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Atiyah-Singer Index Theorem Seminar.

Traces & Eigenvalue Asymptotics

(Cagri Karakurt).

Overview.

Let: M – closed Riem. mfld
Δ – Laplace on $L^2(M)$

Last time: $e^{-t\Delta}$ is a smoothing operator,

$$e^{-t\Delta}f(x_1)=\int_M \underbrace{k_t(x_1,x_2)}_{\text{smooth on } M\times M} f(x_2)\,dx_2$$

Asymptotic expansion

$$k_t(x_1,x_2)\sim\frac{1}{(4\pi t)^{n/2}}\left(\Theta_0(x_1,x_2)+t\,\Theta_1(x_1,x_2)+\dots\right),$$

Along the diagonal, $\Theta_i(x,x)$ are alg. expressions of metric and connection coefficients, e.g.

$$\Theta_0(x,x)=1,\quad \Theta_1(x,x)=\frac{1}{6}\underbrace{\kappa(x)}_{\text{scalar curvature}}+\underbrace{\cancel{K}}_{\text{Clifford contracted curvature operator}}$$

Today: Use the asymptotic expansion to study the spectrum of Δ.

Recall: Spectrum of Δ is a discrete set $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$.

Want to say $\quad \mathrm{Tr}(e^{-t\Delta}) = \sum_{i=0}^{\infty} e^{-t\lambda_i}$. $\quad (*)$

We will see

$$\mathrm{Tr}(e^{-t\Delta}) = \int_M k_t(x,x)\,dx. \quad (**)$$

Use the asymptotic expansion, combine $(*)$ with $(**)$, to get

$$(4\pi t)^{\frac{n}{2}} \sum_i e^{-t\lambda_i} \sim a_0 + a_1 t + a_2 t^2 + \cdots \quad (***),$$

$$a_i = \int_M \Theta_i(x,x)\,dx.$$

$(***)$ tells us that the spectrum of Δ and the set $\{n, a_0, a_1, a_2, \ldots\}$ determine eachother.

Example: $a_0 = \int_M 1\,dx = \mathrm{Vol}(M)$; $a_1 = \frac{1}{6}\int_M K(x)\,dx =$ "total curvature".

Corollary:

For $n=2$, the spectrum determines the topology of M.

Application: Weyl's Asymptotics.

Define $n(\lambda)=\max\{j \mid \lambda_j \leq \lambda\}$.

Thm:

$$n(\lambda) \sim \frac{1}{(4\pi)^{n/2}(\Gamma(\frac{n}{2}+1))} \mathrm{vol}(M)\, \lambda^{n/2} \quad \text{as } \lambda \to \infty.$$

Here Γ is Euler's Gamma function $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$ $\left(\Gamma(n)=(n-1)!\right)$

$$A(\lambda)\sim B(\lambda) \text{ if } \lim_{\lambda\to\infty}\frac{A(\lambda)}{B(\lambda)}=1$$

"Crude" estimate for $n(\lambda)$.

Let $j=n(\lambda)$, $s_1,\dots,s_j$ ON eigenf^{ncs} with eigenvalues less than λ. Let

$$s=\sum_{i=1}^{j}\alpha_i s_i \quad (\text{for some } \alpha_i),$$

and let $k=\min\{\tilde{k}\in 2\mathbb{Z} \mid \tilde{k}>n\}$. Fix $x\in M$.

$$|s(x)| \leq \|s\|_{C^0} \underset{\text{Sobolev embedding}}{\leq} C_1\|s\|_{W^{k/2}} \leq C_2\left(\|s\|_{W^{\frac{k}{2}-1}}+\|\Delta s\|_{W^{\frac{k}{2}-1}}\right)$$

$$\leq C_2(1+\lambda)\|s\|_{W^{\frac{k}{2}-1}} \underbrace{\leq \cdots \leq}_{k/2 \text{ times}} C(1+\lambda)^{\frac{k}{2}}\|s\|_{L^2}$$

$$\leq C(1+\lambda)^{\frac{k}{2}}\left(\sum|\alpha_i|^2\right)^{\frac{1}{2}}$$

Choose $\alpha_i = \overline{s_i}(x)$, so

$$\sum_{i=1}^{j} |s_i(x)|^2 \leqslant C(1+\lambda)^{\frac{k}{2}} \left(\sum_{i=1}^{j} |s_i|^2\right)^{\frac{1}{2}}.$$

Divide by $\left(\sum |s_i|^2\right)^{\frac{1}{2}}$ then square both sides.

$$\underbrace{\int_M \sum |s_i(x)|^2}_{= \sum_{i=1}^{j} \|s_i\|_{L^2} = j = n(\lambda).} \leqslant \int_M C(1+\lambda)^k \leqslant C^2(1+\lambda)^k \operatorname{Vol}(M)$$

<u>So:</u> $n(\lambda) \leqslant C^2(1+\lambda)^k \operatorname{Vol}(M)$. //

Trace Class Operators.

Slogan: Smoothing operators are trace class.

countable dense subset

Let H, H' be separable Hilbert spaces, $A: H \to H'$ a bounded operator.

Represent A by an infinite matrix: fix bases $\{e_i\}$ for H, $\{e_j'\}$ for H',

and let $\quad c_{ij}(A) = \langle Ae_i, e_j' \rangle$.

Definition: The Hilbert-Schmidt norm of A is defined by

$$\|A\|_{HS}^2 = \sum_{i,j} |c_{ij}(A)|^2 \in [0, \infty].$$

A is called a Hilbert-Schmidt operator if $\|A\|_{HS} < \infty$.

Proposition:

$\|A\|_{HS}$ is independent of the choice of $\{e_i\}$ and $\{e_j'\}$.

Proof:

Fix i, write $Ae_i = \sum_{j=1}^{\infty} \langle Ae_i, e_j' \rangle e_j'$.

Parseval's Thm $\Rightarrow \|Ae_i\|^2 = \sum_j |\underbrace{\langle Ae_i, e_j' \rangle}_{c_{ij}}|^2$

$$\Rightarrow \underbrace{\sum_i \|Ae_i\|^2}_{\text{indep. of choice of } \{e_j'\}} = \|A\|_{HS}^2$$

For converse, replace A with A^* and observe $\|A\|_{HS} = \|A^*\|_{HS}$. $\square$

Break time

Proposition:

(1) $\|\cdot\|_{HS}$ is induced by an inner product

$$\langle A, B\rangle_{HS} = \sum_{i,j} \overline{c_{ij}}(A)\, c_{ij}(B).$$

(2) The space of HS operators with $\langle\ ,\ \rangle_{HS}$ is a Hilbert space.

(3) $\|\cdot\| \leq C \|\cdot\|_{HS}$

(4) HS operators are compact.

(5) If A, B are HS and C is bounded, then $A+B$, $A\circ C$, $C\circ A$ are HS.

Remark: Trace Class $\subseteq$ HS $\subseteq$ Compact $\subseteq$ Bounded

c.f.

$$\ell_1 \subseteq \ell_2 \subseteq c_0 \subseteq \ell_\infty$$

Def$^{\underline{n}}$: $T: H \to H$ is said to be of trace class if there exist HS operators A, B s.t. $T = AB$. In this case we define

$$Tr(T) = \langle A^*, B\rangle_{HS}.$$

Fact: $Tr(T)$ is independent of the decomposition $T = AB$.

Proposition:

If T is self-adjoint and trace class then $Tr(T)$ is the sum of eigenvalues of T.

Proposition:

Let $T, B : H \to H$ be bounded operators, and suppose

(a) T is trace class or (b) both T and B are HS.

Then: (i) Both TB and BT are trace class.
(ii) $\operatorname{Tr}(TB) = \operatorname{Tr}(BT)$.

Proof (of (ii)):

Choose an ON basis,

$$\operatorname{Tr}(TB) = \sum \langle TBe_i, e_i \rangle = \sum \langle Be_i, T^* e_i \rangle$$

$$= \sum_{i,j} \overline{c_{ij}}(B)\, \overline{c_{ij}}(T) \quad (\text{by Parseval's Thm}),$$

sum is abs. convergent & symmetric in T and B. $\square$

Proposition (cts kernel $\Rightarrow$ HS):

Let A be the bounded operator on $L^2(M)$ defined by

$$Au(x_1) = \int_M k(x_1, x_2) u(x_2)\, dx_2$$

where k is continuous on $M \times M$. Then A is HS and

$$\|A\|_{HS} = \int_M \int_M |k(x_1, x_2)|^2 \, dx_1 \, dx_2 .$$

Proof:

Choose ON basis $\{e_j\}$ for $L^2(M)$.

$$\|A\|_{HS}^2 = \sum_j \|Ae_j\|^2 = \sum_j \int_M |A e_j(x)|^2 dx$$
$$= \sum_j \int_M \left| \int_M k(x_1,x_2) e_j(x_2) dx_2 \right|^2 dx_1$$
$$= \int_M \sum_j \Big| \underbrace{\int_M k(x_1,x_2) e_j(x_2) dx_2}_{\langle k(x_1,\cdot), e_j \rangle} \Big|^2 dx_1$$

Parseval's Thm

$$= \int_M \int_M |k(x_1,x_2)|^2 dx_2 dx_1$$

$\square$

Theorem (smooth kernel $\Rightarrow$ Trace Class):

Let A be a bounded operator on $L^2(M)$,

$$Au(x_1) = \int_M k_t(x_1,x_2) u(x_2) dx_2$$

with k_t smooth on $M \times M$. Then A is trace class and

$$\mathrm{Tr}(A) = \int_M k_t(x,x) dx.$$

Proof:

Assume A is of trace class with $A = BC$, B, C are HS operators with continuous kernels k_B, k_C. Then

$$k(x_1,x_3) = \int_M k_B(x_1,x_2)\, k_C(x_2,x_3)\, dx_2 .$$

$\mathrm{Tr}(A) = \langle B^*, C\rangle_{HS}$ and $\langle\, ,\rangle_{HS}$ is determined by $\|\cdot\|_{HS}$ and the polarization identity

$$\langle A, B\rangle = \frac{1}{4}\left(\|A+B\|^2 + \|A-B\|^2\right).$$

So,

$$\mathrm{Tr}(A) = \iint_{M\,M} k_B(x_1,x_2)\, k_C(x_1,x_2)\, dx_1\, dx_2 = \int_M k(x,x)\, dx.$$

Now: why should a smoothing operator be of trace class?

In [Roe; Ch. 5] we saw $B = (1+\Delta)^{-N}$ has continuous kernel (hence Hilbert-Schmidt). Hence, write

$$A = BC, \quad C = (1+\Delta)^N A$$ smoothing operator (in particular has cts kernel $\Rightarrow$ HS)

So A is a product of two HS operators, thus is trace class.

$\square$

Remark:

If general Laplacian on $\underset{M}{\overset{S}{\downarrow}}$, $e^{-t\Delta}$ has smooth kernel $k \in \Omega^0(\underset{M\times M}{\underbrace{S \boxtimes S^*}})$.

$$S \boxtimes S^* = \pi^* S \otimes \pi^*(S^*)$$

$$\begin{aligned} \mathrm{Diag}: M &\longrightarrow M\times M \\ x &\longmapsto (x,x) \end{aligned}$$

$$\mathrm{Diag}^*(S\boxtimes S^*) = S\otimes S^* = \mathrm{End}(S).$$

Theorem:

If A is a smoothing operator on $L^2(S)$ with kernel $k\in\Omega^0(S\boxtimes S^*)$,

$$\begin{aligned} \mathrm{Tr}\,A &= \int_M \mathrm{tr}\left(\mathrm{Diag}^*(k)\right)dx \\ &= \int_M \mathrm{tr}\left(k(x,x)\right)dx. \end{aligned}$$