18/07/13

Atiyah-Singer Index Theorem Seminar.

The Harmonic Oscillator and Non-Compact Manifolds (Richard Hughes).

Motivation.

Classically: The harmonic oscillator describes simple oscillatory motion (particle on spring, for instance). Such systems are bound to oscillate near a stable equilibrium state.

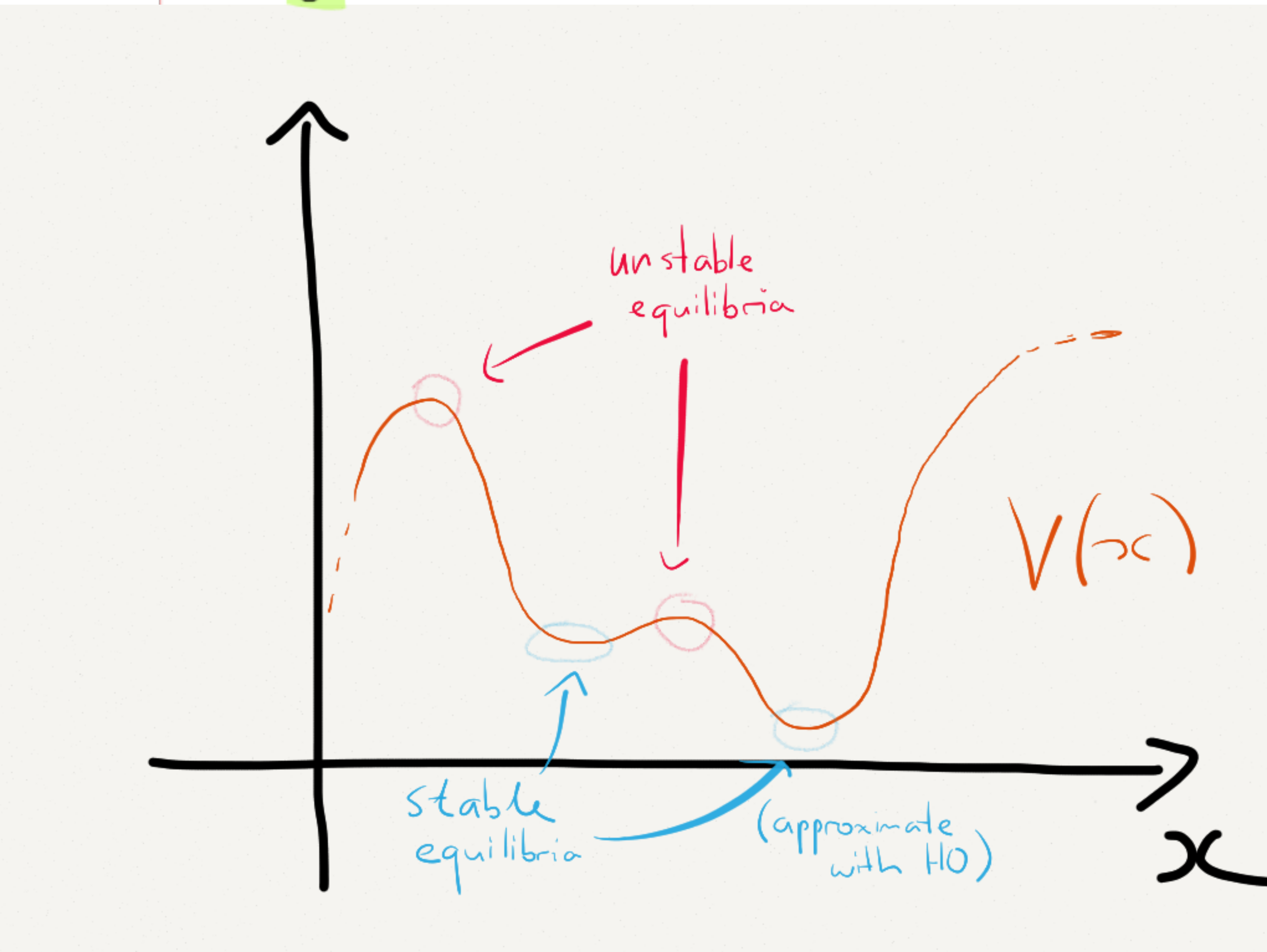
The potential energy function for the harmonic oscillator is quadratic in position, $V \propto x^2$. (V: potential energy)

Since:
- Dynamics of a system are unchanged by adding a constant term (i.e. there is no objective "zero potential energy"; cf vector space $\leftrightarrow$ affine space); and,
- physical equilibrium states correspond to critical points of the potential energy, V,

we can appoximate physical systems near stable equilibria using the harmonic oscillator, by taking a 2^{nd} order Taylor expansion:

$$V(x) \simeq \underbrace{V(x_{eq})}_{\text{arbitrary constant}} + \underbrace{\frac{dV}{dx}(x_{eq})}_{=0 \text{ at equilibrium}}(x - x_{eq}) + \underbrace{\frac{d^2V}{dx^2}(x_{eq})(x-x_{eq})^2}_{\text{First non-trivial term!}} + O(x^3).$$

Visually:



Thus, by studying the harmonic oscillator, we learn a great deal about the behaviour of any physical system that is near a stable equilibrium.

Similarly: When we come to proving the index theorem, a key step will involve reducing certain global calculations to the study of local harmonic oscillator type model. Thus, it is important that we study the harmonic oscillator.

The Harmonic Oscillator.

Definition: The <u>harmonic oscillator</u> is the name given to the unbounded operator

$$H = -\frac{d^2}{dx^2} + a^2x^2 \quad (a>0), \qquad \text{(H.O)}$$

on $L^2(\mathbb{R})$.

Def$^{\underline{n}}$:
- The <u>annihilation operator A</u> is defined by $A = ax + \frac{d}{dx}$.
- The <u>creation operator A^*</u> is $A^* = ax - \frac{d}{dx}$.

Proposition:

(1) A^* is the $L^2(\mathbb{R})$-adjoint operator of A.
(2) The following identities hold:

- $AA^* = H + a$,
- $A^*A = H - a$,
- $[A, A^*] = 2a$,
- $[H, A] = -2aA$,
- $[H, A^*] = 2aA^*$.

Proof:

For (1), use integration by parts, and the fact that for $L^2(\mathbb{R})$ integrable functions, $\lim_{x\to\pm\infty} f(x) = 0$.

For (2), calculate A^*A and AA^* using test functions; the other identities are formal consequences of those first two identities. $\square$

Def$^{\underline{n}}$: A function f is <u>rapidly decreasing</u> if $|f(\lambda)| = O(|\lambda|^{-k})$ for each $k \in \mathbb{Z}_{>0}$.

Def$^{\underline{n}}$: The <u>Schwartz space</u> $\mathcal{S}(\mathbb{R})$ is the space of C^∞ functions on $\mathbb{R}$ which are rapidly decreasing and all of whose derivatives are also rapidly decreasing.

Observation: A, A^* and H map the Schwartz space to itself ($\mathcal{S}(\mathbb{R})$ closed under poly$^{\underline{n}}$ mult. & differentiation).

Now: In what follows, we are (from a rep. thy. point of view) just constructing a highest weight rep. of the Weyl algebra on $L^2(\mathbb{R})$. However, we are interested in the analytic properties of our eigenfunctions, not just their rep. theoretic properties.

Def$^{\underline{n}}$: The <u>ground state</u> of H is the function $\psi_0 \in L^2(\mathbb{R})$ satisfying the differential eq$^{\underline{n}}$

$$A\psi_0 = 0, \text{ and such that } \|\psi_0\| = 1.$$

Observe: $H\psi_0 = (A^*A + a)\psi_0 = A^*(A\psi_0) + a\psi_0 = a\psi_0$, i.e. ψ_0 is an eigenfunction of H (ψ_0 is highest weight vector).

Proposition:

$$\psi_0(x) = a^{\frac{1}{2}} \pi^{\frac{1}{4}} e^{\frac{-ax^2}{2}}.$$

Proof:

$$0 = A\psi_0 = \frac{d\psi_0}{dx} + ax\psi_0, \quad \text{so}$$

$$\underbrace{-a\int x\,dx}_{=-a\left(\frac{1}{2}x^2\right)+K} = \int \frac{1}{\psi_0}\frac{d\psi_0}{dx}\,dx = \int \frac{d\psi_0}{\psi_0} = \log(\psi_0),$$

K ↑ const.

$$\text{so, } \psi_0(x) = Ce^{-\frac{ax^2}{2}}.$$

Imposing $\|\psi_0\| = 1$ determines the normalizing constant C to be $C = a^{\frac{1}{2}}\pi^{\frac{1}{4}}$.

$\square$

Def$^{\underline{n}}$: For $k \geq 1$, define the excited states of H inductively by

$$\psi_k = \frac{1}{\sqrt{2ka}} A^* \psi_{k-1}.$$

Lemma:

ψ_k belongs to $S(\mathbb{R})$, and is a normalized eigenfunction of H with eigenvalue $(2k+1)a$.

Proof:

First, since $A^*: S(\mathbb{R}) \to S(\mathbb{R})$, it is sufficient to show that $\psi_0 \in S(\mathbb{R})$. In particular, since the space of rapidly decreasing f$^{\text{ncs}}$ is closed under mult. by poly$^{\underline{ns}}$ and the derivatives of ψ_0 are of the form $(\text{poly}^{\underline{n}}) \cdot \psi_0$, it is sufficient to show that ψ_0 is rapidly decreasing.

So we wish to show that for every $k \in \mathbb{Z}_{\geq 0}$, $|\psi_0(x)| = O(|x|^{-k})$.
I.e., we want

$$\left|\frac{\psi_0(x)}{x^{-k}}\right| = \left|x^k \psi_0(x)\right| \leq \text{const. as } x \to \pm\infty.$$

Since ψ_0 is even, sufficient to consider $x \geq 0$. Then $x^k\psi_0(x) \geq 0$ for all x. Now,

$$\frac{d}{dx}\left(x^k \psi_0\right) = kx^{k-1}\psi_0 + x^k \frac{d\psi_0}{dx}$$

$$= kx^{k-1}\psi_0 - ax^{k+1}\psi_0 = \underbrace{x^{k-1}\psi_0}_{\geq 0 \text{ for } x \geq 0}\left[k - ax^2\right].$$

Now, $k - ax^2 < 0 \Leftrightarrow k < ax^2 \Leftrightarrow \frac{k}{a} < x^2$ $(a > 0)$, so

$$\frac{d}{dx}\left(x^k\psi_0\right) < 0 \text{ for } x > \sqrt{\frac{k}{a}}.$$

Using even/odd symmetry of $x^k\psi_0$, get

$$0 \leq |x|^k \psi_0(x) < \left(\frac{k}{a}\right)^{\frac{k}{2}} \psi_0\left(\sqrt{\frac{k}{a}}\right) = \left(\frac{k}{a}\right)^{\frac{k}{2}} a^{\frac{1}{2}} \pi^{\frac{1}{4}} e^{-\frac{a}{2}\frac{k}{a}} = \sqrt{\frac{k^k \pi^{\frac{1}{2}} e^{-k}}{a^{k-1}}} \text{ for } |x| > \sqrt{\frac{k}{a}}.$$

Thus, $\psi_0(x) = O(|x|^{-k})$ for all $k \in \mathbb{Z}_{\geq 0}$, and so $\psi_l \in \mathcal{S}(\mathbb{R})$ for all $l \in \mathbb{Z}_{\geq 0}$.

As for the rest of the lemma, we already have the base case ψ_0.
So by induction:

$$H\psi_k = \frac{1}{\sqrt{2ka}} HA^* \psi_{k-1} = \frac{1}{\sqrt{2ka}}\left(A^*H + 2aA^*\right)\psi_{k-1} = \frac{1}{\sqrt{2ka}} A^*\left[\overbrace{(2(k-1)+1)a}^{\text{induction step}} + 2a\right]\psi_{k-1} = (2k+1)a\psi_k,$$

and,

$$\|\psi_k\|^2=\frac{1}{2ka}\langle A^*\psi_{k-1},A^*\psi_{k-1}\rangle=\frac{1}{2ka}\langle AA^*\psi_{k-1},\psi_{k-1}\rangle$$

$$=\frac{1}{2ka}\langle (H+a)\psi_{k-1},\psi_{k-1}\rangle$$

$$=\frac{1}{2ka}\langle (2k-1+1)a\psi_{k-1},\psi_{k-1}\rangle$$

induction assumption

$$=\frac{1}{2ka}\langle 2ka\psi_{k-1},\psi_{k-1}\rangle=\|\psi_{k-1}\|^2=1.$$

$\square$

Lemma:

$\psi_k(x)=h_k(x)e^{\frac{-ax^2}{2}}$, where h_k is a polynomial of degree k with positive leading coefficient.

Proof:

Let $h_k(x)=\psi_k(x)e^{\frac{ax^2}{2}}$; first, we will determine a recurrence relation for h_k. We have

$$h_k(x)=\psi_k(x)e^{\frac{ax^2}{2}}=e^{\frac{ax^2}{2}}\frac{1}{\sqrt{2ka}}A^*\psi_{k-1}(x)=e^{\frac{ax^2}{2}}\frac{1}{\sqrt{2ka}}A^*\left(h_{k-1}(x)e^{\frac{-ax^2}{2}}\right)$$

$$=\frac{e^{\frac{ax^2}{2}}}{\sqrt{2ka}}\left(ax-\frac{d}{dx}\right)\left(h_{k-1}(x)e^{\frac{-ax^2}{2}}\right)$$

$$=\frac{e^{\frac{ax^2}{2}}}{\sqrt{2ka}}\left[axh_{k-1}(x)e^{\frac{-ax^2}{2}}-\frac{d}{dx}\left(h_{k-1}(x)e^{\frac{-ax^2}{2}}\right)\right]$$

$$=\frac{e^{\frac{ax^2}{2}}}{\sqrt{2ka}}\left[axh_{k-1}(x)e^{\frac{-ax^2}{2}}-\frac{dh_{k-1}}{dx}e^{\frac{-ax^2}{2}}-h_{k-1}(x)\frac{d}{dx}\left(e^{\frac{-ax^2}{2}}\right)\right]$$

$$= \frac{e^{\frac{ax^2}{2}}}{\sqrt{2ka}}\left[axh_{k-1}(x)e^{\frac{-ax^2}{2}} - \frac{dh_{k-1}}{dx}e^{\frac{-ax^2}{2}} - h_{k-1}(x)\left(\frac{-a(2x)}{2}\right)e^{\frac{-ax^2}{2}}\right.$$

$$= \frac{1}{\sqrt{2ka}}\left[axh_{k-1}(x) - \frac{dh_{k-1}}{dx} + axh_{k-1}(x)\right].$$

Thus, we obtain the recurrence relation for h_k:

$$\boxed{h_k(x) = \frac{1}{\sqrt{2ka}}\left(2axh_{k-1}(x) - \frac{dh_{k-1}}{dx}\right).}$$

Now,

$$h_0(x) = a^{\frac{1}{2}}\pi^{\frac{1}{4}} \quad (\text{constant}),$$

i.e. h_0 is a degree zero polynomial with positive leading coefficient. Thus, since

$$h_k(x) \propto xh_{k-1}(x),$$

$h_k(x)$ is a degree k polynomial, and the coefficient of x^k in $h_k(x)$ is

$$[h_k(x)]_{x^k} = \frac{2a}{\sqrt{2ka}}[h_{k-1}(x)]_{x^{k-1}} = \underbrace{\sqrt{\frac{2a}{k}}}_{>0}\underbrace{[h_{k-1}(x)]_{x^{k-1}}}_{>0 \text{ by induction assumption}} > 0.$$

$\square$

Remark:

Up to normalization, the h_k are the Hermite polynomials. If I have time, I'll put up some extra notes on these guys.

From the previous lemma we have:

Corollary:

$$\mathcal{P} = \mathbb{R}\text{-span}\{\psi_k\}_{k=0}^{\infty} = \left\{ f(x) = p(x) e^{-\frac{ax^2}{2}} \,\middle|\, p(x) \in \mathbb{R}[x] \right\} \leq L^2(\mathbb{R}).$$

Proof:

From previous lemma, since we are taking $\mathbb{R}$-span $\{(\deg k \text{ poly}^{\text{n}}) e^{-\frac{ax^2}{2}}\}_{k=0}^{\infty}$. $\square$

Proposition:

$\mathcal{P}$ is dense in $L^2(\mathbb{R})$.

Proof:

Assume $a = 1$ (this is sufficient - general a comes from

$$p(x) e^{-\frac{ax^2}{2}} = p(x) e^{-\frac{(x\sqrt{a})^2}{2}} = p\left(\frac{y}{\sqrt{a}}\right) e^{-\frac{y^2}{2}} = \tilde{p}(y) e^{-\frac{y^2}{2}}$$

where $y = x\sqrt{a}$ and $\tilde{p}$ is the poly$^{\text{n}}$ obtained by letting $\tilde{p}(x) = p\left(\frac{x}{\sqrt{a}}\right)$).

So,

$$\psi_0(x) = \sqrt[4]{\pi}\, e^{-\frac{x^2}{2}}, \quad \psi_k(x) = \frac{1}{\sqrt{2k}} A^* \psi_{k-1}(x).$$

Let $f_j(x) = x^j e^{-\frac{x^2}{2}}$. We calculate,

$$\|f_j\|^2 = \int_{-\infty}^{+\infty} x^{2j} e^{-x^2} dx = 2\int_0^{+\infty} x^{2j} e^{-x^2} dx \overset{\substack{x=\sqrt{y} \\ dx = \frac{dy}{2\sqrt{y}}}}{=} 2\int_0^{+\infty} y^j e^{-y} \frac{dy}{2\sqrt{y}} = \int_0^{+\infty} y^{(j+\frac{1}{2})-1} e^{-y} dy = \Gamma\left(j+\tfrac{1}{2}\right) \left(= \frac{(2j)!}{4^j j!}\sqrt{\pi}\right),$$

Γ increasing f^{n} on $\mathbb{R}_{>0}$.

so $\|f_j\|^2 = \Gamma(j+\frac{1}{2}) \leqslant \Gamma(j+1) = j!$.

Now,

$$e^{i\lambda x - \frac{x^2}{2}} = e^{-\frac{x^2}{2}} e^{i\lambda x} = e^{-\frac{x^2}{2}} \sum_{j=0}^{\infty} \frac{(i\lambda x)^j}{j!} = \sum_{j=0}^{\infty} \frac{(i\lambda)^j}{j!} x^j e^{-\frac{x^2}{2}} = \sum_{j=0}^{\infty} \frac{(i\lambda)^j}{j!} f_j(x),$$

so,

$$\left\| e^{i\lambda x - \frac{x^2}{2}} \right\| = \left\| \sum_{j=0}^{\infty} \frac{(i\lambda)^j}{j!} f_j(x) \right\| \leqslant \sum_{j=0}^{\infty} \frac{|i\lambda|^j}{j!} \|f_j\| \leqslant \sum_{j=0}^{\infty} \frac{|\lambda|^j (j!)^{\frac{1}{2}}}{j!},$$

i.e. $\|e^{i\lambda x - \frac{x^2}{2}}\| \leqslant \sum_{j=0}^{\infty} \frac{|\lambda|^j}{\sqrt{j!}}$. Let $a_j = \frac{|\lambda|^j}{\sqrt{j!}}$; and calculate

$$\left| \frac{a_{j+1}}{a_j} \right| = \frac{|\lambda|^{j+1}}{\sqrt{(j+1)!}} \cdot \frac{\sqrt{j!}}{|\lambda|^j} = |\lambda| \cdot \frac{1}{\sqrt{j+1}} \longrightarrow 0 \quad \text{as } j \to +\infty \text{ (indep. of } \lambda).$$

Let $F_\lambda(x) := e^{i\lambda x - \frac{x^2}{2}}$, so $\|F_\lambda\| \leqslant \sum_{j=0}^{\infty} \frac{|\lambda|^j}{\sqrt{j!}} < +\infty$. Since each $f_j \in \mathcal{P}$, we also have

$$F_\lambda(x) \in \overline{\mathcal{P}}.$$

Now, suppose that $f \in L^2$ is orthogonal to $\mathcal{P}$. Then

$$\int_{-\infty}^{+\infty} f(x) e^{i\lambda x - \frac{x^2}{2}} dx = 0 \quad \forall \lambda \in \mathbb{R}.$$

I.e. the Fourier transform $\widehat{\left(f(x) e^{-\frac{x^2}{2}}\right)}$ is identically zero. So by Plancherel's theorem,

$$f(x) e^{-\frac{x^2}{2}} = 0 \text{ a.e., so since } e^{-\frac{x^2}{2}} \neq 0, \; f(x) = 0 \text{ a.e.}$$

$\square$

Thus, $L^2(\mathbb{R})$ admits a complete orthogonal decomposition into (1D) eigenspaces for H, with discrete spectrum tending to infinity. We compare this with Thm 5.27 of Roe...

Decomposition result from Ch. 5.

Let $H = L^2(S)$, $\begin{matrix} S \\ \downarrow \\ M \end{matrix}$ a Clifford bundle with Dirac operator D.

Thm: There is a direct sum decomposition of H into a sum of countably many orthogonal subspaces H_λ. Each H_λ is a finite dimensional space of smooth sections, and is an eigenspace for D with eigenvalue λ. The eigenvalues of D form a discrete subset of $\mathbb{R}$.

Decomposition result for H.O.

There is a decomposition

$$L^2(\mathbb{R}) = \overline{\bigoplus_{k=1}^{\infty} \mathbf{H}_k},$$

where $\mathbf{H}_k = \mathbb{R}\psi_k$, the $\{\psi_k\}_{k=0}^{\infty}$ being (orthogonal) eigenfunctions of the harmonic oscillator H, and with (discrete) spectrum $\{(2k+1)a\}_{k=1}^{\infty}$; i.e. $H\psi_k = (2k+1)a\psi_k$.

Thus, we have an analogy of behaviour.

H, the harmonic oscillator operator, associated to (noncompact) $\mathbb{R}$.

$\updownarrow$ is spectrally like

D, the Dirac operator, associated to a compact manifold.

Functional Calculus of the Harmonic Oscillator.

Lemma:

Let $u \in L^2(\mathbb{R})$. Then $u \in \mathcal{S}(\mathbb{R})$ if and only if the "Fourier coefficients" $a_k = \langle \psi_k, u \rangle$ are rapidly decreasing in k.

Proof:

$u \in \mathcal{S}(\mathbb{R}) \Rightarrow a_k$ rapidly decreasing.

Let $u \in \mathcal{S}(\mathbb{R})$ and write $u(x) = \sum_{k \geq 0} a_k \psi_k$ so that

$$H^l u = \sum_{k \geq 0} a_k H^l \psi_k = \sum_{k \geq 0} ((2k+1)a)^l a_k \psi_k \in \mathcal{S}(\mathbb{R}).$$

since $H : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$

Since this holds for all $l \in \mathbb{Z}_{>0}$, the a_k must be rapidly decreasing (else the expression $\propto$ to $\sum k^l a_k \psi_k$ would fail to be $\mathcal{S}(\mathbb{R})$ in finite l – ψ_k is rapidly decreasing in x, not k.

Rapidly decreasing $a_k \Rightarrow u \in S(\mathbb{R})$.

Conversely, suppose that the Fourier coefficients are rapidly decreasing,

$$a_k = \langle \psi_k, u \rangle = O(k^{-\alpha}) \text{ for all } \alpha \in \mathbb{Z}_{>0}.$$

Then for all l, $(A^*)^l u$ and $A^l u$ have rapidly decreasing Fourier coefficients ⊛.

Calculations to prove ⊛:

$\underline{A^* u}$: $\displaystyle (A^*)^l u = \sum_{k\geq 0} \langle \psi_k, u \rangle (A^*)^l \psi_k$

$$= \sum_{k \geq 0} \langle \psi_k, u \rangle \sqrt{\frac{2^l a^l (k+l)!}{k!}}\, \psi_{k+l}.$$

So, $$\langle \psi_\alpha, (A^*)^l u \rangle = \sum_{k\geq 0} a_k \sqrt{\frac{2^l a^l (k+l)!}{k!}} \underbrace{\langle \psi_\alpha, \psi_{k+l} \rangle}_{= \delta_{\alpha, k+l}}$$

$$= a_{\alpha - l} \sqrt{\frac{2^l a^l \alpha!}{(\alpha - l)!}} \quad (\text{for } \alpha \geq l),$$

i.e., $$\langle \psi_\alpha, (A^*)^l u \rangle = \begin{cases} a_{\alpha - l} \sqrt{\frac{2^l a^l \alpha!}{(\alpha-l)!}}, & \alpha \geq l, \\ 0, & \alpha < l. \end{cases}$$

Since $a_{\alpha - l}$ is rapidly decreasing in α and $\sqrt{\frac{2^l a^l \alpha!}{(\alpha - l)!}}$ is poly$^{\underline{n}}$ in $\sqrt{\alpha}$, $\langle \psi_\alpha, (A^*)^l u \rangle$ is rapidly decreasing in α for all l.

$\underline{Au}$: $\displaystyle \langle \psi_\alpha, A^l u \rangle = \langle (A^*)^l \psi_\alpha, u \rangle = \sqrt{\frac{2^l a^l (\alpha + l)!}{\alpha!}} \langle \psi_{\alpha + l}, u \rangle = \underbrace{\sqrt{\frac{2^l a^l (\alpha+l)!}{\alpha!}}}_{\text{poly}^{\underline{n}} \text{ in } \alpha^{\frac{1}{2}}} \underbrace{a_{\alpha + l}}_{\text{rapidly decreasing in } \alpha}$, which is rapidly decreasing as above.

Let D denote the differentiation operator $\frac{d}{dx}$, M the multiply by x operator, so

$$\left.\begin{aligned} A &= aM + D \\ A^* &= aM - D \end{aligned}\right\} \Rightarrow \begin{aligned} D &= \tfrac{1}{2}(A - A^*), \\ M &= \tfrac{1}{2a}(A + A^*), \end{aligned}$$

and since $A^l u$ and $(A^*)^l u$ have rapidly decreasing Fourier coeff^s, so must $D^l u$ and $M^l u$ for all $l \in \mathbb{Z}_{\geq 0}$. Thus, for any (noncommuting) poly^n p in M and D, $p(M,D)$, we have

$$p(M,D)u \in L^2(\mathbb{R}).$$

In particular, consider $M^\alpha D^\beta u = x^\alpha \frac{d^\beta u}{dx^\beta}$. Since this is in $L^2(\mathbb{R})$ for all α & β, we have that

$$x^\alpha \frac{d^\beta u}{dx^\beta} \longrightarrow 0 \text{ as } |x| \longrightarrow +\infty \text{ for every } \alpha \,\&\, \beta.$$

So $|u^{(\beta)}(x)| = O(|x|^{-\alpha})$ for all $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, i.e. $u \in \mathcal{S}(\mathbb{R})$.

$\square$

Proposition:

If f is a bounded function on the spectrum of H, then $f(H)$ is defined and is a bounded operator on $L^2(\mathbb{R})$; the map $f \mapsto f(H)$ is a homomorphism from the ring of bounded functions on the spectrum of H to $\mathcal{B}(L^2(\mathbb{R}))$. Moreover, $f(H)$ maps $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$.

Remark: Compare this statement (& proof) to [Roe; Th^m 5.30], the functional calculus proposition for the Dirac operator for compact M.

Proof: Recall that $f(H)u = f(H)\sum_{k\geq 0} \underbrace{\langle \psi_k, u\rangle}_{a_k} \psi_k = \sum_{k\geq 0} f(\underbrace{(2k+1)a}_{\equiv \lambda_k \text{ (e-val of } \psi_k)}) a_k \psi_k$

$f(H)$ is a bounded operator on $L^2(\mathbb{R})$.

$f : \sigma(H) \longrightarrow \mathbb{R}$ is bounded, so $|f(\lambda)| \leq M < +\infty$ for some $M \geq 0$ and all $\lambda \in \sigma(H)$.

$\sigma(H) \subset \mathbb{R}$ discrete, countable

So,
$$\|f(H)u\| = \left\|\sum_{k\geq 0} f(\lambda_k) a_k \psi_k\right\| \leq \sum_{k\geq 0} |f(\lambda_k)| |a_k| \|\psi_k\|$$
$$\leq M \sum_{k\geq 0} |a_k| = M\|u\| < +\infty.$$

O.N. of the ψ_k

Thus, for $u \in L^2(\mathbb{R})$, $f(H)u \in L^2(\mathbb{R})$, and moreover,
$$\|f(H)\| = \sup_{\|u\|=1} \|f(H)u\| \leq \sup_{\|u\|=1} M\|u\| = M,$$
so $f(H)$ is a bounded operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Homomorphism.

Let $f, g \in B(\sigma(H), \mathbb{R})$. Then
$$(f+g)(H)u = (f+g)(H)\sum_{k\geq 0} \langle \psi_k, u\rangle \psi_k$$
$$= \sum_{k\geq 0} (f+g)(\lambda_k) \langle \psi_k, u\rangle \psi_k$$
$$= \sum_{k\geq 0} f(\lambda_k)\langle \psi_k, u\rangle \psi_k + \sum_{k\geq 0} g(\lambda_k)\langle \psi_k, u\rangle \psi_k$$
$$= f(H)u + g(H)u = (f(H) + g(H))u,$$

and,

$$(fg)(H)u = (fg)(H)\sum_{k\geq 0}\langle \psi_k, u\rangle \psi_k = \sum_{k\geq 0}(fg)(\lambda_k)\langle \psi_k, u\rangle \psi_k$$

$$= \sum_{k\geq 0} f(\lambda_k) g(\lambda_k)\langle \psi_k, u\rangle \psi_k$$

$$= f(H)\sum_{k\geq 0} g(\lambda_k)\langle \psi_k, u\rangle \psi_k$$

$$= f(H)g(H)\sum_{k\geq 0}\langle \psi_k, u\rangle \psi_k = \big(f(H)g(H)\big)u.$$

Thus, $(f+g)(H) = f(H)+g(H)$ and $(fg)(H) = f(H)g(H)$, so

$f \mapsto f(H)$ is a ring homomorphism $\mathcal{B}(\sigma(H),\mathbb{R}) \longrightarrow \mathcal{B}(L^2(\mathbb{R}))$.

<u>$f(H): \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$.</u>

If $u \in \mathcal{S}(\mathbb{R})$, then $\langle \psi_k, u\rangle$ is rapidly decreasing in k. So,

$$\langle \psi_\alpha, f(H)u\rangle = \Big\langle \psi_\alpha, \sum_{k\geq 0} f(\lambda_k)\langle \psi_k, u\rangle \psi_k\Big\rangle = \sum_{k\geq 0} f(\lambda_k)\langle \psi_k, u\rangle\langle \psi_\alpha, \psi_k\rangle = \underbrace{f(\lambda_\alpha)}_{\text{bounded}}\ \underbrace{\langle \psi_\alpha, u\rangle}_{\text{rapidly decreasing in } \alpha}$$

is rapidly decreasing in α, and so $f(H)u \in \mathcal{S}(\mathbb{R})$.

$\square$

Harmonic Oscillator Heat Equation.

We can now carry over our previous discussion of the heat eq$^{\underline{n}}$ to the "harmonic oscillator heat equation",

$$\frac{\partial u}{\partial t} + Hu = 0 \qquad \text{(HOHE)}.$$

Suppose that $u_0(x) = u(x,0) \in L^2(\mathbb{R})$ is the initial condition given for the HOHE.

Claim: $u_t(x) = e^{-tH} u_0(x)$ is a sol$^{\underline{n}}$ to the HOHE with initial condition $u_0(x)$.

Proof: That e^{-tH} is a well-defined operator on $L^2(\mathbb{R})$ comes from the functional calculus proposition; the rest is "plug in and compute". $\square$

The solution operator e^{-tH} is a smoothing operator, so there is a heat kernel $k_t^H \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$ such that

$$e^{-tH}u(x) = \int k_t^H(x,y)\,u(y)\,dy.$$

Remark (via Roe):

The heat kernel is completely characterized by the fact that (i) it is Schwartz class, (ii) it satisfies the heat eq$^{\underline{n}}$ in the first variable, and (iii) as $t \to 0$, $k_t^H(x,y) \to \delta(x-y)$ (c.f. the characterization of the heat kernel in the compact manifold case).

Our next task is to find an explicit expression for the heat kernel k_t^H.

Case 1 ($y=0$): Omit this case from talk - skip to general case.

Ansatz: $k_t^H(x,0) = U(x,t) = \alpha(t) e^{-\frac{1}{2}\beta(t)x^2}$, where α, β are f^{ncs} to be determined.

Then: • $HU = \left(a^2x^2 - \frac{d^2}{dx^2}\right)\left(\alpha(t)e^{-\frac{1}{2}\beta(t)x^2}\right) = \alpha(t)\left[x^2\left(a^2 - \beta(t)^2\right) + \beta(t)\right]e^{-\frac{1}{2}\beta(t)x^2}$

• $\frac{\partial U}{\partial t} = \frac{\partial \alpha}{\partial t} e^{-\frac{1}{2}\beta(t)x^2} + \alpha(t)\frac{\partial}{\partial t}\left(e^{-\frac{1}{2}x^2\beta(t)}\right) = \left(\dot{\alpha} - \frac{x^2\alpha\dot{\beta}}{2}\right)e^{-\frac{1}{2}\beta x^2}$

Plug into (HOHE):

$$0 = \frac{\partial U}{\partial t} + HU = \left(\dot{\alpha} - \frac{x^2\alpha\dot{\beta}}{2}\right)e^{-\frac{1}{2}\beta x^2} + \alpha\left(x^2\left(a^2 - \beta^2\right) + \beta\right)e^{-\frac{1}{2}\beta x^2}, \text{ so,}$$

$$0 = \dot{\alpha} - \frac{x^2\alpha\dot{\beta}}{2} + \alpha x^2\left(a^2 - \beta^2\right) + \alpha\beta = \underbrace{\left[\dot{\alpha} + \alpha\beta\right]}_{=0} + x^2\underbrace{\left[\alpha\left(a^2 - \beta^2\right) - \frac{\alpha\dot{\beta}}{2}\right]}_{=0}$$

I.e.,

(I) $\dot{\alpha} = -\alpha\beta$, and

(II) $\dot{\beta} = 2\left(a^2 - \beta^2\right)$.

We want a solution for β that behaves as $\frac{1}{t}$ as $t \to 0$. Take

$$\beta(t) = a\coth(2at).$$

Now, $\dot{\alpha} = -\alpha\beta$, so $\frac{\dot{\alpha}}{\alpha} = -a\coth(2at)$.

So, $$\int -a\coth(2at)\,dt = \int \frac{1}{\alpha}\frac{\partial \alpha}{\partial t}\,dt = \int \frac{d\alpha}{\alpha} = \log(\alpha),$$

thus,

$$\alpha(t) = e^{-a\int \coth(2at)\,dt} = e^{-\frac{1}{2}\log(\sinh(2at)) + (\mathrm{Const.})}$$

const.

$$= C\left(\sinh(2at)\right)^{-\frac{1}{2}} = \frac{C}{\sqrt{\sinh(2at)}}.$$

For small t,

$$\sinh(2at) \sim 2at, \quad \cosh(2at) \sim 1 + 2a^2t^2, \quad \coth(2at) \sim \frac{1}{2at},$$

so in the limit $t \to 0$,

$$U(x,t) \sim \frac{C}{\sqrt{2at}}\, e^{-\frac{x^2}{4t}},$$

so if we choose $C = \sqrt{\frac{a}{2\pi}}$,

Euclidean heat kernel

$$U(x,t) \sim \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^3}{4t}} \quad \text{for small } t.$$

Thus, from the known properties of the Euclidean heat kernel, for any function $s \in \mathcal{S}(\mathbb{R})$,

$$\int_{-\infty}^{+\infty} U(x,t)\,s(x)\,dx \sim \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{4t}} s(x)\,dx \longrightarrow s(0) \quad \text{as } t \to 0,$$

i.e. $U(x,t) \to \delta(x)$ as $t \to 0$, and $U(x,t)$ satisfies (HOHE).
So by uniqueness, $U(x,t)$ is equal to the heat kernel $k_t^H(x,0)$.

Proposition:

The harmonic oscillator heat kernel $k_t^H(x,0) = U(x,t)$ satisfies

$$U(x,t) = \sqrt{\frac{a}{2\pi \sinh(2at)}}\, e^{\frac{-ax^2\coth(2at)}{2}}$$

$\boxed{\text{(Mehler's Formula)}}$

Case 2 (general y).

Ansatz: $k_t^H(x,y) = \alpha(t) e^{-\frac{1}{2}\beta(t)(x^2+y^2) - \gamma(t)xy}$.

Why this ansatz? C.f. the standard heat kernel k_t on $\mathbb{R}$,

$$k_t(x,y) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-y)^2}{4t}} = \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x^2+y^2)}{4t} + \frac{xy}{2t}}.$$

We base our guess on the reasonable supposition that for small t, $k_t^H \sim k_t$, so that k_t^H is of the form given above and that

$$\alpha(t) \sim t^{-\frac{1}{2}},\ \beta(t) \sim \frac{1}{t},\ \gamma(t) \sim -\frac{1}{t} \quad \text{as } t \to 0.$$

Solving for appropriate α, β and γ, we obtain...

Proposition:

$$k_t^H(x,y) = \sqrt{\frac{a}{2\pi\sinh(2at)}}\, e^{-\frac{a\coth(2at)}{2}(x^2+y^2) + \frac{a}{\sinh(2at)}xy}$$

$$= \sqrt{\frac{a}{2\pi\sinh(2at)}}\, e^{-\frac{a}{2}\left(\coth(2at)(x^2+y^2) - 2\operatorname{cosech}(2at)xy\right)}.$$

General version of Mehler's Formula

Functional Calculus on Open Manifolds.

Let: • M be a complete Riemannian manifold, and
• D be a Dirac operator on a Clifford bundle $\begin{smallmatrix} S \\ \downarrow \\ M \end{smallmatrix}$.

We want to develop a functional calculus for D, i.e. a ring hom. $f \mapsto f(D)$ having properties analogous to the compact case.

Method: We will use the finite propagation speed of solutions to the Dirac W.E. to construct the functional calculus directly.

Proposition:

$\boxed{\frac{\partial s}{\partial t} = iDs}$ has a unique solution for smooth, compactly supported initial data s_0 on M, and the solution s_t is smooth and compactly supported for all times t.

Proof:

Uniqueness - from the energy estimate $\|s_t\|^2 \leq \|s_0\|^2$.

Existence - construct a compact manifold M' that contains an open subset isometric to a neighbourhood of $\mathrm{supp}(s_0) \subseteq M$, then use the results from [Roe; Ch.7] (Val's talk) for compact manifolds. $\square$

Remark: The solⁿ operator e^{itD} is defined and unitary on the dense subspace $C_c^\infty(S) \subseteq L^2(S)$, so extends to a unitary op. on $L^2(S)$ by continuity.

Let $f \in S(\mathbb{R})$ and define $f(D)$ by the Fourier integral

$$f(D) = \frac{1}{2\pi}\int \hat{f}(t) e^{itD} dt$$

where e^{itD} is the unitary sol^{n} operator to the Dirac W.E. described above. Since $f \in S(\mathbb{R})$, $\hat{f} \in S(\hat{\mathbb{R}})$ (so is rapidly decreasing), and so this integral converges (in the weak sense).

Proposition:

(1) The mapping $f \mapsto f(D)$ is a ring homomorphism $S(\mathbb{R}) \to B(L^2(S))$.
(2) $\|f(D)\| \leq \sup|f|$.
(3) If $f(x) = xg(x)$, then $f(D) = Dg(D)$.

Proof:

All three results follow by reducing to the compact case and applying the results we have already obtained there.

For instance, first consider $f \in S(\mathbb{R})$ such that $\hat{f}$ is compactly supported and $s \in C_c^\infty(S)$. Construct a compact manifold M' isometric to t_0-nbhd U of K, where

$$\mathrm{Supp}(\hat{f}) \subset [-t_0, t_0], \quad \mathrm{Supp}(s) \subset K.$$

Then by the finite propagation speed result, $\mathrm{Supp}(f(D)s) \subset U$ and $f(D)s = f(D')s$ on U. Apply the compact case, and then use that $\overline{C_c^\infty(S)} = L^2(S)$ and that $\overline{\{f \in S(\mathbb{R}) \mid \mathrm{Supp}(\hat{f}) \text{ compact}\}} = S(\mathbb{R})$. $\square$

Remark:

The closure of $S(\mathbb{R})$ in the sup norm is

$$C_0(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \text{ cts} \;\middle|\; \lim_{x \to \pm\infty} f(x) = 0 \right\}.$$

Since $\|f(D)\| \leq \sup|f|$ for $f \in S(\mathbb{R})$, can extend by continuity to a map

$$\begin{aligned} C_0(\mathbb{R}) &\longrightarrow \mathbf{B}(L^2(S)) \\ f &\longmapsto f(D) \end{aligned}$$

with the same properties as above.