09/07/13

Atiyah-Singer Index Theorem Seminar.

Spin groups III: Beyond Thunderdome.

(Rahul Shah)

Characteristic classes; Spin bundles.

I. Chern-Weil approach.

Def$^{\underline{n}}$: $P : \mathfrak{gl}_n \longrightarrow \mathbb{C}$ is invariant if $P(XY) = P(YX)$ for all $X, Y \in \mathfrak{gl}_n$.

Prop: P is invariant $\Leftrightarrow P(XYX^{-1}) = P(Y)$ $\forall X \in GL_n, Y \in \mathfrak{gl}_n$.

Examples:

- $P = \det, \operatorname{tr}$
- $P = \operatorname{tr}(\Lambda^k X)$; $(\Lambda^k X)(v_1 \wedge \cdots \wedge v_k) = (Xv_1 \wedge \cdots \wedge Xv_k)$.

Now: Let $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ v.b. with con ∇ & curv K.

Define $\widetilde{P}(E) := P(K) = \widetilde{P}(E, \nabla)$.

Thm: 1) $P(K)$ is closed.
2) If ∇' is a different conn., K' is curv., then $P(K')-P(K)$ is exact.

Under pullbacks: $\tilde{P}(f^*E) = f^*\tilde{P}(E)$.

Example: Let E be a $\mathbb{C}$ v.b., and let

$$c(E) = \det\left(I + \frac{i}{2\pi}K\right) = \sum_k \left(\frac{i}{2\pi}\right)^k \mathrm{tr}(\Lambda^k K) = \sum_k c_k(E).$$

Exercise: $c_1(\det(E)) = c_1(E)$.

What about $\mathbb{R}$ v.b.?

Let E be a $\mathbb{R}$ v.b. with complexification $E_{\mathbb{C}}$, and let

$$p_k(E) = (-1)^k c_{2k}(E_{\mathbb{C}}) \text{ (p_k are Pontrjagin classes).}$$

<u>Note:</u> $c_{2k+1}(E_{\mathbb{C}}) = 0$ since $(-1)^k tr(\Lambda^k K) = tr(\Lambda^k K)$ in this case.

Genera.

Let E be a $\mathbb{C}$ v.b., $f(z)$ be any formal power series (hol. f^{ns} near $z=0$).

<u>Def^n:</u> Chern f-genus: $\prod_f(E) = \det\left(f\left(\frac{i}{2\pi}K\right)\right)$.

<u>We have:</u> • $\prod_f(E_1 \oplus E_2) = \prod_f(E_1)\prod_f(E_2)$.

• If K has eigenvalues $\{\lambda_j\} \in \Omega^2$, then $\prod_f(E) = \prod_j f(\lambda_j)$.

<u>Now:</u> Let E be a $\mathbb{R}$ v.b., $g(z)$ a hol. f^{ns} s.t. $g(0)=1$. Let

$$f(z) := \sqrt{g(z^2)},\ f(0) = 1.$$

<u>Def^n:</u> The <u>Pontrjagin g-genus</u> of E is $\prod_f(E_{\mathbb{C}})$.

<u>Lemma:</u> g-genus $= \prod_j g(y_j)$, y_j are formal variables s.t. $p_i(E) = i^{th}$ symm. f^{ns} in y_j.

<u>Finally:</u> Let $ch(E) = tr\left(\exp\left(\frac{i}{2\pi}K\right)\right)$. Then: • $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$

• $ch(E_1 \otimes E_2) = ch(E_1)ch(E_2)$.

Associated bundles.

$$\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array} \leftarrow \text{principal bundle}, \quad \rho : G \longrightarrow \mathrm{Aut}(F)$$

$\leftarrow$ G-rep

Def$^{\underline{n}}$: $\begin{array}{c} P\times_\rho F \\ \downarrow \\ M \end{array}$ $\leftarrow$ associated bundle, is $P\times F/\sim$, $(p,f)\sim(p\cdot g, g^{-1}\cdot f)$.

If H acts on F and the action commutes with the G-action then $P\times_\rho F$ carries an H-action $h\cdot(p,f) = (p, h\cdot f)$.

If F is a right torsor (and action commutes w/ a left G-action), then $P\times_\rho F$ is a principal H-bundle.

Reduction of structure groups.

$$\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array}, \quad \rho : \tilde{G} \longrightarrow G$$

Def$^{\underline{n}}$: A <u>reduction of structure group</u> of P to $\begin{array}{c} \tilde{P} \circlearrowleft \tilde{G} \\ \downarrow \\ M \end{array}$ is an iso. $\varphi : \tilde{P}\times_\rho G \xrightarrow{\sim} P$.

Example: $TM \underset{\text{can.}}{\cong} \overset{\text{frame bundle}}{P} \times_{id} \mathbb{R}^n$, $id : GL_n \to GL_n = \mathrm{Aut}(\mathbb{R}^n)$,
$P_x = \{e : \mathbb{R}^n \to T_xM\}$,
$(e, \vec{v}\in\mathbb{R}^n) \mapsto e(\vec{v})$.

Example: Choose a Riem. metric on TM. This <u>is</u> a reduction of structure group from the frame bundle to orth. frame bundle

$$\begin{array}{l} \tilde{P} \circlearrowleft O(n), \text{ orth frame bundle} \\ \downarrow \\ M \end{array}$$

$$\begin{array}{l} P \circlearrowleft GL_n, \text{ frame bundle} \\ \downarrow \\ M \end{array}$$

$$i : O(n) \hookrightarrow GL(n)$$

$$\begin{aligned} \varphi : \tilde{P} \times_i GL_n &\xrightarrow{\sim} P \\ (\tilde{e}, g) &\longmapsto \tilde{e} \circ g. \end{aligned}$$

Spin bundles.

Let M be a spin mfld of even dimension (in particular, M is oriented, Riemanian).

Def^{n}: A <u>spin structure</u> on M is a reduction of structure group from the oriented orth. frame bundle to a principal $Spin(n)$ bundle.

$$\begin{array}{ccc} P \circlearrowleft SO(n) & \qquad & \tilde{P} \circlearrowleft Spin(n) \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

$$\rho : Spin(n) \longrightarrow SO(n)$$

$$\varphi : P \times_i \mathbb{R}^n \xrightarrow{\sim} TM$$

defining rep ($SO(n) \circlearrowright \mathbb{R}^n$)

$$\tilde{P} \times_\rho SO(n) \longrightarrow P.$$

$$TM \simeq (\tilde{P} \times_\rho SO(n)) \times_i \mathbb{R}^n.$$

Recall: There is a unique $Cl(\mathbb{R}^n)$ $\mathbb{C}$ irrep Δ.

Δ has a hermitian structure under which $Spin(n)$ acts unitarily.

$E_n = \{ \pm e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n} \mid i_k = 0,1 \}$ on which -1 acts as $-id$.

$$\delta(e_i) = \delta(-e_i^*).$$

Abuse notation by letting: $\Delta : \tilde{P} \times_\gamma \Delta$.

Prop: Δ is a Clifford bundle.

Reminder: $Cl(TM) \curvearrowright \Delta$ s.t. $g(\vec{v}\cdot\vec{\omega}_1, \vec{\omega}_2) + g(\vec{\omega}_1, \vec{v}\cdot\vec{\omega}_2) = 0$,
$$\nabla_X(\vec{v}\cdot\vec{\omega}) = \vec{v}\cdot\nabla_X\vec{\omega} + (\nabla_X^{LC}\vec{v})\cdot\vec{\omega}.$$

Proof:

$TM \curvearrowright \Delta$, $(\tilde{e}' = \tilde{e}\,\tilde{g})$.

$$((\hat{e}, g), \vec{v})\cdot(\hat{e}, \vec{\omega}) = (\tilde{e}, (\rho(\tilde{g})(g\vec{v}))\cdot\vec{\omega}) = (\tilde{e}, \hat{g}\cdot(g\vec{v})\cdot\tilde{g}^{-1}\vec{v}).$$

$$Spin(n) \curvearrowright Cl(\mathbb{R}^n) \quad \text{via} \quad \tilde{g} \mapsto (\vec{v} \mapsto \tilde{g}\cdot\vec{v}\,\tilde{g}^{-1})$$

Cliff. mult.

Exercise: Check this is well-defined.

Now, $((\tilde{e}, 1), \vec{v})^2\cdot(\tilde{e}, \vec{\omega}) = (\hat{e}, \vec{v}\cdot\vec{v}\cdot\vec{\omega}) = (\tilde{e}, -\|\vec{v}\|^2\vec{\omega})$.

$\blacksquare$

modulo "do the calculations and such yourself".

Break Time!

Back from break, to calculate!

Recall that M is an even dim spin manifold (oriented, Riem., w/spin structure). $Cl(TM) \curvearrowright \Delta$ (and $Cl(V) \curvearrowright \Delta(V)$).

Theorem:

$ch(\Delta) = 2^m \mathcal{G}(TM)$, where $\mathcal{G}(V)$ is the Pontrjagin g-genus associated to $g(z) = \cosh(\frac{1}{2}\sqrt{z})$.

Warning: Roe says that if $K = \begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}$ is skew self-adjoint then there is a basis in which

$$K = \begin{pmatrix} 0 & \omega_1 & & \\ -\omega_1 & 0 & & \\ & & 0 & \omega_2 \\ & & -\omega_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$ — this is not true when the entries are 2-forms.

How to fix this: Splitting principal.

(1) Complex v.b. injective $\mathbb{Z}$ cohom.
(2) $\mathbb{R}$ v.b inj. on $\mathbb{Z}/2\mathbb{Z}$ cohom.
(3) oriented $\mathbb{R}$ even rank v.b.; then you can pull back to a direct sum of 2-plane bundles, injective on $\mathbb{Z}$ cohom.

We will use version (3) of the splitting principal.

Proof of Thm:

W.l.o.g. assume V is a rank 2 bundle (use splitting principle). Can do this and notice that then if $V = V_1 \oplus V_2$,

$$\Delta(V) = \Delta(V_1) \otimes \Delta(V_2),$$

$$ch(\Delta(V)) = ch(\Delta(V_1))\, ch(\Delta(V_2)).$$

V is an oriented ON 2-plane bundle, so V carries a complex structure given by $*$, and

$$\tilde{V} \simeq_{\mathbb{R}} V$$

complex line

$i^m \omega \in Cl(V)$, & $i^m\omega$ splits Δ into 2 $Spin(n)$ invariant subspaces

$$\Delta_+ \oplus \Delta_- = \Delta.$$

Lemma (proof delayed): $\left.\begin{array}{l} \Delta^+ \otimes \Delta^+ \simeq \tilde{V} \\ \Delta^- \otimes \Delta^- \simeq \tilde{V}^* \end{array}\right\}$ as $Spin(2)$ bundles.

Assuming the lemma,

$$\begin{array}{l} c_1(\tilde{V} \oplus \tilde{V}^*) = c_1(\tilde{V} \oplus \bar{\tilde{V}}) = c_1(V_{\mathbb{C}}) = 0 \\ \quad \| \\ c_1(\Delta^+ \otimes \Delta^+) + c_1(\Delta^- \otimes \Delta^-) = 2c_1(\Delta^+) + 2c_1(\Delta^-). \end{array}$$

Let $c_1(\Delta^+) = x$, so $c_1(\Delta^-) = -x$. Then

$$ch(\Delta) = ch(\Delta^+) + ch(\Delta^-) = e^x + e^{-x} = 2\cosh(x).$$

Now,

$$p_1(V) = -c_2(V \otimes \mathbb{C}) = -c_2(\tilde{V} \oplus \tilde{V}^*)$$
$$= -c_1(\tilde{V}) \cdot c_1(\tilde{V}^*) = 2x(-2x) = 4x^2.$$

Now, $G(V) = \prod_j g(y_j)$, $p_i(V)$ is i^{th} symm f^{ns} in y_j.

So, $p_1(V) = 4x^2 = y$, $G(V) = \cosh(\frac{1}{2}\sqrt{4x^2}) = \cosh(x)$.

So, modulo proving that lemma, we are done.

Reminder: • $Cl(\mathbb{R}^2) \simeq \mathbb{H}$

• $Cl(\mathbb{R}^2) \otimes \mathbb{C} \simeq M_2(\mathbb{C})$

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e_1 \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; e_2 \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$e_1 e_2 \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$Spin(2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (\text{take norm 1 even el}^{ts}).$$

Then, $\mathbb{R}^2 \ni \vec{v} = v^1 e_1 + v^2 e_2 = i\begin{pmatrix} v^1 & v^2 \\ v^2 & -v^1 \end{pmatrix}$. We act by conj., so

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} i \begin{pmatrix} v^1 & v^2 \\ v^2 & -v^1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}.$$

Now, $ie_1e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, so

$$\Delta_+ = \left\{ \begin{pmatrix} x \\ ix \end{pmatrix} \middle| x \in \mathbb{C} \right\}, \quad \Delta_- = \left\{ \begin{pmatrix} x \\ -ix \end{pmatrix} \middle| x \in \mathbb{C} \right\}.$$

$\mathrm{Spin}(2) \circlearrowright \Delta_+$,

$\begin{pmatrix} (\cos\theta + i\sin\theta)x \\ (\cos\theta + i\sin\theta)ix \end{pmatrix}$ acts by rot by θ; so $\Delta_+ \otimes \Delta_+$ acts by 2θ,

and the lemma is proved. $\square$

Let $S = \Delta \otimes V$, recall $R^S(X,Y) = \sum_{i,j} e_i e_j R(X,Y)$.
Clifford bundle

Then K of S can be written as $K = R^S + F^S$, $[F^S, c] = 0$.
Clifford mult.

Let $ch(S/\Delta) := tr^{S/\Delta}\left(\exp\left(\frac{i}{2\pi} F^S\right)\right)$.

Then

$$ch(S) = ch(\Delta \otimes V) = ch(\Delta) \cdot ch(V)$$
$$= 2^m \hat{G}(TM)\, ch(S/\Delta).$$