13/06/13

Atiyah-Singer Index Theorem Seminar.

Clifford algebras & Dirac operators (Tom Mainiero).

Motivation:

Consider $\mathbb{E}^n$ equipped with orthogonal coords $x^1, \dots, x^n$.
Then we have the Laplace operator:

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2} \quad \text{(minus sign is convention).}$$

Q: Can we find a 1^{st} order op. D s.t. $D^2 = \Delta$?

Take $D = \gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \dots + \gamma^n \frac{\partial}{\partial x^n}$.

Then: $D^2 = \Delta \iff \begin{cases} (\gamma^i)^2 = -1 \\ \gamma^i\gamma^j + \gamma^j\gamma^i = 0, \; i \neq j \end{cases}$

$$\gamma^i\gamma^j + \gamma^j\gamma^i = g^{ij}$$

metric

E.g. for $n=2$ could take the matrix rep

$$\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We want to construct some algebra A with a map of vector spaces $\varphi : V \hookrightarrow A$, $V \cong \mathbb{R}^n$, such that

$$\varphi(v)^2 = -(v,v)\cdot 1_A \qquad (*)$$

$(*)$ is equivalent to $\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = -2(v,w)\cdot 1_A$.

Def$^{\underline{n}}$:

Let V be a v.s. over $K = \mathbb{R}$ or $\mathbb{C}$ w/ a symmetric (or hermitian) bilinear form $(\cdot,\cdot)$, and

$$\bigotimes V = K \oplus \bigoplus_{n=1}^{\infty} V^{\otimes n}.$$

Then define $\mathrm{Cliff}(V) := \bigotimes V \big/ (v\otimes v + (v,v)\cdot 1)$, as an algebra over K.

Def$^{\underline{n}}$:

A Clifford Algebra for V is a pair (A, φ) where

(1) A is a unital algebra,
(2) $\varphi : V \hookrightarrow A$ is a map of v.s. such that $\varphi(v)^2 = -(v,v)\cdot 1$,
(3) If there exists (A', φ') satisfying (1) and (2), then

$$\begin{tikzcd} V \arrow[r, hook, "\varphi"] \arrow[dr, "\varphi'"'] & A \arrow[d, dashed, "\exists!\eta"] \arrow[dl, phantom, "\circlearrowleft"] \\ & A' \end{tikzcd}$$

Prop: $\mathrm{Cliff}(V)$ is a Clifford Algebra, and it is unique up to isomorphism.

Proof: Check existence, uniqueness by abstract nonsense. $\square$

Ref: Atiyah, Bott, Shapiro, "Clifford Modules..." (google it)

$$\mathrm{Cliff}(\mathbb{R}^1) \cong \mathbb{C}, \qquad \mathrm{Cliff}(\mathbb{R}^2) \cong \mathbb{H}$$

$$\mathrm{Cliff}(\mathbb{R}^3) \cong \mathbb{H} \oplus \mathbb{H}.$$

Remarks:
- If $(\cdot,\cdot) \equiv 0$, $\mathrm{Cliff}(V) = \otimes V / (v \otimes v) \cong \Lambda^\bullet V$
- In general, $\mathrm{Cliff}(V) \xrightarrow{\sim} \Lambda^\bullet V$ as vector spaces, $\dim(\mathrm{Cliff}(V)) = 2^{\dim V}$.

The degree Filtration on $\bigotimes V$

$\wr$ induces

A Filtration on $\mathrm{Cliff}(V)$.

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n$$

$$F_q \cdot F_r \subset F_{q+r} \; ; \quad Gr_q \mathrm{Cliff}(V) = F_q / F_{q-1} = \Lambda^q V$$

$$Gr_\bullet \mathrm{Cliff}(V) \cong \Lambda^\bullet V$$

$$Gr_\bullet \mathrm{Cliff}(V) = \bigoplus_{q=0}^{n} Gr_q \mathrm{Cliff}(V).$$

Dirac Operators.

Fix V to be a $\mathbb{R}$ v.s. (w/ inner product), let S be a v.s. over $K=\mathbb{R}$ or $\mathbb{C}$ that is also a left module for $Cliff(V)$.

$$\underset{\text{module}}{Cliff(V)} \longleftrightarrow \begin{array}{l} c: V \longrightarrow End_K(S) \\ \mathbb{R}\text{-linear and } c(v)^2 = -(v,v)\cdot \mathbb{1}_S \\ c(v\otimes w) = c(v)c(w) \end{array}$$

Remark: In the case $K=\mathbb{C}$, can extend the action to a $Cliff_{\mathbb{C}}(V) := Cliff(V)\otimes_{\mathbb{R}} \mathbb{C}$ action.

Example:

$\Lambda^{\bullet} V$ is a $Cliff(V)$-mod when equipped with

$$c(v) = \epsilon(v) + \imath(v),$$

where for $w \in V$,

$$\epsilon(v)w = v\wedge w \quad \text{(exterior product)}$$

$\imath(v)$ is "adjoint" to $\epsilon(v)$,

$$(\imath(v)q, r) = (q, \epsilon(v)r)$$

Can take: $\iota(v)w = \langle v, w\rangle$ for $w \in \Lambda^1 V = V$, and extend by
$\iota(v)(w_1\wedge w_2) = (\iota(v)w_1)\wedge w_2 + (-1)^{\deg(w_1)} w_1 \wedge \iota(v) w_2$.

Exercises: Check that $c(v)^2 = -(v,v)$.

Let $e_1, \dots, e_n$ be a basis for V, and define

$$D(\cdot) = \sum_{i=1}^{n} c(e_i) \cdot \left[\frac{\partial}{\partial x_i}(\cdot)\right] : C^\infty(V;S) \to C^\infty(V;S).$$

Then
$$\begin{aligned} D^2 s &= \sum_{i,j} c(e_j)\, \partial_j [c(e_i) \partial_i s] \\ &= \sum_{i,j} c(e_j) c(e_i) \partial_i \partial_j s \\ &= \sum_i c(e_i)^2 \partial_i^2 s + \underbrace{\sum_{i<j} [c(e_i)c(e_j) + c(e_j)c(e_i)] \partial_i \partial_j s}_{0} \\ &= -\sum_i \partial_i^2 s \end{aligned}$$

Think: this is the Dirac operator for triv. bundle $\begin{matrix} V \times S \\ \downarrow \\ V \end{matrix}$ with trivial conn.

$$\nabla = dx_i \wedge \frac{\partial}{\partial x_i} = d.$$

Def^{n}: Let S be a bundle of Cliff-mod over a Riemannian manifold M. S is a Clifford bundle if it is equipped with $h(\cdot,\cdot)$ and a compatible connection ∇^S s.t.

S_m is a $\mathrm{Cliff}(T_m^* M)$-mod

Hermitian (on each fibre)

(1) The Clifford action $\mathrm{Cliff}(T_m^* M)$ is skew-adjoint for all m:

$\alpha \in T_m^* M$ and $s_1, s_2 \in S_m$, $h_m(c(\alpha)s_1, s_2) + h_m(s_1, c(\alpha)s_2) = 0$.

(2) ∇^S is compatible with the Levi-Civita (LC) connection on T^*M:

$$\nabla^S[c(\alpha)s] = c[\nabla^{LC}(\alpha)]s + c(\alpha)\nabla^S s.$$

Here: • $\alpha \in \Gamma(T^*M)$
• $s \in \Gamma(S)$
• $\nabla^{LC}: \Gamma(T^*M) \longrightarrow \Gamma(T^*M^{\otimes 2})$,
• $\nabla^S: \Gamma(S) \longrightarrow \Gamma(S \otimes T^*M)$.

Def^n: The Dirac operator D on a Clifford bundle S is the first order operator

$$D: \Gamma(S) \xrightarrow{\nabla^S} \Gamma(S \otimes T^*M) \xrightarrow{c} \Gamma(S).$$

Let $\alpha_i : U \longrightarrow T^*M$, $i = 1, \dots, n$, $(e_i : U \to TM)$

$U \subset M$

α_i are dual to e_i, i.e. $\alpha_i(e_j) = \delta_{ij}$

(or $\alpha_i(e_j) = g_{ij}$)

Then $Ds = \sum_{i=1}^{n} c(\alpha_i)\nabla^S_{e_i} s$.

Break time!

Weitzenböck/Lichnerowicz Formula.

We wish to compute D^2 and see how it is related to the Hodge Laplacian.

Let $\{e_i\}_{i=1}^n$, $\{\alpha_i\}_{i=1}^n$ be dual orthonormal local coordinate systems.

Assume these are "synchronous" coords at a point $m \in M$:

$$(1)\ \nabla_i^{LC}\alpha_j = 0,\ \nabla_i^{LC} e_j = 0,\ (2)\ [e_i, e_j] = 0 \ \text{(at a point } m\text{)}.$$

Compute $D^2 s$ at $m \in M$ in these coordinates:

$$D^2 s = \sum c(\alpha_j)\nabla_j^S\left[c(\alpha_i)\nabla_i^S s\right]$$

$$\overset{(1)}{=} \sum c(\alpha_j)c(\alpha_i)\nabla_j^S\nabla_i^S s$$

$$\overset{(2)}{=} -\sum_i \nabla_i^2 s + \overbrace{\sum_{j<i} c(\alpha_j)c(\alpha_i)\underbrace{\left[\nabla_j^S\nabla_i^S - \nabla_i^S\nabla_j^S\right]}_{=\Omega_{\nabla^S}(e_j,e_i),\ \text{curvature of } \nabla^S}}^{\text{call this "}K\text{"}}$$

$\Omega_{\nabla^S}(e_j,e_i) \in \operatorname{End}(S)$? (no need to worry about $\nabla_{[i,j]}$ term due to synchronous coords)

Remark: $-\sum_i \nabla_i^2$ is the coordinate expression for the Laplacian $\nabla^*\nabla$, where

$$\nabla^* : \Gamma(S \otimes T^*M) \to \Gamma(S)$$

is the "formal" adjoint of ∇^S:

$$\langle s, r\rangle_S := \int_M h(r,s)\,\mathrm{vol}$$

Let $s \in \Gamma(S)$, $r \in \Gamma(S \otimes T^*M)$, then

$$\langle \nabla s, r \rangle_{S \otimes T^*M} = \langle s, \nabla^* r \rangle_S ,$$

and define

$$h(\nabla s, r)_{S\otimes T^*M} \, vol = h(s, \nabla^* r)_S \, vol + \underbrace{d(\Omega_{n-1})}_{\text{some exact } n\text{-form.}}$$

<u>So:</u> in coordinate free notation, write

$$\boxed{D^2 = \nabla^* \nabla + K} \quad \text{W/2 formula}$$

<u>Theorem (Bochner):</u>

Let M be compact. If $K_m \in End(S_m)$ has least eigenvalue > 0 for all $m \in M$, then there are no non-trivial sol$\underline{^{ns}}$ to $D^2 s = 0$.

<u>Proof:</u>

$\langle Ks, s \rangle_S \geq c \langle s, s \rangle_S$ for some $c > 0$.

$\langle Ks, s \rangle = \langle \cancelto{0}{D^2 s}, s \rangle - \|\nabla s\|^2 \leq 0$ — contradiction.

$\square$

From Roe:

$$K = \overbrace{\underbrace{R^S}_{\text{Riemann-curv. of } M}}^{= \frac{1}{4} R \cdot \mathbb{1}_S} + F^S \leftarrow \text{Twisting curvature}$$

When M is Spin, then $F^S = 0$.

next week

So, $D^2 = \nabla^* \nabla + \frac{1}{4} R \cdot \mathbb{1}_S$.

Theorem:

If $R > 0$ and M is compact, there are no non-trivial solutions to $D^2 s = 0$.

Example:

Let
$$\begin{array}{c} S = \Lambda^{\bullet} TM \otimes \mathbb{C} \\ \downarrow \\ M \end{array}$$

The $Cliff_{\mathbb{C}}(T^{*}M)$ structure is given fibrewise by

$$c(\alpha) = \epsilon(\alpha) + \iota(\alpha), \quad \alpha \in T^{*}_{m} M.$$

Calculations (exercises) show this is a Clifford bundle.

What is the Dirac operator for this Clifford bundle?

$$D\omega = \sum_i c(\alpha_i) \nabla_i \omega, \quad \omega \in \Gamma(\Lambda^{\bullet} TM \otimes \mathbb{C})$$
$$= \sum_i \alpha_i \wedge \nabla_i \omega + \sum_i \iota(\alpha_i) \nabla_i \omega.$$

If ∇ is a torsion-free connection, $\nabla_{[\mu} \omega_{\nu]} = \partial_{\mu} \omega_{\nu}$.

So, $\boxed{d = \epsilon \nabla}$ $d : \Lambda^{\bullet} TM \to \Lambda^{\bullet+1} TM$.

$\nabla \omega \in \Gamma(\Lambda^{\bullet} TM \otimes T^{*}M) \to \Gamma(\Lambda^{\circ} TM)$

Then $\boxed{D = d + d^{*}}$ where $d^{*} = * d *$.