20/06/13

Atiyah-Singer Index Theorem Seminar.

Spin groups II (Richard Hughes).

The Lie structure of Spin.

Let $Cl(k)$ denote the Clifford algebra of $\mathbb{R}^k$ with the standard positive definite form. We recall:

Def^n: (i) $Pin(k)$ is the multiplicative subgroup of $Cl(k)$ generated by the unit vectors $v \in \mathbb{R}^k$.

(ii) $Spin(k) = Pin(k) \cap Cl(k)_0$, where $Cl(k)_0$ is the even part of $Cl(k)$, i.e.

$$Cl(k) = \underbrace{Cl(k)_0}_{\text{even graded}} \oplus \underbrace{Cl(k)_1}_{\text{odd graded.}}$$

Example: $Cl(1) = \mathbb{R}[x] / (x^2+1)\mathbb{R}[x] \cong \mathbb{C}$.

$Pin(1) = \langle x \mid x^4 = 1 \rangle \cong \{\pm 1, \pm i\} \leq \mathbb{C}^\times$.

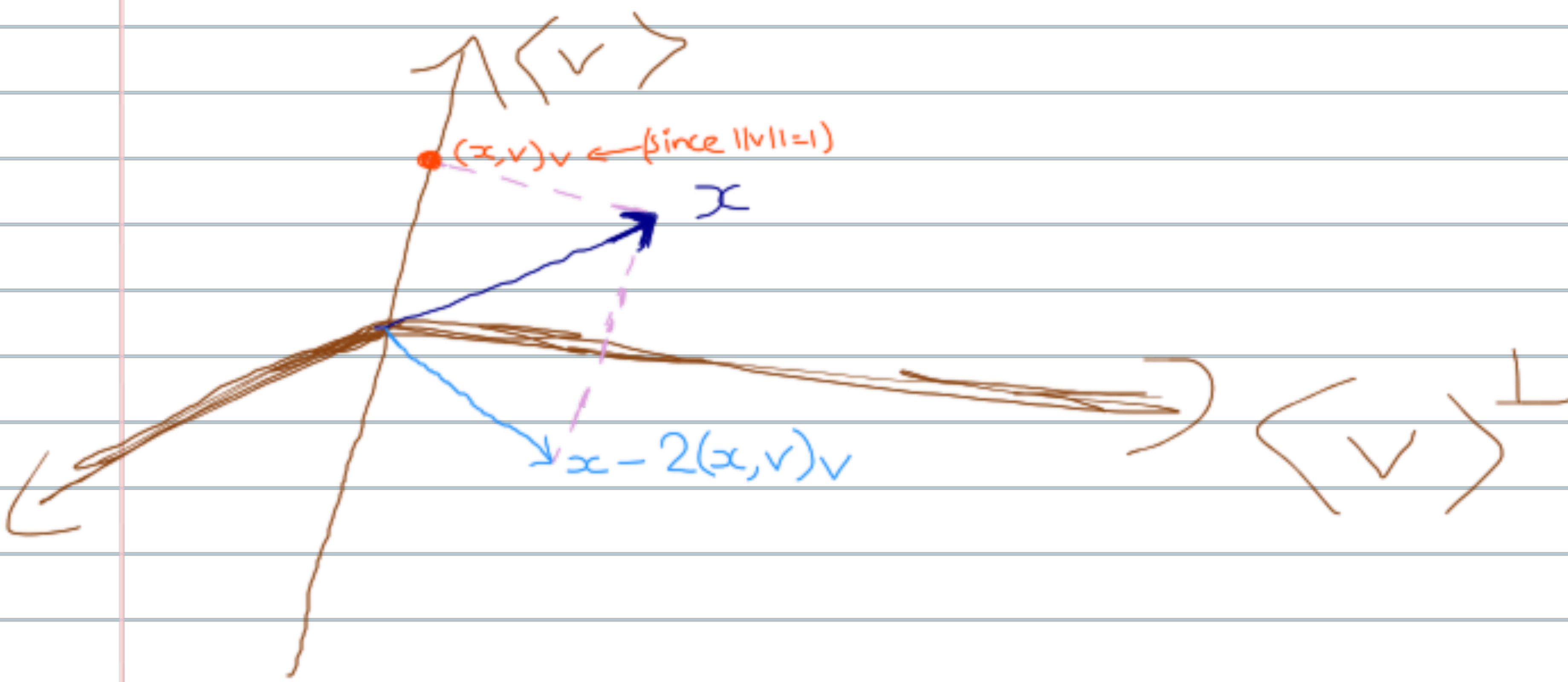
$Spin(1) = \{1, x^2\} \cong \{\pm 1\}$

$\mathbb{R}^k \subseteq Cl(k)$, so we can look for actions of $Cl(k)$ on itself which preserve $\mathbb{R}^k$.

Let $v \in \mathbb{R}^k$ with $\|v\| = 1$. Then $v^{-1} = -v \in Cl(k)$. For $x \in \mathbb{R}^k$, consider

$$-vxv^{-1} = vxv = \big(-xv - 2(x,v)\big)v = x - 2(x,v)v,$$

since by the defining relations of $Cl(k)$ we can deduce the anticommutation relation $vw + wv = -2(v,w)$ for $v, w \in \mathbb{R}^k$. Geometrically, we have



i.e., $-vxv^{-1} = x - 2(x,v)v$ is the reflection of x in the hyperplane perpendicular to v.

Let $\varepsilon : Cl(k) \to Cl(k)$ be the grading automorphism $\varepsilon(\underset{\text{even}}{v_e} + \underset{\text{odd}}{v_o}) = v_e - v_o$. Then since reflections are elements of $GL_k(\mathbb{R})$, and the unit vectors in $\mathbb{R}^k$ generate $Pin(k)$, we can extend the above to a representation of Pin:

Def$^{\underline{n}}$: The twisted adjoint representation of $\mathrm{Pin}(k)$ is

$$\rho : \mathrm{Pin}(k) \longrightarrow GL(\mathbb{R}^k)$$

$$\rho(y)x = yx\mathcal{E}(y^{-1}).$$

Note that by the above,

$$yx\mathcal{E}(y^{-1}) = \pm yxy^{-1} = \pm \underbrace{(u_1 \cdots u_l)}_{\text{string of unit vectors}} x \underbrace{(u_l^{-1} \cdots u_1^{-1})}$$

$$= \pm R_1 \cdots R_l(x)$$

reflections in hyperplanes $\perp$ to the u_i,

so that in fact, $\rho(\mathrm{Pin}(k)) \subseteq O(k)$.

Since $\mathrm{Spin}(k)$ is the even part of $\mathrm{Pin}(k)$, the image of an element of $\mathrm{Spin}(k)$ is the product of an even number of reflections, and thus is in $SO(k)$. So we have:

$$\rho : \mathrm{Spin}(k) \longrightarrow SO(k)$$

$$\rho(y)x = yxy^{-1} \quad (\text{since } \mathcal{E}(y) = +y).$$

Proposition:

There is an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(k) \xrightarrow{\rho} SO(k) \to 0, \text{ where } \mathbb{Z}/2\mathbb{Z} = \underset{\mathrm{Spin}(k)}{\{\pm 1\} \cap}$$

Proof:

Since $\mathrm{Pin}(k)$ is generated by all the unit vectors in $\mathbb{R}^k$, we can reflect in every hyperplane in $\mathbb{R}^k$, so ρ is surjective.

It can be shown that $\ker(\rho) \subseteq Z_s(Cl(k))$, the super-center of $Cl(k)$. By [Roe; Lemma 4.3] this is just $\mathbb{R}$, and a further computation yields that $\mathbb{R} \cap \mathrm{Spin}(k) = \{\pm 1\}$.

$\square$

Take home message: By the above, $\mathrm{Spin}(k)$ is a compact Lie group (it is a double cover of $SO(k)$).

Recall: Last time used a homotopy LES argument to prove that for $k \geq 3$, $\mathrm{Spin}(k)$ is the universal cover of $SO(k)$.

Because of this, we can identify

$$\underbrace{\mathfrak{spin}(k)}_{\mathrm{Lie}(\mathrm{Spin}(k))} \cong \underbrace{\mathfrak{so}(k)}_{\mathrm{Lie}(SO(k))} = \{A \in M_k(\mathbb{R}) \mid A^t = -A\}$$

But since we also have that $\mathrm{Spin}(k)$ is a submanifold of $Cl(k)$, we should be able to identify $\mathfrak{spin}(k)$ with a vector subspace of $Cl(k)$.

Lemma:

The Lie algebra $\mathrm{spin}(k)$ may be identified with the vector subspace of $Cl(k)$ spanned by the products $e_i e_j$, $i \neq j$. The map $\mathrm{so}(k) \longrightarrow \mathrm{spin}(k) \leqslant Cl(k)$ is given by

$$\begin{aligned} \mathrm{so}(k) &\longrightarrow Cl(k) \\ (a_{ij}) &\longmapsto \frac{1}{4}\sum_{i,j} a_{ij} e_i e_j \, . \end{aligned}$$

Proof:

Proceeds by calculation, and comparing dimensions. $\square$

Classification of $Cl(k)\otimes\mathbb{C}$ irreps.

For $Cl(k)$, we can calculate explicitly that:

- $Cl(0)\cong\mathbb{R}$
- $Cl(1)\cong\mathbb{C}$
- $Cl(2)\cong\mathbb{H}$
- $Cl(3)\cong\mathbb{H}\oplus\mathbb{H}$
- $Cl(4)\cong M_2(\mathbb{H})$
- $Cl(5)\cong M_4(\mathbb{C})$
- $Cl(6)\cong M_8(\mathbb{R})$
- $Cl(7)\cong M_8(\mathbb{R})\oplus M_8(\mathbb{R})$
- $Cl(8)\cong M_{16}(\mathbb{R})$

There is then a version of <u>Bott periodicity</u> that states

$$Cl(n+8)\cong Cl(n)\otimes_{\mathbb{R}}Cl(8)\cong Cl(n)\otimes M_{16}(\mathbb{R}),$$

completely describing the Clifford algebras $Cl(k)$.

Let $Cl_k(\mathbb{C}) := Cl(k)\otimes_{\mathbb{R}}\mathbb{C}$. Then we have (will provide a reference later):

- $Cl_0(\mathbb{C})\cong\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}$
- $Cl_1(\mathbb{C})\cong\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C}$
- $Cl_2(\mathbb{C})\cong\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}\cong M_2(\mathbb{C})$

low dim results

- $Cl_{k+2}(\mathbb{C})\cong Cl_k(\mathbb{C})\otimes_{\mathbb{C}}Cl_2(\mathbb{C})\cong Cl_k(\mathbb{C})\otimes_{\mathbb{C}}M_2(\mathbb{C})$ periodicity

Therefore, in general:

$$Cl_n(\mathbb{C})\cong\begin{cases} M_{2^{\frac{1}{2}n}}(\mathbb{C}) & \text{for } n \text{ even,}\\ M_{2^{\frac{1}{2}(n-1)}}(\mathbb{C}) & \text{for } n \text{ odd.}\end{cases}$$

Thus, for n even, $\mathcal{C}l(n)\otimes\mathbb{C}$ has a unique irrep given by the action of the matrices of $M_{2^{\frac{n}{2}}}(\mathbb{C})$ on the vector space $\mathbb{C}^{2^{\frac{n}{2}}}$.

For n odd, there are two distinct irreps,

$$\rho_n^{(i)}: \mathcal{C}l_n(\mathbb{C}) \overset{\Phi}{\cong} M_{2^{\frac{n-1}{2}}}(\mathbb{C})^{\oplus 2} \longrightarrow M_{2^{\frac{n-1}{2}}}(\mathbb{C})$$

$$x \longmapsto \Phi(x) = (X_1, X_2) \longmapsto X_i$$

For $i = 1, 2$.

From now on, we consider only the even case in these notes.

Defn: We have a unique irrep of $\mathcal{C}l(2m)\otimes\mathbb{C}$ which we will denote by Δ and call the spin representation.

Note: $\dim(\Delta) = \dim(\mathbb{C}^{2^{\frac{2m}{2}}}) = 2^m$.

Classification of finite dimensional $C\ell(k)$-reps.

By semisimplicity of $M_n(\mathbb{C})$ (matrix algebra), any finite dim complex $C\ell(k)$-rep must be a direct sum of copies of Δ.

We can write this as

$$W = \Delta \otimes_{\mathbb{C}} V$$

auxilliary "coefficient" vector space

We can recover V from W as

$$V \cong (\Delta^* \otimes_{C\ell(k)\otimes\mathbb{C}} \Delta) \otimes_{\mathbb{C}} V \cong \Delta^* \otimes_{C\ell_k(\mathbb{C})} (\Delta \otimes_{\mathbb{C}} V) = \mathrm{Hom}_{C\ell_k(\mathbb{C})}(\Delta, W).$$

Moreover,

$$\mathrm{End}_{\mathbb{C}}(W) = \underbrace{C\ell(k)}_{\text{action on } \Delta} \otimes_{\mathbb{C}} \underbrace{\mathrm{End}_{\mathbb{C}}(V)}_{\text{action on } V} = C\ell(k) \otimes_{\mathbb{C}} \mathrm{End}_{C\ell_k(\mathbb{C})}(W).$$

Def^{n}: Let $F \in \mathrm{End}_{C\ell_k(\mathbb{C})}(W)$ where W is a complex $C\ell(k)$-rep,

$$W = \Delta \otimes V.$$

The *relative trace* of F, $\mathrm{tr}^{W/\Delta}(F)$, is defined to be the trace of the element $\tilde{F} \in \mathrm{End}_{\mathbb{C}}(V)$ corresponding to F via

$$\mathrm{End}_{C\ell_k(\mathbb{C})}(W) \cong \mathrm{End}_{\mathbb{C}}(V),$$
$$F \longmapsto \tilde{F}.$$

Irreps of Spin(k).

Since the elements of Pin(k) generate $\mathcal{C}\ell(k)$, Δ is also an irrep of Pin(k).

Now, $\mathrm{Spin}(k) \trianglelefteq \mathrm{Pin}(k)$ of index 2, so either:

- Δ is an irrep of Spin(k); or,
- Δ splits as the direct sum of two inequivalent irreps of Spin(k).

Proposition: Δ splits as a rep of Spin(k).

Proof:

Let $\omega = e_1 \cdots e_k \in \mathcal{C}\ell(k)$ be the <u>volume element</u>. This satisfies (for $k=2m$ even),

$$\omega^2 = (-1)^m \quad \text{and} \quad \omega x = \varepsilon(x)\omega.$$

ω is a product of unit vectors, so $\omega \in \mathrm{Pin}(k)$, and since we are considering k even, $\omega \in \mathrm{Spin}(k)$.

Suppose $\omega v = \lambda v$; then $\omega^2 v = (-1)^m v = \lambda^2 v$. So consider the <u>grading operator</u> $i^m\omega$. We have

$$(i^m\omega)v = \lambda v \Rightarrow (i^m\omega)^2 v = \lambda^2 v = v, \text{ so } \lambda = \pm 1.$$

Let Δ_+ and Δ_- be the ± 1 eigenspaces of $i^m\omega$ acting on Δ, and consider the action of

$$Cl(k)\otimes\mathbb{C} \curvearrowright \Delta = \Delta_+ \oplus \Delta_- .$$

Let $x\in Cl(k)$, $v=v_+ + v_- \in \Delta_+\oplus\Delta_-$. Then

$$i^m\omega(xv_\pm) = \varepsilon(x)(i^m\omega v_\pm) = \pm\varepsilon(x)v_\pm .$$

So if x is even, $xv_\pm \in \Delta_\pm$ (preserves decomp.), and if x is odd, $xv_\pm \in \Delta_\mp$ (reverses decomp.).

Thus, Spin(k) preserves Δ_+ and Δ_-, so

$$\Delta_+ \,\&\, \Delta_- \text{ are Spin}(k)\text{-reps},$$

and since $\Delta = \Delta_+\oplus\Delta_-$ we have that $\Delta_\pm$ are irreps of Spin(k) and $\dim(\Delta_+) = \dim(\Delta_-) = \frac{1}{2}\dim(\Delta) = 2^{m-1}$.

Def$^{\underline{n}}$:
- Δ_+ is the positive half-spin representation of Spin(2m).
- Δ_- is the negative half-spin representation of Spin(2m).

Remark: Can also say that the super vector space $\Delta = \Delta_+ \oplus \Delta_-$ becomes a graded representation of $Cl(2m)$.

Spin structures on manifolds.

Initial data:

Let (M, g) be an oriented Riemannian manifold of dimension $n = 2m$ (even), and let E be the principal $SO(n)$-bundle of oriented orthonormal frames for the tangent bundle TM:

$$\begin{tikzcd} SO(n) \arrow[r] & E \arrow[d, "\pi"'] \\ & M \arrow[u, dashed, bend right] \end{tikzcd}$$

local (on $U \subset M$) section of E = choice of positively oriented orthonormal basis for T_pM at each pt $p \in U$, varying smoothly with p.

Def^{n}: A *spin structure on* M is a principal $\mathrm{Spin}(n)$-bundle $\tilde{E}$ over M which is a double covering of E such that the restriction to each fibre of the double covering $\tilde{E} \to E$ is the double covering $\rho : \mathrm{Spin}(n) \to SO(n)$.

If M admits a spin-structure, it is called a *spin manifold*.

So: A spin-structure is a principal bundle

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r] & \widetilde{E} \arrow[d, "\widetilde{\pi}"] \\ & M \end{tikzcd}$$

where $\widetilde{E} \xrightarrow{R} E$ is a double covering, and such that on each fibre,

$$\widetilde{E}_p = \mathrm{Spin}(n) \xrightarrow{R|_{\widetilde{E}_p} = \rho} SO(n) = E_p,$$

where $\widetilde{E}_p = \widetilde{\pi}^{-1}(p)$, $E_p = \pi^{-1}(p)$.

Proposition: If M is 2-connected, then it admits a unique spin-structure.

partial result on existence & uniqueness of spin-structures

Proof: See [Roe].

Example: For $n \geq 3$, $\mathbb{R}^n$ and S^n (with a given metric g) have unique spin-structures.

Def$^{\underline{n}}$: If M is a spin manifold, then its <u>spin bundle</u> Δ is the vector bundle associated to the principal spin bundle by means of the spin representation.

Def^n: • The *spin connection* on the principal Spin(n)-bundle $\tilde{E}$ over a spin manifold M is defined to be the lifting to $\tilde{E}$ of the principal $SO(n)$ connection on E induced by the Levi-Civita connection on TM.

• The *spin connection on Δ* is the connection on Δ associated (via the spin representation) to the spin connection on $\tilde{E}$.

Proposition: The spin bundle Δ can be equipped with a natural hermitian metric and compatible spin connection, making $\Delta \to M$ a Clifford bundle.

Recall: • For a Clifford bundle S, the *Riemann endomorphism* $R^S \in \Omega^2(\mathrm{End}(S))$ is

$$R^S(X,Y) = \frac{1}{4}\sum_{k,l}\big(\underbrace{R(X,Y)}_{\text{Riemann curvature operator}}e_k, e_l\big)\underbrace{c(e_k)c(e_l)}_{\text{Clifford action}}.$$

• The curvature 2-form K of a Clifford bundle S can always be written as

$$K = R^S + F^S \quad (\in \Omega^2(\mathrm{End}(S))),$$

where the *twisting curvature* F^S commutes with the action of the Clifford algebra.

Proposition:

The twisting curvature of the spin bundle associated to a spin structure is zero.

Proof:

Let $\{e_k\}$ be a local ON frame for TM, and recall that the connection & curvature forms for TM have their values in the Lie algebra $so(n)$.

In particular $K \in \Omega^2(so(n)) = \Gamma(T^*M \otimes T^*M \otimes so(n))$, has matrix entries are $(R(\cdot,\cdot)e_k, e_l)$.

By our identification of $spin(n)$ as a subspace of $Cl(n)$, the corresponding $spin(n)$-valued 2-form (which gives the curvature of the connection) is

$$\frac{1}{4}\sum_{k,l}(R(\cdot,\cdot)e_k, e_l)e_k e_l,$$

which acts on the spin rep. by

$$\frac{1}{4}\sum_{k,l}(R(\cdot,\cdot)e_k, e_l)c(e_k)c(e_l) = R^S(\cdot,\cdot).$$

So $K = R^S = R^S + F^S$, i.e. $F^S \equiv 0$.

$\square$