chapter 13!

Let M be compact, even-dim, spin,
let Δ be its associated spin bundle,
and let D be the Dirac operator.

Last time: $\mathrm{Ind}(D) = \int_M \hat{A}(TM) \wedge ch(S/\Delta)$.

Remark: $\hat{A}$-genus is the Pontrjagin genus $z \overset{g(z)}{\longmapsto} \dfrac{\frac{\sqrt{z}}{2}}{\sinh(\frac{\sqrt{z}}{2})}$

$$\hat{A}_4 = \frac{-p_1}{24}, \quad \hat{A}_8 = \frac{-4p_2 + 7p_1^2}{5760}, \ldots$$

Easy application (Lichnerowicz):

Let M-spin and $\int_M \hat{A}(TM) \neq 0$. Then M has no metric of strictly positive curvature.

Proof:

Weitzenbock: $D^2 = \nabla^* \nabla + \frac{1}{4}K$. So if $K > 0$, Bochner arg implies

$$\ker D = \ker D^2 = 0 \Rightarrow \mathrm{Ind}(D) = 0. \qquad \square$$

The Signature Theorem.

Let M be oriented, compact, $2m$-dim; D the de Rham operator and $i^m\omega$ grading op (ω - vol form).

Let D be the associated Dirac op, called the Signature operator.

$$\varepsilon = i^m\omega = i^{m+p(p-1)} * \text{ on } p\text{-forms.}$$

Prop: The index of the sig. op. is equal to the topological index. That is, #pos e-vals − #neg e-vals on the sig form on $H^m(M,\mathbb{R})$.

Proof:

Write $\overset{D^2}{\Delta} = \Delta^+ \oplus \Delta^-$ rel. to ε. Then $\mathrm{Ind}(D) = \dim(\ker(D^+))$
$-\dim(\ker(D^-))$

Let Δ_l^+, Δ_l^- denote the restriction of Δ^+, Δ^- to ε-invariant subspaces $C^\infty(\Lambda^l T^*M \oplus \Lambda^{2m-l} T^*M)$, $0 \le l < m$ and $C^\infty(\Lambda^m T^*M)$ for $l = m$.

If $l < m$ and $\alpha \in \ker(\Delta_l^+)$ then $\alpha = \beta + \varepsilon(\beta)$ where β is a monic l-form. Then $\beta - \varepsilon(\beta) \in \ker(\Delta_l^-)$, so

$$\ker(\Delta_l^+) \cong \ker(\Delta_l^-) \quad \text{for } l < m.$$

So, $\mathrm{Ind}(D) = \dim(\ker(\Delta_m^+)) - \dim(\ker(\Delta_m^-))$
$= \dim \mathcal{H}^+ - \dim \mathcal{H}^-$

where $\mathcal{H}^\pm$ are ± 1 e-spaces of $\varepsilon = *$ on harmonic m-forms.

The quadratic form $\int \alpha \wedge \alpha$ is pos def on $\mathcal{H}^+$ and neg def on $\mathcal{H}^-$. So $\mathrm{Ind}(D)$ is topological signature.

$\square$

Aux calc: $\Delta_\ell^+(\beta + \varepsilon(\beta)) = \Delta_\ell^+(\beta) + \Delta_\ell^+(i^{m+\ell(\ell-1)} * \beta) = \Delta_\ell^+(\beta) + \Delta_\ell^+(*\beta) = 0$

$$\Rightarrow \beta + \varepsilon(\beta) \in \ker(\Delta_\ell^+).$$

Now, calculate $\mathrm{Ind}(D)$ of the signature operator.
Let $S = \Lambda^\bullet T^* M$ on which D acts.

Lemma: $\mathrm{ch}(S/\Delta) = 2^m \widetilde{\mathcal{G}}(TM)$.
pont. genus of $z \mapsto \cosh(\frac{1}{2} z)$.

Proof: $S \cong Cl(TM) = \Delta \otimes \Delta^*$ locally. Thus,

$$\mathrm{ch}(\Delta^m) = 2^m \widetilde{\mathcal{G}}(TM) \text{ by previous talk.}$$

$\square$

Def$^{\underline{n}}$: $\mathcal{L}$-class is Pont. genus associated to $z \mapsto \frac{\sqrt{z}}{\tanh(\sqrt{z})}$.

Signature Theorem:

$$\mathrm{Ind}(D) = \mathrm{Sign}(M) = \int_M \mathcal{L}(TM).$$

Here $\dim M = 2m$, m is even.

Proof:

$$\mathrm{Ind}(D) = 2^m \int_M \hat{A}(TM)G(TM),$$

$$\frac{\frac{\sqrt{z}}{2}}{\sinh\left(\frac{\sqrt{z}}{2}\right)} \cdot \cosh\left(\frac{\sqrt{z}}{2}\right) = \frac{\frac{\sqrt{z}}{2}}{\tanh\left(\frac{\sqrt{z}}{2}\right)},$$ change of vars cancels the 2^m.

$\square$

Remark: $\mathcal{L}_4 = \frac{p_1}{3}$, $\mathcal{L}_8 = \frac{7p_2 - p_1^2}{45}$, ...

So $\mathcal{L}_4 = -8\hat{A}_4$.

So the signature of a spin 4-mfld is divisible by 16.

Hirzebruch-Riemann-Roch Theorem.

M^n - complex manifold, $TM\otimes\mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ and $T^{1,0}M \underset{\mathbb{C}\text{-bdl}}{\cong} TM$.

$S = \Lambda^*(T^{0,1}M)^*$ carries spin rep. of $Cl(TM)$.

M also has a spin^c-structure w/ fundamental line bundle given by

$$L = \mathrm{Hom}_{Cl}(\bar{S}, S) = \mathrm{Hom}_{Cl}(\Lambda^\bullet(T^{1,0})^*, \Lambda^\bullet(T^{0,1}M)^*).$$

$1 \in \Lambda^0(T^{1,0})^*$ is mapped to a top form. So $L \cong \Lambda^n(T^{0,1}M)^*$.

Recall: $Spin^c(k) \subset Cl(k)\otimes\mathbb{C}$ generated by $Spin(k)$ and $S^1 \subset \mathbb{C}$.

$$Spin^c(k) \cong Spin(k) \times_{\{\pm 1\}} S^1.$$

$$0 \to \mathbb{Z}/2\mathbb{Z} \to Spin^c(k) \to SO(k)\times S^1 \to 1$$

Def: A $Spin^c$-structure on M is a principal $Spin^c(n)$ bundle $\tilde{E}$ which is a double covering of $E\times L$ where L is a principal S^1-bundle, thought of as a Hermitian line bundle.

Claim: Now if $\begin{smallmatrix} S \\ \downarrow \\ M \end{smallmatrix}$ is Clifford s.t. each fibre is a copy of spin rep., then M admits a $Spin^c$-structure where $L = \mathrm{Hom}_{Cl}(\bar{S}, S)$.

Lemma: For the Clifford bundle S, the rel. Chern character $Ch(S/\Delta)$ is equal to the Chern genus of $T^{1,0}M$ associated to $z \mapsto e^{-z/2}$.

Proof:

$c_1(L) = c_1((T^{0,1}M)^*) = c_1(T^{1,0}M)$, so $ch(L) = e^{-c_1}$.

Also, the twisting curvature of S is $\frac{1}{2}$ the twisting curvature of L — why? — and so the relative Chern char of S is $e^{-c_1/2}$.

$\square$

Hirzebruch-Riemann-Roch Theorem:

Let W be a holomorphic vector bundle on $M_{\mathbb{C}}$. Then

$$\sum_k (-1)^k \dim H^{0,k}(W) = \int_M Td(T^{1,0}) ch(W).$$

Td is Chern genus associated to $z \mapsto \frac{z}{e^z - 1}$.

"Proof":

Give W a Hermitian metric and compatible connection. Consider Clifford bundle $S \otimes W$. If M is Kähler, $D = \sqrt{2}(\bar\partial + \bar\partial^*)$ and its Index is $\chi(\Omega^{0,\bullet}(W))$.

In general, $\sqrt{2}(\bar\partial + \bar\partial^*) = D + A$ where $A \in End(S)$. The homotopy $D + tA$ shows index of D is still equal to $\chi(\Omega^{0,\bullet}(W))$.

$$\operatorname{Ind}(D) = \int_M \hat{A}(TM)\,\mathrm{ch}(S/\Delta)\,\mathrm{ch}(W)$$

and $\hat{A}(TM)\,\mathrm{ch}(S/\Delta)$ is Chern genus associated to

$$\frac{\left(\frac{z}{2}\right)}{\sinh\left(\frac{z}{2}\right)}e^{-z/2} = \frac{z}{e^z - 1}.$$

<u>Remark:</u> Any metric on a compact Riemann surface is Kähler.

Break time!

The analytic index of a divisor on a compact Riemann surface.

Def$^{\underline{n}}$: Let D be a divisor on X, $D = \sum n_p \cdot p$. Associate line bundle $\mathcal{L}_D$ to D. Choose some $\{U_i\}$ an open cover of X and g_i on U_i such that

$$\mathrm{ord}_{g_i}(p) = n_p \quad \forall p \in U_i.$$

The $\{g_i\}$ have the same zeros and poles on $U_i \cap U_j$, $\frac{g_i}{g_j}$ is a nonzero holomorphic functions on $U_i \cap U_j$. So $\{\{U_i\}, \{g_i\}\}$ give the data of a line bundle.

Remark: Let $\mathcal{O}_D$ be a sheaf,

$$\mathcal{O}_D(U) = \{\phi \text{ meromorphic} \mid D|_U \pm \mathrm{div}(\phi) \geq 0\}.$$

sgn?

Given $\begin{array}{c}\mathcal{L}_D \\ \downarrow \\ X\end{array}$ we can form Dolbeault complex $\mathcal{E}^{p,q}(\mathcal{L}_D)$ and $\bar\partial_D$.

$$\bar\partial_D\left(\sum f_i \otimes \beta_i\right) = \sum_i f_i \otimes \bar\partial(\beta_i)$$ where β_i is a (p,q)-form and f_i is a section of $\mathcal{L}_D$.

Claim: $\bar\partial_D$ are elliptic.

Observe,

$$0 \to \mathcal{O}_D \to \underset{\mathcal{E}_D^{0,0}}{\underset{\parallel}{\mathcal{E}^{0,0}(\mathcal{L}_D)}} \xrightarrow{\bar\partial_D} \underset{\mathcal{E}_D^{0,1}}{\underset{\parallel}{\mathcal{E}^{0,1}(\mathcal{L}_D)}} \to 0$$ is exact by Dolb't lemma.

By LES argument and vanishing of higher cohom's,

$$0 \to \Gamma(\mathcal{O}_D) \to \Gamma(\mathcal{E}_D^{0,0}) \to \Gamma(\mathcal{E}_D^{0,1}) \to H^1(X,\mathcal{O}_D) \to \underset{\substack{\parallel \\ 0}}{H^1(X,\mathcal{E}_D^{0,0})} \to$$

$$\ker(\bar\partial_D) = H^0(X,\mathcal{O}_D)$$
$$\operatorname{coker}(\bar\partial_D) = H^1(X,\mathcal{O}_D).$$

Upshot: $\operatorname{Ind}(\bar\partial_D) = h^0(X,\mathcal{O}_D) - h^1(X,\mathcal{O}_D)$.

Theorem (Riemann-Roch):

$$h^0(X,\mathcal{O}_D) - h^1(X,\mathcal{O}_D) = 1 - g + \deg D.$$

Want to show RHS is topological information.

Goal: $\int_X \left(1 + c_1(\mathcal{L}_D) + \frac{1}{2}c_1(TX)\right) = 1 - g + \deg D.$

Note: $Ch(\mathcal{L}_D)Td(TX) = (1 + c_1(\mathcal{L}_D))(1 + \frac{1}{2}c_1(TX))$
$\qquad = 1 + c_1(\mathcal{L}_D) + \frac{1}{2}c_1(TX).$

This really is our Hirzebruch-Riemann-Roch.

1. $\int_X c_1(\mathcal{L}_D) =: \deg(\mathcal{L}_D) = \deg(D)$ ref: p. 141 of Griffiths & Harris

   Idea: $c_1(\mathcal{L}_D) \in H^2(X,\mathbb{Z}) \xrightarrow{[X]} \mathbb{Z}$

   ↳ Poincaré dual to D thought of as $H^0(X;\mathbb{Z})$

② $\deg(TX) = 2-2g \Rightarrow \int_X \frac{1}{2} c_1(TX) = 1-g$.

So why is $\deg(TX) = 2-2g$ or why is $\deg(T^*X) = 2g-2$?

① Cheating (Gauss-Bonnet)

$$\deg(TX) = \frac{1}{4\pi}\int_X K \cdot dA = \chi(X) = 2-2g$$

BUT THIS IS THE INDEX THM

② Otherway: Riemann-Hurwitz formula.

Let $f: S \to S'$ be a holomorphic map b/w compact Rieman surfaces of degree n.

For any $p \in S$, take coordinate z near p and w near $f(p)$ such that f is given by $w = z^\nu$.

- ν is called ramification index of f at p
- p is called a branch point if $\nu(p) > 1$.
- The branch locus of f is the divisor

$$B = \sum_{p \in S} (\nu(p)-1) \cdot p \quad \text{or its image} \quad B' = \sum_{p \in S} (\nu(p)-1) f(p).$$

For any $p \in S'$, $f^*(p) := \sum_{q \in f^{-1}(p)} \nu(q) \cdot q$ and $\deg f^*(p) = \sum_{q \in f^{-1}(p)} \nu(q) = n$.

Upshot: Away from Branch locus f is a covering map, at a branch pt $p \in S$ of ramification index k, k-sheets of the covering space come together.

Can relate genus of S to genus of S'.

Take a triangulation of S' in which every branched pt lies in a vertex of our triangulation. Let c_0, c_1, c_2 be # 0 cells, 1 cells, 2 cells.

Then we have in S nc_1 1-cells, and nc_2 2-cells, but

$$n \cdot c_0 - \sum_{q \in S}' (v(q)-1) \text{ is } \# \text{0-cells in } S.$$

Then $\chi(S) = n\chi(S') - \sum_{q\in S}' (v(q)-1)$,

$$\Rightarrow g(S) = n(g(S')-1) + 1 + \frac{1}{2}\sum_{q \in S}(v(q)-1).$$

$\boxtimes$

Now, let $\mathbf{W}$ be a 1-form on S', $\mathbf{W} = \frac{g(w)}{h(w)}\,dw$.

For $p \in S$ of index v, f is $w = z^v$.

Thus $f^*\mathbf{W} = \frac{g(z^v)}{h(z^v)}\,dz^v = v z^{v-1} \frac{g(z^v)}{h(z^v)}\,dz$.

So $\operatorname{ord}_p(f^*\mathbf{W}) = v \cdot \operatorname{ord}_{f(p)}(\mathbf{W}) + (v-1)$.

Thus, for our divisor, $(f^*\mathbf{W}) = f^*\omega + \sum_{p \in S} (v(p)-1) \cdot p$.

Thus, $K_S = f^* K_{S'} + B$, $\deg(K_S) = n \deg K_{S'} + \sum_{p \in S} (v(p)-1)$.

Recall: Any Rieman surface has a holomorphic map to $\mathbb{P}^1$.

Why? If f is a global meromorphic f^{nc} written locally as g/h. Then

$$p \mapsto [g(p) : h(p)] \text{ is such a map.}$$

Let $f: S \to \mathbb{P}^1$ be such a map. Since $\chi(\mathbb{P}^1) = 2 = -\deg(K_{\mathbb{P}^1})$,

$$\begin{aligned}
\chi(S) &= n\chi(\mathbb{P}^1) - \sum_{p \in S} (v(p)-1) \\
&= -n \deg(K_{\mathbb{P}^1}) - \sum_{p} (v(p)-1) \\
&= -\deg K_S \quad \text{by our triangulation argument.}
\end{aligned}$$

Thus, for any S, $\deg K_S = -\chi(S) = 2g-2$.

$\boxed{KZ}$