08/08/13

Atiyah-Singer Index Theorem Seminar.

The Index Theorem (Valentin Zakharevich).

We begin with the following lemma, which we will prove later:

Lemma:

$$h_t\left(\sigma_0(\Theta_0)+t\sigma_2(\Theta_1)+\cdots+t^{\frac{n}{2}}\sigma_n(\Theta_{\frac{n}{2}})\right)$$
$$=(4\pi t)^{\frac{-n}{2}}\det{}^{\frac{1}{2}}\left(\frac{tR/2}{\sinh\left(\frac{tR}{2}\right)}\right)\exp\left(\frac{-1}{4t}\left\langle\frac{tR}{2}\coth\left(\frac{tR}{2}\right)x,x\right\rangle\right)\exp(-tF)$$

where

$$h_t=\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{r^2}{4t}\right)\in\mathbb{C}[[TM]]\otimes\Lambda^{\bullet}TM\otimes\mathrm{End}(S)$$

$$r^2=x_1^2+\cdots+x_n^2.$$

Proof of index theorem assuming the above lemma.

Theorem (Atiyah-Singer Index Theorem):

Let M be a compact oriented even dimensional manifold, and let S be a Clifford bundle over M with canonical grading. Then

$$\mathrm{Ind}(D) = \int_M \hat{A}(TM) \wedge ch(S/\Delta)$$

where

$$ch(S/\Delta) = tr^{S/\Delta} \exp\left(\frac{-F^S}{2\pi i}\right)$$

and $\hat{A}$ is the Pontrjagin class associated to

$$z \longmapsto \frac{\left(\frac{\sqrt{z}}{2}\right)}{\sin\left(\frac{\sqrt{z}}{2}\right)}.$$

Proof:

$$\mathrm{Ind}(D) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_M tr_s \Theta_{\frac{n}{2}} \, vol$$

If $a = c \otimes F \in Cl(T_x M) \otimes \mathrm{End}_{Cl}(S_x)$, then

$$tr_s(a) = \tau_s(c) \, tr^{S/\Delta} F$$

and $\tau_s(c) = (-2i)^{\frac{n}{2}} c_{12\ldots n}$. So

$= \sigma_n(\Theta_{\frac{n}{2}})$ picks up the top Clifford degree of $\Theta_{\frac{n}{2}}$

$$tr_s(\Theta_{\frac{n}{2}}) vol = (-2i)^{\frac{n}{2}} tr^{S/\Delta}\left(\sigma^0_n(\Theta_{\frac{n}{2}})\right).$$

h_t has terms of degree ≤ 0, so we are interested only in its zeroth term. Claim that

$$\sum_{j=0}^{\frac{n}{2}} \sigma_{2j}^{0}(\Theta_j) = \det^{\frac{1}{2}}\left(\frac{\frac{R}{2}}{\sin(\frac{R}{2})}\right)\exp(-F).$$

Do this by setting $t=1$ and comparing sides in our lemma, taking only the zeroth terms of the $\sigma_{2j}(\Theta_j)$.

Each $\sigma_{2j}(\Theta_j)$ has exterior degree $2j$, so $\sigma_n^0(\Theta_{\frac{n}{2}})$ is the only n-form in the expression, i.e.,

$$\sigma_n^0(\Theta_{\frac{n}{2}}) \text{ is the } n\text{-form part of } \det^{\frac{1}{2}}\left(\frac{(\frac{R}{2})}{\sinh(\frac{R}{2})}\right)\exp(-F).$$

Let's look at $\hat{A}(TM)\wedge ch(S/\Delta)$. Claim that

$$n\text{-form of } \hat{A}(TM)\wedge ch(S/\Delta) \text{ is } \frac{1}{(2\pi i)^{\frac{n}{2}}}\det^{\frac{1}{2}}\left(\frac{(\frac{R}{2})}{\sinh(\frac{R}{2})}\right) tr^{S/\Delta}(\exp(-tF))$$

which follows from staring and numerology. Thus,

$$\mathrm{Ind}(D) = \frac{(-2i)^{n/2}(2\pi i)^{n/2}}{(4\pi)^{n/2}}\int_M \hat{A}(TM)\wedge ch(S/\Delta) = \int_M \hat{A}(TM)\wedge ch(S/\Delta).$$

$\square$

Proposition:

The terms Θ_j have Getzler order $\leq 2j$ and the "heat symbol"

$$W_t = h_t\left(\sigma_0(\Theta_0) + \dots + t^{\frac{n}{2}}\sigma_n(\Theta_{\frac{n}{2}})\right)$$

satisfies the equation

$$\frac{\partial W}{\partial t} + \sigma_2(D^2)W = 0$$

and is unique up to some extra conditions.

"Proof:" (sketch)

Pick $q \in M$. h is the function

$$h = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-d(q,p)^2}{4t}},$$ and s is (any) Clifford section of $S \boxtimes S^*$

Locally,

$$\frac{1}{h}\left[\frac{\partial}{\partial t} + D^2\right](hs) = \frac{\partial s}{\partial t} + D^2 s + \frac{r}{4gt}\frac{\partial g}{\partial r} s + \frac{1}{t}\nabla_{r\frac{\partial}{\partial r}} s.$$

Then to get the asymptotic expansion we wrote $s \sim u_0 + tu_1 + \cdots$.
Get recursion relation

$$\nabla_{r\frac{\partial}{\partial r}} u_j + \left(j + \frac{r}{4g}\frac{\partial g}{\partial r}\right)u_j = -D^2 u_{j-1}.$$

Can determine what the top degree part of this equation by considering the order of each operator, and take symbols to get the relation

$$\left(j+r\frac{\partial}{\partial r}\right)\sigma_{2j}(u_j)=-\sigma(\mathcal{D}^2)\sigma_{2j-2}(u_{j-1}).$$

Suppose we look to solve

$$\frac{1}{h}\left[\frac{\partial}{\partial t}+\sigma(\mathcal{D}^2)\right](hs)=0$$

for hs of the form $h_t(v_0+tv_1+\dots+t^{\frac{n}{2}}v_{\frac{n}{2}})$, v_i degree $2i$.

If we could show

$$\frac{1}{h}\left[\frac{\partial}{\partial t}+\sigma_2(\mathcal{D}^2)\right](hs)=\frac{\partial s}{\partial t}+\sigma(\mathcal{D}^2)s+\frac{1}{t}r\frac{\partial}{\partial r}s$$

then the same recursive relation would hold, and we would be done - this might or might not work.

$\blacksquare$

Recall: $\sigma_2(D^2) = -\sum_i \left(\partial_i + \frac{1}{4}\sum_j R_{ij} x^j\right)^2 + F^S.$

Lemma:

Let R_{ij} be a skew-symmetric matrix of real scalars and F is a real scalar. Then the differential equation

$$\frac{\partial w}{\partial t} - \sum\left(\frac{\partial}{\partial x_i} + \frac{1}{4}R_{ij}x^j\right)^2 w + Fw = 0$$

has solution for small t which is analytic in R_{ij}, F. Explicitly,

$$w_t = (4\pi t)^{-\frac{n}{2}} \det{}^{\frac{1}{2}}\left(\frac{\left(\frac{tR}{2}\right)}{\sinh\left(\frac{tR}{2}\right)}\right) \exp\left(\frac{-1}{4t}\left\langle \frac{tR}{2}\coth\left(\frac{tR}{2}\right)x, x\right\rangle\right)\exp(-tF).$$

Now, generally, write w_t as a series in x^α, R_{ij}, F. The lemma above tells us that certain combinatorial relations are satisfied by the solution as a series, and since x^α, R_{ij}, and F all commute, the same relations are s satisfied in the 2-form case.

This proves our original lemma, and we are done!