11/06/13

Atiyah-Singer Index Theorem Seminar.

Characteristic classes (Lee Cohn).

Main ref: Milnor & Stasheff.

Goal: What is an Euler class?

Will need orientation of vector bundles.

Review (orientation): (Orientation class)

Let V be an oriented v.s. of dim n, V_0 = set of nonzero vectors

Choose an orientation preserving embedding $\sigma : \Delta^n \to V$ such that $\sigma(0) = 0$.

Then $\sigma =: u_V \in H_n(V, V_0; \mathbb{Z}) = \mathbb{Z}$. relative homology

Similarly, $\mu_V \in H^n(V, V_0; \mathbb{Z}) = \mathbb{Z}$.

u_V, μ_V are canonical generators for n^{th} (co)homology, that come from the orientation.

Moral: Choosing an orientation is equivalent to choosing a generator for top (co)homology.

Theorem:

Let E be an oriented n-plane bundle; then $\exists!\ \mu \in H^n(E, E_0; \mathbb{Z})$ such that

$$\mu\big|_{(F,F_0)} \in H^n(F, F_0; \mathbb{Z}) = \mathbb{Z} \text{ is a generator for each fiber } F.$$

Furthermore, $y \longmapsto y \cup \mu$ gives an isomorphism

$$H^k(E, \mathbb{Z}) \longrightarrow H^{k+n}(E, E_0; \mathbb{Z}) \text{ for all } k.$$

In other words, $H^\bullet(E; E_0; \mathbb{Z}) = H^\bullet(E; \mathbb{Z})[\mu]$, $\deg(\mu) = n$.

Since, $H^\bullet(E; \mathbb{Z}) \cong H^\bullet(B; \mathbb{Z})$,

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \pi \\ & & B \end{array}$$

($B \hookrightarrow E$ as zero section, then deformation retract onto zero section).

we have the Thom Isomorphism

$$\varphi : H^k(B; \mathbb{Z}) \longrightarrow H^{k+n}(E; E_0; \mathbb{Z})$$

$$\varphi(x) = (\pi^* x) \cup \mu.$$

can pull back x to E, cup with $\mu \to$ shifts deg up by n

We have an inclusion of relative pairs

$$(E; \emptyset) \hookrightarrow (E; E_0)$$

which induces the restriction,

$$H^*(E, E_0; \mathbb{Z}) \xrightarrow{\text{res}} H^*(E; \mathbb{Z})$$
$$y \longmapsto y|_E$$

In particular, $\mu \mapsto \mu|_E$ is called the Thom class.

Definition:

The Euler Class $e(E) \in H^n(B, \mathbb{Z})$ is the el$^{\pm}$ corresponding to $\mu|_E$.

In other words, if $i : B \hookrightarrow E$, then $e(E) = i^*(\mu|_E)$.

Property:

orient. pres. n-plane bundles

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & f & \downarrow \\ B & \longrightarrow & B' \end{array}$$

then $e(E) = f^* e(E')$

$f^* : H^n(B'; \mathbb{Z}) \to H^n(B; \mathbb{Z})$

Application: If E is trivial bundle then $e(E) = 0$. $\blacksquare$

Property: If n odd, then $2e(E) = 0$.

Proof:

Thom iso. says $\varphi(x) = (\pi^* x) \cup \mu$.

$\pi^* e(E) \cup \mu = (\mu|_E) \cup \mu = \mu \cup \mu$.

So $e(E) = \varphi^{-1}(\mu \cup \mu)$, and $\mu \cup \mu$ is of order 2. since cup product is graded commutative

$\square$

Property: $e(E\times E') = e(E)\overset{\text{cross prod.}}{\times} e(E')$ & $e(E\oplus E') = e(E)\cup e(E')$

Remark:
$$\begin{array}{ccc} E\times E' & \text{and} & E\oplus E' \\ \downarrow & & \downarrow \\ B\times B' & & B \end{array}$$

Proof:

$$\mu(E\times E') = (-1)^{mn}\mu(E)\times\mu(E') \quad \text{(apply restriction)}$$

Reminder: $\det(A^n\times B^m) = \det(A)^n\det(B)^m$.

$$H^{m+n}(E\times E', (E\times E')_0) \longrightarrow H^{m+n}(E\times E') \cong H^{m+n}(B\times B').$$

It follows that $e(E\times E') = (-1)^{mn} e(E)\times e(E')^k$ [not odd]

Pulling back along $\Delta: B\to B\times B$ gives the corresponding statement for $E\oplus E'$ (pulling back cross product gives cup product). $\boxtimes$

Property: If $\overset{\text{section}}{S}: B\longrightarrow E_0 \subset E$ then $e(E)=0$.

Proof:

$$\begin{tikzcd} B \arrow[r, "S"] \arrow[rrr, bend right, "id"'] & E_0 \arrow[r, hook] & E \arrow[r, "\pi"] & B \end{tikzcd} \Rightarrow \begin{tikzcd} H^n(B) \arrow[r, "\pi^*"] \arrow[rrr, bend left, "id"] & H^n(E) \arrow[r] & H^n(E_0) \arrow[r, "S^*"] & H^n(B) \end{tikzcd}$$

$$u\ e(E) \mapsto \mu|_E \mapsto (\mu|_E)_{E_0} = 0 \mapsto 0$$

Why? ~~Relative~~ Restriction gives exact sequence

$$\underbrace{H^n(E,E_0) \to H^n(E) \to H^n(E_0)}_{0}$$

Next Goal: If M^n – smooth, compact oriented manifold, then
$$\langle e(TM), \mu_M \rangle = \chi(M).$$

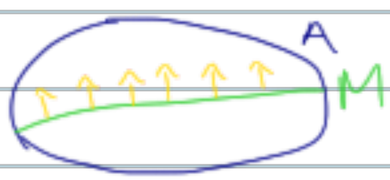
Fact: If $M^n \subset A^{n+k}$ as Riemannian mflds and $E \to M$ is oriented normal bundle of M in A, then
$$H^\bullet(E, E_0; \mathbb{Z}) \cong H^\bullet(A, A-M; \mathbb{Z})$$
$$\mu \longleftrightarrow \mu'$$

not so secretly is the tubular nbhd thm

Corollary:

If $M \subset A$, then
$$H^k(A; A-M) \longrightarrow H^k(A) \longrightarrow H^k(M)$$
$$\mu \longmapsto e(\nu^k) = e(E)$$
normal bdle rank k

Picture:



Application: If $A = \mathbb{R}^{n+k}$ then $H^k(A) = 0$, so $e(\nu^k) = 0$.

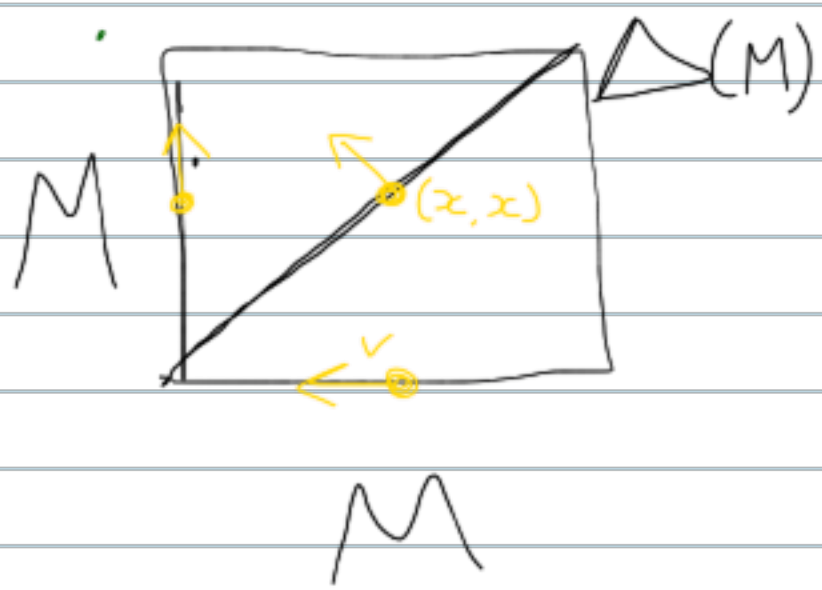
Remark: The image of μ' in $H^k(A)$ is the cohom. class dual to submfld M of codim k.

Lemma:

The normal bundle ν^n of $\Delta: M \longrightarrow M\times M$ is $\cong$ to TM.

$$TM \longrightarrow \nu^n$$

$(x,v) \longmapsto ((x,x),(-v,v))$ gives an $\cong$ of bundles



Now, let $\mu' \in H^n(M\times M, M\times M - \Delta)$ be an orientation class.

The restriction of μ' to Δ is $e(\nu^n) \cong e(TM)$ by the lemma.

Also, $\mu'' := \mu'|_{M\times M} \in H^n(M\times M)$ is "dual" to the Δ.

Property:

For $a \in H^{\cdot}(M)$, $(a\times 1) \cup \mu'' = (1\times a)\cup \mu''$.

Meaning: μ'' is supported on the diagonal.

See [Milnor] for proof.

There is a slant product/cap product,

$$H^{p+q}(X\times Y)\otimes H_q(Y) \longrightarrow H^p(X),$$

given by $a\times b/\beta = a\langle b,\beta\rangle$. (hom-cohom pairing).

For fixed $\beta \in H_{\bullet}(Y)$, the map $p \mapsto p/\beta$ is $H^{\cdot}(X)$-linear:

$$((a\times 1)\cup p)/\beta = a\cup p/\beta \text{ for every } a\in H^{\cdot}(X) \text{ and } p\in H^{\cdot}(X\times Y).$$

Lemma:

Given $\mu_M \in H_n(M)$ fundamental class, then

$$\mu'' / \mu_M = 1 \in H^0(M),$$

Sketch: $H^n(M \times M) \otimes H_n(M) \xrightarrow{/\mu_m} H^0(M)$

$\underset{pt}{*} \times M \to M \times M \leftarrow$ [Milnor] for proof.

Poincaré Duality:

If $\{b_i\}$ basis for $H^\bullet(M)$ there is a dual basis $\{b_i^\vee\}$ of $H^\bullet(M)$ such that $\langle b_i \cup b_j^\vee, \mu_M \rangle = \delta_{ij}$.

Proof:

$H^\bullet(M \times M) \cong H^\bullet(M) \otimes H^\bullet(M)$ [Assume no tor for simplicity]

The diagonal class $\mu'' = b_1 \times c_1 + \cdots + b_r \times c_r$ for some c_r, $\deg b_i + \deg c_i = n$.

Apply $/\mu_m$ to $(a \times 1) \cup \mu''$ to get

LHS: $(a \times 1) \cup \mu'' / \mu_M = a \cup \mu'' / \mu = a$

RHS: $(1 \times a) \cup \mu'' / \mu_M = \sum (-1)^{\bar{a}\bar{b}_j} (b_j \langle a \cup c_j, \mu_m \rangle)$ Now Plug in b_i for a. $\bar{a} = \deg(a)$

This implies the coeff of $b_j = \delta_{ij}$. This simplifies $b_i^\vee = (-1)^{\bar{b}_i} c_i$. $\square$

Corollary: $\mu'' = \sum_i (-1)^{\bar{b}_i} b_i \times b_i^\vee$.

Corollary:

If M is smooth, cmpt, oriented, then

$$\langle e(TM), \mu_M \rangle = \chi(M).$$

Proof:

Since $e(TM) = \Delta^* \mu''$, by $\mu'' = \sum (-1)^{\bar{b}_i} b_i \times b_i^\vee$, we get

$$e(TM) = \sum (-1)^{\bar{b}_i} b_i \cup b_i^\vee.$$

Apply $\langle\ , \mu_M \rangle$ to both sides

$$\langle e(TM), \mu_M \rangle = \sum (-1)^{\bar{b}_i} = \sum (-1)^k \dim H^k(M) = \chi(M).$$

$\square$

We have our first index theorem!

Time for a break...

Application: $\chi(S^n) = \begin{cases} 2 & n\text{-even} \\ 0 & n\text{-odd} \end{cases}$

$\Rightarrow e(TS^n) \neq 0$ if n is even.

$\Rightarrow S^n$ is not parallelizable if n-even.

Goal: What is the Todd Class? [Important for future lectures]

Let $\begin{array}{c} L \\ \downarrow \\ B \end{array}$ be a complex line bundle.

Def$^{\underline{n}}$: $c_1(L) \in H^2(B)$, the first chern class, is the Euler class of the corresponding rank 2 vector bundle.

Remark: L is oriented.

Theorem: $\left\{ \begin{array}{l} \text{Iso classes of} \\ \text{cmplx line bundles} \\ L \text{ on } B \text{ a curve} \end{array} \right\} \Big/ \left(\begin{array}{l} \text{topological} \\ \text{equivalence} \end{array} \right) \xleftrightarrow{\text{bijection}} \{ c_1(L) \in H^2(B) \}$

Remark: Divisors up to linear equivalence are in correspondence with holomorphic line bundles.

Technical Lemma:

$\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ rank n complex v.b. Then $\exists\, p: Y \longrightarrow X$ such that

1) $p^*: H^\circ(X) \longrightarrow H^\circ(Y)$ is injective & $H^\circ(Y,\mathbb{Z}) = H^\circ(X,\mathbb{Z})[1, y, \dots, y^{n-1}]$
$\deg(y) = 2$

2) $\begin{matrix} p^*E \\ \downarrow \\ Y \end{matrix} \cong L_1 \oplus L_2 \oplus \cdots \oplus L_n$ as bundles on Y, where each L_i is a line bundle.

Remark: Injectivity of p^* is important because any eq^{n} which holds in $H^\circ(Y)$ also holds in $H^\circ(X)$.

Def^{n}: The $c_1(L_i)$ are called the Chern Roots of E.

There is a series

$$Q(x) = \frac{x}{1 - e^{-x}}$$

Def^{n}: $Td(E) = \prod_i Q(c_1(L_i))$ is called the Todd Class.

Since $y^k \in H^{2k}(Y, \mathbb{Z})$ satisfies some poly^{n} eq^{n}:

$$y^k + c_1(E) y^{k-1} + c_2(E) y^{k-2} + \dots + c_k(E) = 0$$

for some coefficients $c_i(E) \in H^{2i}(X, \mathbb{Z})$.

These are called the i^{th} Chern class.

$$Td(E) = 1 + \frac{c_1}{2} + \frac{(c_1^2 + c_2)}{12} + \frac{c_1 c_2}{24} + \dots$$

Chern Class:

Let L be a line bundle $\rightsquigarrow c_1(L)$.

$$Ch(L) = e^{c_1(L)} = \sum \frac{c_1(L)^k}{k!}.$$

Example:

$$c_i(T\mathbb{CP}^n) = \binom{n+1}{i}\alpha^i \text{ for } 1 \leq i \leq n.$$

$$\alpha \in H^2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$$

For future talks:

For $\mathcal{F}$ a coherent sheaf, $\chi(M, \mathcal{F}) = \int_X Ch(\mathcal{F}) Td(TM)$.